Inner radiation emitted during $\beta^{\pm}$ decay and related problems

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In the present study we consider the radiation emitted during $\beta^{\pm}$ decay and a few related problems of importance in applications. The so-called initial inner bremsstrahlung is emitted by the accelerated $\beta^{\pm}$ particle at very short distances from the maternal nucleus. Then, the $\beta^{\pm}$ particle moves with constant velocity through the electronic shells of the maternal atom (ion) and emits the inner bremsstrahlung. Later the $\beta^{\pm}$ particle leaves the maternal atom (ion) and begins to pass (with the same constant velocity) through the surrounding atoms and emits the usual bremsstrahlung. Likewise, both the maternal and surrounding atoms can be excited and even ionized either by the instantaneous change of the nuclear charge or by such a penetrating $\beta^{\pm}$ particle. Then the excited atoms transit to the ground state emitting the so-called atomic transition radiation, which may be observed during or after $\beta^{\pm}$ decay. Our present analysis includes the inner bremsstrahlung and inner atomic transition radiation, i.e., both continuous and discrete radiation from the maternal $\beta^{\pm}$-decaying atom (ion). [S1050-2947(98)04108-0]

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I. INTRODUCTION

In the present paper we consider processes related to nuclear $\beta^{\pm}$ decay. As is well known, β decay is of great interest in various applications related to modern technology. Let us mention only a few widely known β^{-}-decaying nuclei, which are of paramount importance for technology: ^{3}H, ^{233}Th, ^{239}U, and ^{247}Pu. Tritium is of interest for thermonuclear fusion and some other problems, while these three heavy isotopes are used in production of the so-called well-fissionable nuclei: ^{233}U, ^{239}Pu, and ^{247}Cm [1], respectively. A large number of the β^{-} nuclei can be found among the fission fragments, which are produced in nuclear reactors and nuclear explosions (see, e.g., [3] and [4] and references therein). Most such nuclei, in particular ^{90}Sr and ^{137}Cs, are extremely dangerous for people and nature. Also, there are quite a few purely scientific problems where $\beta^{\pm}$ decay is the leading process. For instance, the neutron's β^{-} decay is very important for the theory of weak interactions.

Actually, $\beta^{\pm}$ decay is a quite complicated phenomenon, which includes a number of different steps, e.g., formation of the $\beta^{\pm}$ particle inside of a nucleus, its escape from the nucleus, motion of the emitted $\beta^{\pm}$ particle outside the nucleus, interaction with electronic shells of the maternal atom, penetration through the electronic shells of other atoms, etc. In the present study we do not wish to discuss the nuclear part of the problem, and restrict ourselves to analysis of the emitted radiation and related atomic problems. In principle, there are two different types of radiation related with the $\beta^{\pm}$ decay. The interaction between the emitted $\beta^{\pm}$ particles and atomic nuclei produces the so-called continuous radiation or bremsstrahlung. The discrete radiation is related mainly with changes in the atomic electronic shells arising either during or after the nuclear $\beta^{\pm}$ decay.

The continuous radiation analysis includes the following two steps. The first is to solve the equations of motion, and the second is to consider the radiation emitted during such motion. Since the velocity of the emitted particle changes along the trajectory, the radiation analysis contains also a few different stages, e.g., the initial inner bremsstrahlung, the inner bremsstrahlung, and the usual bremsstrahlung. Below, the inner radiation means always the radiation that is emitted by the $\beta^{\pm}$ particle inside the maternal atom. Our present main goal is to develop the theory that can represent quite well the experimental results known for the inner bremsstrahlung emitted during the $\beta^{\pm}$ decay [5]. This problem is solved in the next two sections. Note that in the present work the inner bremsstrahlung caused by the nuclear $\beta^{\pm}$ decay is studied in both relativistic and nonrelativistic approximations. Also, we consider the two cases when positive and negative particles are emitted from the positively charged nucleus. In other words, the found results can be applied to nuclear α decay, nuclear fission, and other nuclear processes as well.

The discrete radiation or atomic (ionic) transition radiation is discussed in detail in the fourth section. The main problem here is to calculate the respective excitation probabilities for ions produced during the $\beta^{\pm}$ decay. Indeed, if such probabilities are known, then the spectrum and intensities of the emitted radiation can be easily predicted. Here the so-called sudden approximation is extensively used. Such an approximation works very well if the perturbation, which may have a quite large amplitude, acts only for a very short time. In this section some examples are considered (the tritium atom, He-like ions, etc. with $\beta^{\pm}$-decaying nuclei) where either analytical or numerical solutions can be found. The concluding remarks are given in the final section.

II. EQUATIONS OF MOTION

Let us consider a nucleus that has a perfect spherical form with radius R and is placed at the origin point $x=0$, $y=0$, and $z=0$. At the initial time $t=0$, a charged particle with charge e and mass m is emitted from this nucleus. Its mass m is significantly less than the mass of the nucleus M, i.e., $m \ll M$. Without loss of generality, we may assume that the emitted particle begins to move along the z axis. The remaining nucleus can be considered as immovable, since its mass

is significantly larger than the mass of the emitted particle. The charge of this remaining (or ''final'') nucleus is designated below as Q. The principal question is the formation of the charged $\beta^{\pm}$ particle inside of the nucleus and its escape from this (i.e., maternal) nucleus. We shall assume that the emitted particle appears at the point z_0 ($z_0 \geqslant R$) on the z axis, with velocity vector $\vec{v}=(0,0,\dot{z}_0)$. Actually, this means that the emitted $\beta^{\pm}$ particle moves in the positive z direction with the initial speed $\dot{z}_0$. Below only the frame related with the immovable, heavy nucleus is used. In this frame the electric field acting on the emitted particle (at $z \geqslant z_0$) equals $E_z = -Q/z^2$, while the magnetic field is identically zero, as are the x and y components of the electric field. Now, the appropriate equations of motion can be considered in detail.

A. The relativistic case

The relativistic equations of motion take the form

$$\dot{p}_x=0,\ \dot{p}_y=0,\ \dot{p}_z=eE_z,\quad \dot{\mathcal{E}}=eE_z\dot{z}, \tag{1}$$

where $E_z=-Q/z^2$ and $p_z=\mathcal{E}\dot{z}/c^2$. The initial conditions are $z(t=0)=z_0$ and $\dot{z}(t=0)=\dot{z}_0$. In these equations and everywhere below, $\mathcal{E}$ is the kinetic energy of the emitted particle. The charge of the particle, e, is negative (positive) for β^- (β^+) decay. From the last two equations follows the equation

$$z^2\mathcal{E}\ddot{z}=eQ(\dot{z}^2-c^2)=-eQc^2\left(1-\frac{\dot{z}^2}{c^2}\right)=-eQc^2\gamma^{-2}, \tag{2}$$

where $\gamma=(1-\dot{z}^2/c^2)^{-1/2}$ and $\gamma \geqslant 1$ always. In the ultrarelativistic case the right-hand side of this equation vanishes, and therefore $\ddot{z}=0$, i.e., the particle does move with constant velocity [6]. However, it should be mentioned that for real β decay such an approximation is not quite accurate, since the maximal value of $\gamma=(1-\dot{z}^2/c^2)^{-1/2}$ does not exceed 25–30. Moreover, in many actual cases $\gamma \approx 2-5$. Now, by using $\mathcal{E}=\gamma mc^2$, we reduce the last equation to the form

$$\ddot{z}=-\frac{eQ}{mz^2}\gamma^{-3}=-\frac{eQ}{mz^2}\left(1-\frac{\dot{z}^2}{c^2}\right)^{3/2}. \tag{3}$$

The first integral for this equation takes the form

$$\gamma+\frac{eQ}{mc^2}\frac{1}{z}=\gamma_0+\frac{eQ}{mc^2}\frac{1}{z_0}=b. \tag{4}$$

The last expression has a quite transparent physical sense: the sum of the kinetic and potential energies (divided by mc^2) is a constant ($=b$) upon z. It can be written also in another form:

$$\gamma=\frac{1}{\sqrt{1-\dot{z}^2/c^2}}=\frac{(\gamma_0+eQ/mc^2z_0)z-eQ/mc^2}{z}=\frac{bz+a}{z}, \tag{5}$$

where $a=-eQ/mc^2$. Note that the parameter b is always positive (otherwise the emitted particle cannot move away from the maternal nucleus), while a may have either sign.

After a few simple transformations the last equation is reduced to the following form:

$$\dot{z}=c\sqrt{1-\left(\frac{z}{a+bz}\right)^2}=c\frac{\sqrt{a^2+2abz+(b^2-1)z^2}}{a+bz} \tag{6}$$

or

$$ct=\int_{z_0}^{z}\frac{(a+bz)dz}{\sqrt{a^2+2abz+(b^2-1)z^2}}. \tag{7}$$

The final analytical expression for this integral depends significantly on the following two values: (i) $\Delta=4a^2(b^2-1)-4a^2b^2=-4a^2$, which is always negative, and (ii) b^2-1, which can be either positive or negative. Since $b>0$, the question about the value of b^2-1 is reduced to the comparison of b and 1.

When $eQ>0$ (e.g., β^+ decay), the parameter $b=\gamma_0+eQ/mc^2z_0>1$. Therefore, in this case $b^2-1>0$ always, and the last equation takes the form

$$b\sqrt{R(z)}-a\ \ln[S(z)]=(b^2-1)^{3/2}ct+b\sqrt{R(z_0)}-a\ \ln[S(z_0)], \tag{8}$$

where $R(z)=a^2+2abz+(b^2-1)z^2$ and $S(z)=\sqrt{(b^2-1)R(z)}+(b^2-1)z+ab$. From this equation one finds the explicit dependence $z=z(t)$. For the β^- decay the case when $eQ<0$ and $b>1$ is also possible. But now of the two inequalities $\gamma_0 \geqslant 1$ and $\gamma_0 \geqslant 1-eQ/mc^2z_0$, the last one is stronger, since $eQ<0$. From this inequality after a few simple transformations one finds the so-called threshold condition:

$$\dot{z}_0 \geqslant c\frac{\sqrt{\delta(2+\delta)}}{1+\delta}, \tag{9}$$

where $\delta=-eQ/mc^2z_0=a/z_0$. Actually, this is the relation between $\dot{z}_0$ and z_0, which includes some other physical parameters. Obviously, the right-hand side in the last inequality is always less than c.

For the β^- decay the case when $eQ<0$ and $b<1$ or $1 \leqslant \gamma_0 \leqslant 1-eQ/mc^2z_0$ is also possible. Actually, this inequality means that the particle cannot move to an infinite distance from the maternal nucleus, i.e., the classical or spontaneous β^- decay is impossible. The nonspontaneous or stimulated β^- decay requires a number of very specific conditions and cannot proceed easily. Furthermore, such a process was never observed experimentally. For this reason we restrict ourselves below only to the situation when $b>1$.

B. The nonrelativistic case

The equation of motion for the nonrelativistic case is obvious:

$$\dot{p}_z=eE_z \quad \text{or} \quad m\ddot{z}=-\frac{eQ}{z^2}, \tag{10}$$

where again as before $z(t=0)=z_0$ and $\dot{z}(t=0)=\dot{z}_0$. By multiplying both sides of this equation with $\dot{z}$, and integrating the equation we find the following equation:

$$\dot{z}=\sqrt{\frac{bz+a}{z}}=\sqrt{b+\frac{a}{z}}, \tag{11}$$

where $a=-2eQ$ and $b=2eQ/mz_0+\dot{z}_0^2$. Note that $b>0$ always (otherwise, there is no decay [7]), while a can be either positive or negative. Nevertheless, the sum $bz+a$ is always positive. Finally, after integration the last equation takes the form

$$\sqrt{bz(bz+a)}-\sqrt{bz_0(bz_0+a)} - a\ \ln\frac{\sqrt{bz}+\sqrt{bz+a}}{\sqrt{bz_0}+\sqrt{bz_0+a}}=b\sqrt{b}t. \tag{12}$$

This can be written also in a different form:

$$\sqrt{R_{\rm nr}(z)}-a\ \ln[S_{\rm nr}(z)]=b^{3/2}t+\sqrt{R_{\rm nr}(z_0)}-a\ \ln[S_{\rm nr}(z_0)], \tag{13}$$

where $R_{\rm nr}(z)=bz(bz+a)$ and $S_{\rm nr}(z)=\sqrt{bz}+\sqrt{bz+a}$. The subscript ''nr'' means nonrelativistic. In this form the last equation coincides almost exactly with Eq. (8) given above. Such a coincidence is discussed in detail in the next section.

III. THE ANALYSIS OF CONTINUOUS RADIATION EMITTED DURING $\beta^{\pm}$ DECAY

In this section we consider in detail the continuous electromagnetic radiation emitted during $\beta^{\pm}$ decay. Actually, such radiation can be emitted in at least four different stages of this process. In the first stage, there is radiation that is related with the nuclear process of forming the $\beta^{\pm}$ particle inside of the nucleus. This process needs a separate investigation, and obviously, cannot be studied in terms of classical electrodynamics. In the second stage, there is radiation emitted by the accelerated $\beta^{\pm}$- particle at distances $z\approx z_0$ from the maternal nucleus. Such radiation can be called the initial inner bremsstrahlung. Later radiation is emitted when the $\beta^{\pm}$ particle moves (with almost constant velocity) through the electronic shells of either the maternal atom (the so-called inner bremsstrahlung) or surrounding atoms (i.e., the usual bremsstrahlung). These are the third and fourth stages of the process, respectively. Below we consider only the inner bremsstrahlung.

First, we wish to note that in the asymptotic limit ($z\to\infty$) the equations of motion (relativistic as well as nonrelativistic) take the following form:

$$z-z_0-A\ln\left(\frac{z}{z_0}\right)=Bt, \tag{14}$$

where $z\geqslant z_0$, $B\geqslant 0$, while A can be either positive or negative. Now, if z is really very large in comparison with z_0, then the logarithmic term is negligible and one finds from the last equation $z=Bt+z_0$, i.e., the equation for uniform motion with constant velocity ($\dot{z}=B$). Actually, this follows directly from Eqs. (6) and (11) for the relativistic as well as for the nonrelativistic case. The appropriate asymptotic velocities are (i) $\lim_{z\to\infty}\dot{z}=c\,(\sqrt{b^2-1}/b)$, where $b=\gamma_0+eQ/mc^2z_0$, for the relativistic case; and (ii) $\lim_{z\to\infty}\dot{z}=\sqrt{\dot{z}_0^2+2eQ/mz_0}$, for the nonrelativistic case. This shows that in any case after some time the emitted particle begins to move with constant velocity (actually, this time is quite short, see below).

Let us obtain now explicit formulas for the inner radiation emitted by the accelerated $\beta^{\pm}$ particle, which moves according to the equations of motion found in the preceding section. Since the particle moves always along the z axis and, moreover, its acceleration $\ddot{z}$ is parallel to the velocity $\dot{z}$, we can write for the power radiated per unit solid angle, $(dP(t')/d\Omega)$, the following expression:

$$\frac{dP(t')}{d\Omega}=\frac{e^2\ddot{z}^2}{4\pi c^3}\frac{\sin^2\Theta}{(1-\dot{z}/c\cos\Theta)^5} =\frac{e^4Q^2}{4\pi c^3m^2z^4}\frac{(1-\dot{z}^2/c^2)^3\sin^2\Theta}{(1-\dot{z}/c\cos\Theta)^5} \tag{15}$$

in the relativistic case. In this equation and below, Θ is the angle between the z axis and the direction of observation. Actually, to produce the final formula one needs to use here Eq. (6) for $\dot{z}$. The time t' in this formula designates the time when the radiation was emitted by the charge. Obviously, the time t when the emitted radiation reaches the observation point equals $t=t'+R(t')/c$, where R is the distance between the two points in which radiation was emitted and received, respectively. In the nonrelativistic case $t'=t$ and the analogous formula takes the form

$$\frac{dP}{d\Omega}=\frac{e^2\ddot{z}^2}{4\pi c^3}\sin^2\Theta=\frac{e^4Q^2}{4\pi c^3m^2z^4}\sin^2\Theta. \tag{16}$$

It follows from these two formulas that such radiation losses are negligible ($\simeq m^{-2}$) for heavy particles (e.g., for the emitted α particle). Likewise, they decrease very quickly $\simeq z^{-4}$ when the distance z from the maternal nucleus increases.

The total radiated power $P(t')$ (or P for the nonrelativistic case) is also of interest for applications. It can be easily found by integration of $dP/d\Omega$ over the solid angle Ω ($0\leqslant\Omega\leqslant 4\pi$). The appropriate relativistic result is

$$P(t')=\frac{2e^2}{3c^3}\ddot{z}^2\gamma^6=\frac{2}{3}\frac{e^4Q^2}{c^3m^2z^4}, \tag{17}$$

where $\gamma=(1-\dot{z}^2/c^2)^{-1/2}$. Note that the final result does not depend on the γ value. Furthermore, it coincides exactly with the corresponding nonrelativistic result:

$$P=\frac{2e^2}{3c^3}\ddot{z}^2=\frac{2}{3}\frac{e^4Q^2}{c^3m^2z^4}. \tag{18}$$

In some applications the spectral and angular distribution of the emitted radiation is of great interest. This is given by

$$\frac{dI(\omega)}{d\Omega}=\frac{e^2\omega^2}{4\pi^2c^3}\sin^2\Theta\left|\int_0^{+\infty}\exp\left[\imath\omega\left(1-\frac{\dot{z}}{c}\cos\Theta\right)t\right]\dot{z}\,dt\right|^2, \tag{19}$$

where I is the intensity of the emitted radiation and Θ is the angle between the directions of acceleration and observation, respectively. When the emitted particle begins to move with constant velocity v [where $v=c\,(\sqrt{b^2-1}/b)$], the last expression takes the following well-known form [6]:

$$\frac{dI(\omega)}{d\Omega}=\frac{e^2}{4\pi^2c^3}\frac{v^2\sin^2\Theta}{(1-v/c\cos\Theta)^2}. \tag{20}$$

The total radiation intensity $I(\omega)$ can be easily found from this expression:

$$I(\omega)=\frac{e^2}{\pi c}\left(\frac{b}{\sqrt{b^2-1}}\ln\frac{b+\sqrt{b^2-1}}{b-\sqrt{b^2-1}}-2\right), \tag{21}$$

where b is determined by Eq. (4) in the relativistic case. In the nonrelativistic case $b\gg1$ and one finds $I(\omega)=(2e^2/3\pi c)\,(1-1/b^2)$, which is quite small. This means that the emitted radiation is negligible for slow β particles.

The formulas presented here govern the inner bremsstrahlung emitted during $\beta^{\pm}$ decay. They can be integrated numerically over t (or t') by using, e.g., various stepwise procedures. The found results give an evaluation for the inner continuous radiation emitted by the penetrating $\beta^{\pm}$ particle, when it moves inside the maternal atom. In principle, such radiation may interact with the atomic electrons, and finally, the maternal atom can be found not only in its ground state, but also in excited states as well as in the ionized (or unbound) state. This is the so-called direct interaction between emitted and atomic electrons. Another way to excite the electronic states in the $\beta^{\pm}$-decaying atom is related to the instantaneous change of the nuclear charge. It is shown in the next section that the atomic excitation probabilities due to the direct interaction are negligible in comparison with those caused by the changes in the nuclear charge.

IV. ATOMIC EXCITATION IN NUCLEAR $\beta^{\pm}$ DECAY

Let us discuss now the atomic (and ionic) excitations that are produced by nuclear $\beta^{\pm}$ decays. For convenience, in this section the charge q designates the initial nuclear charge (before $\beta^{\pm}$ decay), while the charge $q\pm1$ stands for the final nuclear charge. It is quite clear that the atomic excitations arise due to the two following reasons: (i) the change in the nuclear charge ($q\rightarrow q\pm1$), and (ii) the interaction between the emitted $\beta^{\pm}$ particle and the electrons of the maternal atom. Now, our main problem is to evaluate and compare the appropriate probabilities. Note that this problem was studied for the first time almost 60 years ago [8] (see also Refs. [9–12]). The approach, developed in [8], is based on the so-called sudden approximation, which means that the $\beta^{\pm}$ decay and the following penetration of the emitted fast electron (i.e., $\beta^{\pm}$ particle) are instantaneous processes in terms of the usual atomic time scale. Indeed, the velocity of the emitted $\beta^{\pm}$ particle is quite comparable with the speed of light c, and it is significantly greater than the usual values for atomic electrons. In terms of this it can be shown (for more detail see Refs. [8–12]) that the atomic excitation probability in the zeroth-order approximation is given by the formula

$$\mathcal{M}_{n_1n_2}=|\langle\Psi_{n_1}(q)|\Psi_{n_2}(q\pm1)\rangle|^2, \tag{22}$$

where $\Psi_{n_1}(q)$ and $\Psi_{n_2}(q\pm1)$ are the stationary wave functions of the initial and final atoms (ions) with the nuclear charges q and $q\pm1$, respectively. Obviously, this part of the transition probability arises due to the instantaneous change of the nuclear charge $q\rightarrow q\pm1$. Actually, since the system (i.e., the nuclear charge) changes instantaneously, its original electronic wave function takes the components (in the new basis set) that are proportional to the wave functions of the new ground and excited states. Briefly, this means that after such an instant change the new system can be found, in principle, not only in its ground state, but also in any of its excited states, including the so-called unbound or continuum states. The contribution of the so-called ''direct'' electron-electron interaction between the $\beta^{\pm}$ particle and atomic electrons is $(q\alpha)^2$ times smaller. Since the fine structure constant $\alpha\approx\frac{1}{137}$, the parameter $(q\alpha)^2$ is really small only for small q, e.g., $1\leqslant q\leqslant10$. Below we restrict our consideration to the case of such q values, and unless otherwise specified below only atomic units ($\hbar=1$, $m_e=1$, and $e=1$) are used.

It should be mentioned also that the matrix element presented above contains the exponential function $\exp(-\imath\cdot\vec{v}_n\cdot\Sigma_i\vec{r}_i)$, where $\vec{v}_n$ is the velocity of the recoiling nucleus and $\vec{r}_i$ are the radius vectors of all of the atomic electrons. However, it can be shown that the $\vec{v}_n$ value is negligible in comparison with the typical velocities of atomic electrons. Indeed, in terms of the momentum conservation we can write $M_nv_n=\gamma_\beta m_ev_\beta$, where M_n and m_e are the nuclear and electron masses, respectively. v_β is the velocity of the emitted $\beta^{\pm}$ particle and $\gamma_\beta^{-1}=\sqrt{1-(v_\beta/c)^2}$. The approximation $\exp(-\imath\cdot\vec{v}_n\cdot\Sigma_i\vec{r}_i)\approx1$ is valid only when the ratio $R=v_n/v_e\ll1$, where v_e is the velocity of atomic electrons. But this expression can be written in the form

$$R=\frac{\gamma_\beta m_ev_\beta}{M_nv_e}\approx\frac{\gamma_\beta}{N(m_p/m_e)q\alpha x}\approx\frac{\gamma_\beta}{13.40Nqx}, \tag{23}$$

where N is the total number of nucleons in the nucleus, while q is the effective nuclear charge that acts on the electron. The m_p is the proton mass and the factor x is of order 1. As we have mentioned above, the γ factor for the emitted $\beta^{\pm}$ particle does not exceed 25–30. Actually, this means that if $x=1$, then $R\ll1$ when $N\geqslant10$ and $q\geqslant5$. But by using the known energies of the $\beta^{\pm}$ particles, one finds that $R\ll1$ also for $N\leqslant10$ and $q\leqslant5$, since in these cases the γ factors are significantly less than 30. Moreover, in all known $\beta^{\pm}$ decays the energy released is shared almost entirely by the electron (positron) and the neutrino (antineutrino). Therefore, in general, the recoiling nucleus takes up momentum which is significantly less than that used in the evaluation made above. Only when the $\beta^{\pm}$ decay is studied in the so-called weakly bound atomic systems (i.e., $x\ll1$) may the ratio R be quite comparable with 1, and the so-called recoiling ionization becomes also possible for such a weakly-bound electron. For instance, the recoiling ionization plays some role, if the

$\beta^{\pm}$-decaying atom initially was in one of its highly-excited Rydberg states. This case requires a separate investigation, but below we consider the $\beta^{\pm}$ decay only from the ground or low-lying excited states when $v_n \approx 0$. It should be mentioned that the immovable nucleus means the conservation of the total angular momentum L for the $\beta^{\pm}$-decaying atom. Moreover, for the hydrogenlike atoms this actually means the conservation of the electron angular momentum l.

Now, we evaluate the $\mathcal{M}_{n_1 n_2}$ probabilities for a number of known atoms and ions with β-active nuclei. There are 30 known β-active nuclei (19 of them emit electrons, while 11 emit positrons) with q values that are less than 10. In some cases two or three different energetical groups of β-particles can be observed, e.g., for the ^{15}C, ^{16}N, or ^{21}F nuclei. However, the sudden approximation is applicable very well for each of these groups, i.e., the formulas presented here can be used in this situation as well. First, consider the situation when the initial and final systems are the one-electron or H-like atoms (ions). In this case analytical formulas for $\mathcal{M}_{in,fi}$ can be found. Indeed, by applying the radial parts of the hydrogenlike wave functions in the form [13]

$$\Psi_{n,l}(q,r) = P_{nl}(r) = \frac{\sqrt{qC_{n+l}^{2l+1}(2l+1)!}}{n}\left(\frac{2qr}{n}\right)^{l+1}\exp\left(-\frac{qr}{n}\right)\times\sum_{k=0}^{n-l-1}\frac{(-1)^{k+1}C_{n-l-1}^{k}(2qr/n)^k}{(2l+k+1)!}, \tag{24}$$

one easily finds an analytical finite sum expression for the transition probability $\mathcal{M}_{n_1,n_2}$, which corresponds to the $\beta^{\pm}$ decay from the initial (q_1,n_1,l_1) to the final $[q_1 \pm 1, n_2, l_2(=l_1)]$ states. In this formula, C_n^k are the appropriate binominal coefficients [14]. The final analytical formula for $\mathcal{M}_{n_1,n_2}$ is quite cumbersome, since it contains a double sum. The explicit form is

$$\mathcal{M}_{n_1,n_2} = \frac{\sqrt{q_1 q_2 C_{n_1+l}^{2l+1} C_{n_2+l}^{2l+1}}(2l+1)!}{n_1 n_2}\left(\frac{4q_1q_2}{n_1n_2}\right)^{(l+1)}\sum_{k_1=0}^{n_1-l-1}\sum_{k_2=0}^{n_2-l-1}(-1)^{k_1+k_2}$$
$$\times\frac{C_{n_1-l-1}^{k_1}C_{n_2-l-1}^{k_2}(2q_1/n_1)^{k_1}(2q_2/n_2)^{k_2}(k_1+k_2+2l+2)!}{(2l+k_1+1)!(2l+k_2+1)!(q_1/n_1+q_2/n_2)^{k_1+k_2+2l+3}}, \tag{25}$$

where $l_1 = l_2 = l$. The results of the numerical calculations for a number of transition probabilities are presented in Table I. In this table the β^- decay in the tritium atom is considered. It follows from these results that the probability to find the resulting He^+ ion in one of its bound states is more than 95–97%. Note that in all previous papers the β^- decay was considered only from the ground state of the tritium atom [15–20]. Furthermore, only the ground $(1s)$ and first excited $(2s)$ states in the final He^+ ion were discussed as the final states. In our present study the appropriate generalization is given for the excited atomic (both initial and final) states as well as for atoms that contain more than one electron.

Another interesting phenomenon should be mentioned, which we discovered by performing the calculations for such transition probabilities in various atomic and ionic hydrogenlike systems. Indeed, as follows from Table I, if the initial tritium atom was in its (5,3) state, then the final ^{3}He ion can be never found, e.g., in its (5,3) or (10,3) states. Actually, such a rule follows immediately from the fact that the radial parts of the hydrogenlike functions Eq. (24) depend on the ratio qr/n, rather than on the radius r only. This means the variation of n may compensate, in principle, the charge change caused by the $\beta^{\pm}$ decay, e.g., $q/n_1 = (q \pm 1)/n_2$. In order to represent such a general situation, let us consider the following example.

The initial state of the tritium atom (^{3}H) is the (2,0) state, while the final state of the $^3He^+$ ion is the (4,0) state. Both the initial and final nuclei are considered as infinitely heavy. The radial part of the tritium wave function takes the form

$$\Psi_{in}(r) = \Psi_{(2,0)}^{(q=1)}(r) = \frac{1}{\sqrt{2}}\left(1-\frac{r}{2}\right)\exp\left(-\frac{r}{2}\right) \tag{26}$$

while the radial part of the $^3He^+$ wave function is

$$\Psi_{fi}(r) = \Psi_{(4,0)}^{(q=2)}(r) = \frac{1}{\sqrt{2}}\left(1-\frac{3r}{2}+\frac{r^2}{2}-\frac{r^3}{24}\right)\exp\left(-\frac{r}{2}\right), \tag{27}$$

where r is exactly the same in both equations. Now, it is easy to see by direct integration that $\Psi_{in}(r)$ and $\Psi_{fi}(r)$ are orthogonal to each other. This means that the $\Psi_{(4,0)}^{(q=2)}(r)$ function lies inside of the functional space $D_{q=1}^{\perp}$, which is orthogonal to the appropriate space of the bound state functions $\overline{D}_{q=1}$ [21–23]. As it follows from the explicit form of the $\Psi_{(4,0)}^{(q=2)}(r)$ function, this function has a finite norm and has the same exponential factor as the $\Psi_{(2,0)}^{(q=1)}(r)$ function. The principal difference can be found in the number of nodes, one vs three [24,25].

Actually, this problem should be discussed by applying the general group theory. As is known (see, e.g., [26,27]) the complete theory of an arbitrary, nonrelativistic, hydrogenlike atom can be given in terms of the noncompact $o(4,2)$ alge-

TABLE I. The ^{3}He ionic excitation probabilities caused by the β^- decay of the tritium nucleus. The initial state (n_1,l) of the β^--decaying tritium atom (^{3}H) is shown in the left column. All other columns contain the final states (n_2,l) of the $^3He^+$ ion (in numerators) and the respective ionic excitation probabilities (in denominators). All nuclei are assumed to be infinitely heavy.

(1,0)	$\frac{(1,0)}{0.702332}$	$\frac{(2,0)}{0.25}$	$\frac{(3,0)}{1.27402\times10^{-2}}$	$\frac{(4,0)}{3.85367\times10^{-3}}$	$\frac{(5,0)}{1.71979\times10^{-3}}$	$\frac{(6,0)}{9.26971\times10^{-4}}$
	$\frac{(7,0)}{5.59880\times10^{-4}}$	$\frac{(8,0)}{3.65204\times10^{-4}}$	$\frac{(9,0)}{2.51893\times10^{-4}}$	$\frac{(10,0)}{1.81284\times10^{-4}}$	$\frac{(11,0)}{1.34919\times10^{-4}}$	$\frac{(12,0)}{1.03179\times10^{-4}}$
(4,0)	$\frac{(1,0)}{4.45710\times10^{-3}}$	$\frac{(2,0)}{2.17433\times10^{-2}}$	$\frac{(3,0)}{1.93388\times10^{-2}}$	$\frac{(4,0)}{1.77829\times10^{-2}}$	$\frac{(5,0)}{0.671364}$	$\frac{(6,0)}{0.262367}$
	$\frac{(7,0)}{1.60923\times10^{-3}}$	$\frac{(8,0)}{0.0}$	$\frac{(9,0)}{1.20443\times10^{-7}}$	$\frac{(10,0)}{5.34943\times10^{-6}}$	$\frac{(11,0)}{8.54473\times10^{-7}}$	$\frac{(12,0)}{9.93393\times10^{-7}}$
(4,3)	$\frac{(4,3)}{0.346439}$	$\frac{(5,3)}{0.578539}$	$\frac{(6,3)}{7.47254\times10^{-2}}$	$\frac{(7,3)}{2.83388\times10^{-4}}$	$\frac{(8,3)}{0.0}$	$\frac{(9,3)}{1.37085\times10^{-8}}$
	$\frac{(10,3)}{5.43315\times10^{-8}}$	$\frac{(11,3)}{8.00246\times10^{-8}}$	$\frac{(12,3)}{8.76175\times10^{-8}}$	$\frac{(13,3)}{8.50932\times10^{-8}}$	$\frac{(14,3)}{7.82034\times10^{-8}}$	$\frac{(15,3)}{6.99720\times10^{-8}}$
(5,3)	$\frac{(4,3)}{0.131413}$	$\frac{(5,3)}{0.0}$	$\frac{(6,3)}{0.557752}$	$\frac{(7,3)}{0.300521}$	$\frac{(8,3)}{1.02980\times10^{-2}}$	$\frac{(9,3)}{6.30432\times10^{-6}}$
	$\frac{(10,3)}{0.0}$	$\frac{(11,3)}{1.52504\times10^{-10}}$	$\frac{(12,3)}{1.82523\times10^{-9}}$	$\frac{(13,3)}{4.71351\times10^{-9}}$	$\frac{(14,3)}{7.33462\times10^{-9}}$	$\frac{(15,3)}{9.08714\times10^{-9}}$
(6,4)	$\frac{(5,4)}{0.134484}$	$\frac{(6,4)}{1.35175\times10^{-2}}$	$\frac{(7,4)}{0.397071}$	$\frac{(8,4)}{0.414525}$	$\frac{(9,4)}{4.00974\times10^{-2}}$	$\frac{(10,4)}{3.04020\times10^{-4}}$
	$\frac{(11,4)}{2.78320\times10^{-8}}$	$\frac{(12,4)}{0.0}$	$\frac{(13,4)}{3.17773\times10^{-13}}$	$\frac{(14,4)}{1.13542\times10^{-11}}$	$\frac{(15,4)}{5.10672\times10^{-11}}$	$\frac{(16,4)}{1.12502\times10^{-10}}$
(7,2)	$\frac{(3,2)}{1.50742\times10^{-2}}$	$\frac{(4,2)}{1.08932\times10^{-2}}$	$\frac{(5,2)}{9.36260\times10^{-4}}$	$\frac{(6,2)}{5.26607\times10^{-2}}$	$\frac{(7,2)}{9.62499\times10^{-2}}$	$\frac{(8,2)}{1.04883\times10^{-3}}$
	$\frac{(9,2)}{0.537746}$	$\frac{(10,2)}{0.273392}$	$\frac{(11,2)}{1.19404\times10^{-2}}$	$\frac{(12,2)}{3.27113\times10^{-5}}$	$\frac{(13,2)}{6.15478\times10^{-10}}$	$\frac{(14,2)}{0.0}$

bra, which contains 15 generators. To compute the appropriate β-decay probabilities the charge q must be considered as a new, additional variable. The complete solution of the problem can be given in terms of the noncompact, 18-generators algebra, which includes 15 generators of the $o(4,2)$ algebra mentioned above, and the following three generators: q, $\partial/\partial q$, and $q\,\partial/\partial q$, where q is the charge of the initial nucleus. We do not wish to present this analytical consideration here, since its subject is far away from our present goals. Moreover, such an analysis has a very restricted scientific sense, since it can be applied only to the infinitely heavy nuclei inside of the hydrogenlike systems. But in any real hydrogenlike system the nuclear recoil cannot be neglected, and in some cases (e.g., the tritium atom) this effect plays a very important role.

To finish the discussion of the selection rules for the β^--decaying hydrogenlike systems $(q\to q+1)$ we present a few simple rules, which have been found either analytically or from the results of numerical calculations. First of all, note that the corresponding probability is proportional to the following amplitude [13]:

$$\mathcal{A}^{l}_{n_1,n_2}=\left\langle \Psi_{n_1 l}\left(\frac{q_1}{n_1}r\right)\middle|\Psi_{n_2 l}\left(\frac{q_2}{n_2}r\right)\right\rangle =\frac{n_2}{q_2}\left\langle \Psi_{n_1 l}\left(\frac{q_1 n_2}{q_2 n_1}r\right)\middle|\Psi_{n_2 l}(r)\right\rangle =\frac{n_1}{q_1}\left\langle \Psi_{n_1 l}(r)\middle|\Psi_{n_2 l}\left(\frac{q_2 n_1}{q_1 n_2}r\right)\right\rangle, \tag{28}$$

where $q_1=q$ and $q_2=q+1$. The following (rational) number

$$\lambda=\max\left(\frac{q_1 n_2}{q_2 n_1},\frac{q_2 n_1}{q_1 n_2}\right) \tag{29}$$

is of specific importance below. For the β^- decay in the tritium atom (i.e., $q_1=1$ and $q_2=2$) the amplitude $\mathcal{A}^{l}_{n_1,n_2}$ vanishes if (but not only if) λ is equal to an integer number. Now, let us consider a few particular cases. First, it is easy to see that when $n_1=1$ the appropriate amplitude has only non-

zero values for arbitrary n_2. This means for instance that there is not an additional selection rule for the β^- decay from the ground state of the tritium atom. In other words the final state of the helium ion can be arbitrary (with $l=0$). Second, only when $n_1>1$ and the ratio λ is an integer number can such an amplitude be equal to zero exactly. The conditions when this amplitude vanishes follow directly from the explicit form of the hydrogenlike radial functions. Some of such additional selection rules are illustrated in Table I. However, it should be noted that Table I corresponds only to the $q_1=1,\ q_2=2$ case and infinitely heavy nuclei.

If the initial and final atoms (ions) are more complicated systems than the hydrogenlike atom or ion, the corresponding many-electron wave functions should be used to compute $\mathcal{M}_{n_1 n_2}$, Eq. (22). For two- and three-electron systems one may apply the completely correlated variational wave function in the relative coordinates (see, e.g., [29,30]). The appropriate expressions for $\mathcal{M}_{in,fi}$ take the form

$$\mathcal{M}_{in,fi}=|\langle\Psi_{in}(r_{32},r_{31},r_{21})|\Psi_{fi}(r_{32},r_{31},r_{21})\rangle|^2 \quad (30)$$

and

$$\mathcal{M}_{in,fi}=|\langle\Psi_{in}(r_{21},r_{31},r_{32},r_{41},r_{42},r_{43})|\Psi_{fi}(r_{21},r_{31},r_{32},r_{41},r_{42},r_{43})\rangle|^2 \quad (31)$$

in the case of the two- and three-electron systems, respectively. The initial and final wave functions can be written in the following general form:

$$\Psi_a(r_{21},r_{31},r_{32})=\sum_i^{N_{\max}} C_i^{(LS)}\mathcal{Y}_{LM}^{l_1,l_2}(\vec{r}_{31},\vec{r}_{32}) \times\phi_i^{(3)}(r_{21},r_{31},r_{32})\chi_i^{(2)} \quad (32)$$

for the two-electron systems, and

$$\Psi_a(r_{21},r_{31},r_{32},r_{41},r_{42},r_{43}) =\sum_i^{N_{\max}} C_i^{(LS)}\mathcal{Y}_{LM}^{(l_1,l_2)l_{12};l_3}(\vec{r}_{41},\vec{r}_{42},\vec{r}_{43}) \times\phi_i^{(6)}(r_{21},r_{31},r_{32},r_{41},r_{42},r_{43})\chi_i^{(3)} \quad (33)$$

for the three-electron systems. In these formulas the subscript ''a'' designates either an ''in'' or ''fi'' state with the total angular momentum L and spin S. Also, the subscripts 1,2 (1,2,3) stand for the electrons, while the subscript 3 (4) means the nucleus for the two- (three-) electron system, respectively. $C_i^{(LS)}$ are the unknown variational parameters. $N_{\max}$ designates the maximal number of terms in the variational expansion. $\chi_i^{(2)}$ and $\chi_i^{(3)}$ stand for the appropriate basis spin functions. Obviously, in an arbitrary two-electron system $\chi_i^{(2)}$ can be chosen in a form that does not depend upon i. The radial functions of the relative coordinates $\{\phi_i^{(3)}(r_{21},r_{31},r_{32})\}$ and $\{\phi_i^{(6)}(r_{21},r_{31},r_{32},r_{41},r_{42},r_{43})\}$ form complete sets in the appropriate three- and six-dimensional spaces of the relative coordinates.

The notations $\mathcal{Y}_{LM}^{l_1,l_2}(\vec{r}_{31},\vec{r}_{32})$ and $\mathcal{Y}_{LM}^{(l_1,l_2)l_{12};l_3}(\vec{r}_{41},\vec{r}_{42},\vec{r}_{43})$ designate vector-coupled products of spherical harmonics for two- and three-electron systems, respectively, to form a state of total angular momentum L. They have the following forms ($\vec{n}_{ij}=\vec{r}_{ij}/r_{ij}$):

$$\mathcal{Y}_{LM}^{(l_1,l_2)}(\vec{r}_{31},\vec{r}_{32})=r_{31}^{l_1}r_{32}^{l_2}\sum_{m_l}\langle l_1,m_1;l_2,m_2|L,M\rangle \times Y_{l_1,m_1}(\vec{n}_{31})Y_{l_2,m_2}(\vec{n}_{32}), \quad (34)$$

and

$$\mathcal{Y}_{LM}^{(l_1,l_2)l_{12};l_3}(\vec{r}_{41},\vec{r}_{42},\vec{r}_{43}) =r_{41}^{l_1}r_{42}^{l_2}r_{43}^{l_3}\sum_{m_l}\langle l_1,m_1;l_2,m_2|l_{12},m_{12}\rangle \times\langle l_{12},m_{12};l_3,m_3|L,M\rangle \times Y_{l_1,m_1}(\vec{n}_{41})Y_{l_2,m_2}(\vec{n}_{42})Y_{l_3,m_3}(\vec{n}_{43}), \quad (35)$$

respectively. In these formulas $\langle l_{12},m_{12};l_3,m_3|L,M\rangle$ are the Clebsh-Gordan coefficients and $Y_{l,m}(\vec{n})$ are the usual spherical harmonics [28]. For more details see, e.g., [31] and [32].

By using these formulas for the β^- decay of the tritium anion ($^3H^-$) from its ground (or *g)* state, we can evaluate the probability for the final He atom to be found in its ground (or $\tilde{g}$) state: $\mathcal{M}_{g,\tilde{g}}=(0.4795)^2$, i.e., approximately 23% of all ^{3}He atoms will be produced in the ground $S(L=0)$ state (here again all nuclear masses are infinite). The analogous result for the β^--decaying ^{9}Li atom is $\mathcal{M}_{g,\tilde{g}}=(0.7601)^2$, where the final state is the ground (or $\tilde{g}$) state of the $^9Be^+$ ion. Actually, for all two- and three-electron atoms and ions the $\mathcal{M}_{in,fi}$ can be computed quite accurately, since recently significant progress was made for such systems. In the case of atoms with four and more electrons, the appropriate many-particle HF or CI wave functions should be used.

V. CONCLUSION

Thus, the inner radiation emitted during $\beta^{\pm}$ decay has been considered in the present study. By using the formulas presented in this work, one may easily compute and represent the qualitative picture for the inner radiation emitted during $\beta^{\pm}$ decay. The quantitative agreement should be quite good unless we use the approximation that the emitted photons have energy small compared to the total particle energy.

In other words, the approach developed above is valid when the quantum nature of the emitted photons can be neglected. Likewise, we did not discuss the radiation reaction effects, which are very difficult to consider analytically.

The atomic excitation analysis was made above in terms of the sudden approximation. Such an approximation can be applied when the parameter $(\alpha q)^2$ is small. In actual atoms this is approximately true only when $q \leqslant 10-20$. Otherwise the complete relativistic approximation is required. It should be mentioned also that we restrict ourselves presently only to the consideration of atomic $\beta^{\pm}$ decays. The analysis of the analogous $\beta^{\pm}$ decays in molecules is significantly more complicated, since the daughter molecule can be left in an excited vibrational or rotational state [33]. In other words, in contrast with atoms, each molecule containing $\beta^{\pm}$ decaying nuclei requires, in principle, a separate investigation. This problem was studied for the first time by Cantell [33]. The emitted radiation analysis becomes more complicated as well, since the emitted $\beta^{\pm}$ particle now moves in the field of many different nuclei. Probably, only the initial inner bremsstrahlung can be considered as for atoms.

For atoms themselves the next step is to consider both adiabatic and nonadiabatic corrections [20] to the appropriate probabilities discussed above. Also, when the fast $\beta^{\pm}$ particle moves through the electronic shells of atoms, it can produce holes in such shells. The resulting evolution and decay of these holes are of increasing theoretical interest in some recent publications (see, e.g., [34] and references therein). In principle, this question is closely related with problems discussed above. However, in the present study we completely neglect the direct electron-electron interaction, which was assumed to be relatively small. This means that in order to discuss the Auger and double Auger effect stimulated by the $\beta^{\pm}$ decay, one needs to consider the appropriate adiabatic, nonadiabatic, relativistic, and radiative corrections.

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[1] Probably, soon ^{247}Cm will be used in all applications related to nuclear fission (and fusion as well [2]) instead of ^{239}Pu. Indeed, the properties of ^{247}Cm with respect to nuclear fission are significantly better than for ^{239}Pu. Furthermore, curium-247 is significantly ($\sim$1000 times) more stable and less toxic than plutonium-239.

[2] The fissionable elements are used for fusion purposes in the three following ways. First, to produce the extremely intense pulse of soft x rays (the so-called Teller's light), which is used to compress the cold thermonuclear fuel (mainly ^{6}LiD) to the densities $\approx 100-1000\ \mathrm{g\times cm^{-3}}$. Second, to heat such a highly compressed fuel to the temperatures $\approx$4–10 keV, and also to support and partially control the conducting thermonuclear reactions by outcoming high intense neutron fluxes. Third, to simplify and stabilize the thermonuclear burning and its propagation. Only for the last purposes are the fissionable elements used as the deuterides that are mixed (in small amounts) with the ^{6}LiD deuteride. This way was found and successfully developed in the middle of the 1950s. Since that time all other approaches to produce self-supported nuclear fusion reactions have failed.

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$$\langle \phi_1 | \phi_2 \rangle = \frac{1}{4\pi} \int_0^{+\infty} \int_{\Omega} \phi_1^*(r,\Omega) \phi_2(r,\Omega) dr\, d\Omega.$$

[14] The binominal coefficients C_n^k used in the text designate the following expression: $C_n^k = n!/k!(n-k)!$.

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[22] For an arbitrary metric (or unitary) space X, the notation $\overline{X}$ means its completion, which is a complete metric (or Hilbert) space (for more detail see [23]).

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[24] In general, $\Psi_{fi}(r)$ corresponds to the so-called tail representations [25] (for the hydrogenlike system with $q=1$), which had never been used previously for the hydrogenlike atoms (ions). The reason why such representations appeared in the present study seems to be quite transparent: the appropriate Migdal amplitude $\mathcal{M}_{in,fi}$ includes the wave functions for the two hydrogenlike systems with different charges. But both of these functions have the same argument, and hence, we may discuss the relation between the first (initial) and second (final) functions.

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