# Quantum and classical descriptions of a measuring apparatus

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A measuring apparatus is described by quantum mechanics while it interacts with the quantum system under observation and then it must be given a classical description so that the result of the measurement appears as objective reality. Alternatively, the apparatus may always be treated by quantum mechanics and be measured by a second apparatus, which has such a dual description. This article examines whether these two different descriptions are mutually consistent. It is shown that if the dynamical variable used in the first apparatus is represented by an operator of the Weyl-Wigner type (for example, if it is a linear coordinate), then the conversion from quantum to classical terminology does not affect the final result. However, if the first apparatus encodes the measurement in a different type of operator (e.g., the phase operator), the two methods of calculation may give different results. [S1050-2947(98)04107-9]

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## I. VON NEUMANN'S CUT

Quantum mechanics provides statistical predictions for the results of measurements performed on physical systems that have been prepared in a specified way. The preparation and measurement are performed by macroscopic devices and these are described in classical terms. The necessity of using a classical terminology was emphasized by Bohr [1], whose insistence on a classical description was very strict. Bohr never considered the measuring process as a dynamical interaction between an apparatus and the system under observation. Any intermediate systems used in that process could be treated quantum mechanically, but the *final* instrument had a purely classical description [2]. Measurement was understood as a primitive notion. Bohr thereby eluded questions which caused considerable controversy among other authors [3,4].

Yet, measuring apparatuses are made of the same kind of matter as everything else and they obey the same physical laws. It therefore seems natural to use quantum theory in order to investigate their behavior during a measurement. This was attempted by von Neumann, in his treatise on the mathematical foundations of quantum theory [5]. In the last section of that book, as in an afterthought, von Neumann represented the apparatus by a single degree of freedom, whose value was correlated to that of the dynamical variable being measured. Such an apparatus is not, in general, left in a definite pure state and does not admit a classical description. Therefore, von Neumann introduced a second apparatus, which observes the first one, and possibly a third apparatus, and so on, until there is a final measurement, which is *not* described by quantum dynamics and has a definite result (for which quantum mechanics can give only statistical predictions). The essential point that was suggested, but not proved by von Neumann, is that the introduction of this sequence of apparatuses is irrelevant: The final result is the same, irrespective of the location of the ''cut'' between classical and quantum physics. (At this point, von Neumann also speculated that a final step would involve the consciousness of the observer, a rather bizarre statement in a mathematically rigorous monograph.)

In the present article we introduce a dual description for the measuring apparatus. It obeys quantum mechanics while it interacts with the system under observation and then it is ''dequantized'' and is described by a classical Liouville density, which provides the probability distribution for the results of the measurement. Alternatively, the apparatus may always be treated by quantum mechanics and measured by a second apparatus that has such a dual description. The question is whether these two different methods of calculation give the same result [6].

We show that a sufficient condition for agreement between the two methods is that the dynamical variable used as a ''pointer'' by the first apparatus be represented by an operator of the Weyl-Wigner type [7]. These ''quasiclassical'' operators are defined as follows. Let a classical dynamical variable $A(q,p)$ be expressed as a Fourier transform

$$A(q,p)=\int\int d\sigma\, d\tau\, e^{i(\sigma q+\tau p)}\alpha(\sigma,\tau). \tag{1}$$

Then the correponding Weyl-Wigner operator is obtained by replacing, in the above expression, the classical variables $q$ and $p$ by the corresponding quantum operators $\hat{q}$ and $\hat{p}$. It can be shown that the expectation value of $\hat{A}(\hat{q},\hat{p})$ for any quantum state, pure or mixed, is equal to the classical expression

$$\langle A\rangle=\int\int W(q,p)A(q,p)dq\, dp, \tag{2}$$

where $W(q,p)$ is Wigner's quasiprobability distribution [7,8]. If the latter is nowhere negative, it can be interpreted as a classical Liouville distribution. In the rest of this paper the same symbols $q$ and $p$ will be used for classical variables and for operators since the meaning of the symbol is always clear from the context and there is no risk of confusion.

We examine two examples. In the simplest one the pointer is described by a linear coordinate $q$, which is an operator of the Weyl-Wigner type. As expected, the conversion from quantum to classical description does not affect the final result. In the second example, the first apparatus en-

 

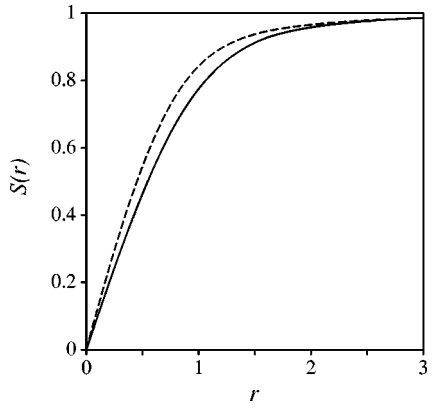


FIG. 1. Function $S(r)$, given by Eq. (41), is the factor by which the mean result of the quantum measurement is reduced, with respect to the classical result. The semiclassical result, given by Eq. (66), is shown by the dotted curve.

codes the measurement in a phase. In that case, the operator that we use is not of the Weyl-Wigner type and the two methods of calculation give different results. It is likely that the validity of these conclusions is not restricted to the particular examples for which we provide detailed calculations.

In both examples the quantum system that we observe is a particle of spin $j$ and we want to measure the $J_z$ component. In Sec. II we couple $J_z$ to the linear position $q$ of a pointer. The latter is then measured by a second pointer, whose linear position is $Q$. The problem is to find the probability distribution of $Q$ for a given initial state of the quantum system. As shown explicitly, it makes no difference to dequantize $q$ after the first measurement and to always treat $Q$ classically.

In Sec. III, on the other hand, we couple $J_z$ to the phase $\theta$ of a harmonic oscillator. The second apparatus (again a linear pointer with position $Q$) measures $\cos\theta$, not $\theta$ itself because the phase is not a well-defined self-adjoint operator in quantum mechanics [9,10]. We then find that in this case the expectation value $\langle Q\rangle$ is not the same when the first apparatus is treated quantum mechanically, or classically, while it is measured by the second one. That is, when we perform the required calculations for such a measuring process, the result depends on the location chosen for the von Neumann cut. Figure 1 encapsulates the difference between the two methods of calculation.

To avoid any misunderstanding, we emphasize that the classical description of a pointer is *not* by means of a point in phase space, but by a Liouville density. Quantum theory makes only statistical predictions and any semiclassical treatment that simulates it must also be statistical. Our approach involves only strictly orthodox quantum mechanics. We never speculate about modifications of the conventional theory, such as those that have been proposed by some authors [4]. In particular, we do not attempt to mix classical and quantum mechanics at any stage of the dynamical evolution [11,12].

The implications of our results on the so-called quantum measurement problem are briefly discussed in Sec. IV. While our work may not satisfy the desiderata of some physicists, it does prove the consistency of those of Bohr and von Neumann, provided that the physical system that is employed as the measuring instrument is indeed suitable for filling that role.

## II. LINEAR POINTER

Let the system under observation be a spin $j$ particle. We want to measure the spin component $J_z$, which satisfies, in natural units ($\hbar=1$),

$$J_z|m\rangle = m|m\rangle, \quad m = j, j-1, \ldots, -j. \tag{3}$$

The initial state of the system is $\Sigma a_m|m\rangle$.

In elementary discussions of quantum measurements, there is no explicit decription of the apparatus. The typical textbook just says that the result of the measurement is $m$, with probability $|a_m|^2$. The reader may imagine a pointer, jumping from $q=0$ to $q=m$ (in suitable units), with probability $|a_m|^2$, as a result of the measuring process. (In the language of statistical mechanics, the Liouville function of the pointer has peaks of size $|a_m|^2$ near $q=m$.) It is then possible to imagine a second apparatus that measures the first one and has its pointer moving from $Q=0$ to $Q=q$. The readings of the two apparatuses of course agree with each other.

In this article we provide a quantum dynamical description for the apparatuses. The initial state of the first pointer is specified by a wave function $\phi(q)$. The position $q$ and its conjugate momentum $p=-i\partial/\partial q$ are linear operators in Hilbert space. Their spectra extend from $-\infty$ to $\infty$. Likewise, the second apparatus is a linear pointer with position operator $Q$, momentum operator $P=-i\partial/\partial Q$, and initial state $\Phi(Q)$.

The joint state of the complete setup is, initially,

$$\psi_0 = \sum_m a_m|m\rangle\otimes\phi(q)\otimes\Phi(Q). \tag{4}$$

The interaction between the system and the first apparatus is represented by the unitary operator

$$U_1 = e^{-iJ_z p} = e^{-J_z(\partial/\partial q)}. \tag{5}$$

This unitary evolution can be generated by a Hamiltonian $H_{\rm int}=J_z p/\epsilon$, acting during a time $\epsilon$, brief enough so that the other parts of the Hamiltonian can be neglected. However, for the present problem, it is simpler to directly use unitary operators, instead of exponentiating a Hamiltonian. If the state of the spin is $|m\rangle$, the operator $U_1$ causes the pointer to move by $m$ length units (with a suitable choice of units). The new state thus is, in general,

$$\psi_1 = U_1\psi_0 = \sum_m a_m|m\rangle\otimes\phi(q-m)\otimes\Phi(Q). \tag{6}$$

Likewise, the second pointer senses the value of $q$ and moves by $q$ units. The interaction of the two pointers is generated by

$$U_2 = e^{-iqP} = e^{-q(\partial/\partial Q)}, \tag{7}$$

so that

$$\psi_2 = U_2\psi_1 = \sum_m a_m|m\rangle\otimes\phi(q-m)\otimes\Phi(Q-q). \tag{8}$$

The probability distribution of $Q$ (while we ignore the other variables) is

$$\int_{-\infty}^{\infty} dq\ \psi_2^{\dagger}\psi_2 = \int_{-\infty}^{\infty} dq\ \sum_m |a_m|^2 |\phi(q-m)|^2 |\Phi(Q-q)|^2. \tag{9}$$

This simply is the convolution of the probability distribution of the first pointer, namely,

$$f(q) = \sum_m |a_m|^2 |\phi(q-m)|^2, \tag{10}$$

with the probability distribution of the second pointer for a given value of $q$,

$$F(Q-q) = |\Phi(Q-q)|^2. \tag{11}$$

It will now be seen that the same result is obtained if the von Neumann cut is placed after the first apparatus. That is, the quantum-mechanical result (10) will be considered as a classical probability distribution for the position of the first pointer. The initial distribution for the second one is $F(Q)$, which is a given non-negative function. The two pointers interact classically in such a way that

$$f(q)F(Q) \to f(q)F(Q-q). \tag{12}$$

The final result for the probability distribution of $Q$ is obviously the same as in the quantum-mechanical calculation above.

However, we still have to formally show that the postulated dynamical evolution (12) is compatible with classical mechanics. Let us thus write $f(q)$ and $F(Q)$ as the marginals of Liouville distributions

$$f(q) = \int L_1(q,p)\ dp, \tag{13}$$

$$F(Q) = \int L_2(Q,P)\ dP. \tag{14}$$

The interaction of the two apparatuses lasts a very brief time $\epsilon$, during which the Hamiltonian is

$$H_{\rm int} = qP/\epsilon. \tag{15}$$

The other parts of the Hamiltonian can be neglected. It follows that $q$ and $P$ remain constant during the measurement and that $\dot{p} = -P/\epsilon$ and $\dot{Q} = q/\epsilon$. When the interaction is concluded after a time $\epsilon$ we have

$$p \to p' = p - P \tag{16}$$

and

$$Q \to Q' = Q + q. \tag{17}$$

It follows that the functional form of the joint distribution evolves as

$$L_1(q,p)L_2(Q,P) \to L_1(q,p+P)L_2(Q-q,P). \tag{18}$$

Note that the $\pm$ signs in Eq. (18) are opposite to those in the two preceding equations. This is because a Liouville distribution flows in phase space as an incompressible fluid and the solution of the Liouville equation is $L'(q',p',Q',P') = L(q,p,Q,P)$.

To get the marginal distributions of $q$ and $Q$, we first integrate the right-hand side of Eq. (18) over $p$ and then over $P$. The dynamical law (12) readily follows, in complete agreement with the quantum calculation. Note that we did not have to assume any particular form for the non-negative functions $L_1(q,p)$ and $L_2(Q,P)$. Only the marginal probabilities (13) and (14) are involved in the final result.

## III. ENCODING A MEASUREMENT IN A PHASE

We shall now measure the same quantum system with a different apparatus. Instead of a linear pointer, we use the phase of a harmonic oscillator, whose Hamiltonian is $H_{\rm osc} = \frac{1}{2}(p^2+q^2)$. In classical mechanics, the phase is given by $\theta = \arctan(p/q)$. In quantum mechanics, the issue is more complicated, as we shall see.

First, let us give, as in the preceding section, an elementary classical description of the quantum measurement (it will later be needed for comparison with the semiclassical and the purely quantum treatments). The final phase of the classical oscillator, which plays the role of a pointer, is given by

$$\theta = \theta_0 - m\chi, \tag{19}$$

with probability $|a_m|^2$. Here $\chi$ is any constant (we shall take $\chi < \pi/2j$, so that there is no overlap in the final values of $\theta$). It will be convenient to take $\theta_0 = \pi/2$.

The second apparatus is, as before, a linear pointer. It is coupled to $\cos\theta$ (not to $\theta$ itself, for reasons that will become clear below). The final position of the second pointer (treated classically) thus is

$$Q = Q_0 + \cos\theta = Q_0 + \sin m\chi. \tag{20}$$

This elementary classical result, for which no dynamical justification was given, will now be compared with the one obtained by treating both apparatuses as quantum systems.

### A. Two quantum apparatuses

The first apparatus is a harmonic oscillator (e.g., one of the modes of an electromagnetic field in a cavity), initially prepared in a coherent state [13],

$$|\alpha\rangle = e^{-r^2/2} \sum_{k=0}^{\infty} (\alpha^k/\sqrt{k!})|k\rangle, \tag{21}$$

where $\alpha$ is a complex number. On the right hand-side of Eq. (21), the orthonormal basis $|k\rangle$ consists of eigenstates of $H_{\rm osc}$,

$$H_{\rm osc}|k\rangle = (k+\tfrac{1}{2})|k\rangle, \tag{22}$$

and $r$ is defined by

$$r^2 \equiv |\alpha|^2 = \langle\alpha|H_{\rm osc}|\alpha\rangle - \tfrac{1}{2}. \tag{23}$$

The coherent states (21) minimize the uncertainty product $\Delta p \Delta q = \hbar/2$ and therefore give results as close as possible to classical physics, in particular when $r \gg 1$.

The second apparatus is, as before, a linear pointer prepared in a state $\Phi(Q)$. The joint state of the complete setup thus is, initially

$$\psi_0 = |m\rangle \otimes |\alpha\rangle \otimes \Phi(Q). \tag{24}$$

Here we have assumed for simplicity that the quantum system is in one of the eigenstates $|m\rangle$ of $J_z$ (the goal of the measurement is to determine $m$). It is obvious that any linear combination $\Sigma a_m |m\rangle$ would give, after the quantum system is traced out, a statistical mixture with weights $|a_m|^2$, as in Eq. (10).

The interaction between the system and the first apparatus is represented by the unitary operator

$$U_1 = e^{-i\chi J_z H_{\rm osc}}. \tag{25}$$

As before, it is easy to write an interaction Hamiltonian that generates this unitary evolution. In the present case, where $m$ has a definite value, we can replace in Eq. (25) $J_z$ by $m$. It then follows from Eqs. (21) and (22) that

$$\psi_1 = U_1 \psi_0 = e^{-im\chi/2} |m\rangle \otimes |e^{-im\chi}\alpha\rangle \otimes \Phi(Q). \tag{26}$$

From this point, we can safely ignore the spin state $|m\rangle$, since we shall not observe again the quantum system itself, and of course we ignore the phase factor $e^{-im\chi/2}$.

If we could now measure the phase of the parameter $e^{-im\chi}\alpha$ in the coherent state on the right-hand side of Eq. (26), this would readily give us the value of $m$. This is of course impossible, because coherent states are not mutually orthogonal [13] and they cannot be distinguished with certainty. At most, we may get probabilistic indications for the value of $m$. Moreover, there is no self-adjoint phase operator [9]. It is however possible to define a self-adjoint operator $C$, which is a legitimate quantum analog of the classical variable $\cos\theta$. (Most authors simply call that operator $\cos\theta$, or $\widehat{\cos\theta}$, instead of $C$ as we do here to avoid ambiguities.) The reader who is not interested in computational details may skip from here to Eq. (40).

The spectrum of $C$ runs from $-1$ to 1, and it is convenient to label the eigenvalues by $\cos\theta$, with $0 \leqslant \theta \leqslant \pi$. The eigenstates of $C$ are given, in terms of the number states $|n\rangle$, by [9]

$$|\cos\theta\rangle = \sqrt{2/\pi} \sum_{n=0}^{\infty} \sin[(n+1)\theta] |n\rangle. \tag{27}$$

They have a delta-function normalization

$$\langle \cos\theta | \cos\theta' \rangle = \delta(\theta - \theta'), \tag{28}$$

and a completeness property

$$\int_0^{\pi} |\cos\theta\rangle\langle\cos\theta| \, d\theta = \mathbf{1}, \tag{29}$$

where $\mathbf{1}$ is the unit operator.

The interaction between the first and the second apparatuses is given, as in Eq. (7), by

$$U_2 = e^{-iCP} = e^{-C(\partial/\partial Q)}. \tag{30}$$

To see how this acts on $\psi_1$, we have to expand $|e^{-im\chi}\alpha\rangle$ into a sum of eigenstates of $C$. For such an eigenstate, the evolution is

$$U_2 |\cos\theta\rangle \otimes \Phi(Q) = |\cos\theta\rangle \otimes \Phi(Q - \cos\theta). \tag{31}$$

By virtue of Eq. (29) we have

$$|e^{-im\chi}\alpha\rangle = \int_0^{\pi} d\theta \, |\cos\theta\rangle\langle\cos\theta|e^{-im\chi}\alpha\rangle. \tag{32}$$

The expression $\langle\cos\theta|e^{-im\chi}\alpha\rangle$ can be evaluated explicitly owing to Eqs. (21) and (27). For brevity, let us write

$$e^{-im\chi}\alpha = e^{i\mu} r, \tag{33}$$

where $\mu = \mu_0 - m\chi$. It will be convenient to take as the initial phase $\mu_0 = \pi/2$.

The next step is to compute $\psi_2 = U_2 \psi_1$. Collecting all the relevant expressions, we obtain from Eq. (31)

$$|\psi_2(Q)\rangle = \sqrt{\frac{2}{\pi}} e^{-r^2/2} \int_0^{\pi} d\theta \sum_{n=0}^{\infty} \sin[(n+1)\theta] \times \frac{r^n e^{in\mu}}{\sqrt{n!}} |\cos\theta\rangle \otimes \Phi(Q - \cos\theta), \tag{34}$$

where we have used a mixed notation, as in the previous equations: The Dirac symbol $|\ \rangle$ is used for the states of the first apparatus and ordinary functions of $Q$ for the second apparatus. With these notations, the probability distribution for $Q$, irrespectively of the value of $\cos\theta$, is given by the diagonal elements of the partly traced density matrix $\mathrm{Tr}_\theta[|\psi_2(Q)\rangle\langle\psi_2(Q)|]$. The result is, owing to the orthogonality relation (28),

$$\frac{2}{\pi} e^{-r^2} \int_0^{\pi} d\theta \sum_{n=0}^{\infty} \sin[(n+1)\theta] \frac{r^n e^{in\mu}}{\sqrt{n!}} \times \sum_{s=0}^{\infty} \sin[(s+1)\theta] \frac{r^s e^{-is\mu}}{\sqrt{s!}} |\Phi(Q - \cos\theta)|^2. \tag{35}$$

This expression is a convolution, just as in Eq. (9). It is difficult to evaluate it explicitly, but the mean value $\langle Q \rangle$ can easily be obtained. Keeping the integration over $\theta$ for the end, we have

$$\int_{-\infty}^{\infty} Q \, dQ |\Phi(Q - \cos\theta)|^2 = \langle Q \rangle_0 + \cos\theta \tag{36}$$

and therefore

$$\langle Q \rangle = \langle Q \rangle_0 + \langle C \rangle, \tag{37}$$

where

$$\langle C\rangle=\frac{2}{\pi}e^{-r^2}\int_0^{\pi}d\theta\,\cos\theta\sum_{n,s=0}^{\infty}\sin[(n+1)\theta]$$

$$\times\sin[(s+1)\theta]\frac{r^{n+s}e^{i(n-s)\mu}}{\sqrt{n!}\sqrt{s!}} \tag{38}$$

is the expectation value of $C$ in the state $|e^{-im\chi}\alpha\rangle$ whose expansion was given in Eq. (32). We now make use of

$$\int_0^{\pi}d\theta\,\cos\theta\,\sin[(n+1)\theta]\sin[(s+1)\theta]$$

$$\equiv\pi(\delta_{n,s+1}+\delta_{s,n+1})/4. \tag{39}$$

This gives, after some rearrangement,

$$\langle Q\rangle-\langle Q\rangle_0=\cos\mu e^{-r^2}\sum_{n=0}^{\infty}r^{2n+1}\Big/\sqrt{n!(n+1)!}. \tag{40}$$

The coefficient $\cos\mu\equiv\sin m\chi$ is the classical result (20) for the displacement of $Q$. The quantum motion of the first apparatus reduces the average value of this displacement by a factor $S(r)$, which depends on the amplitude of the coherent state in which the oscillator was prepared:

$$S(r)=e^{-r^2}\sum_{n=0}^{\infty}r^{2n+1}\Big/\sqrt{n!(n+1)!}. \tag{41}$$

For small $r$, we have $S(r)\to r$. For large $r$, the sum in Eq. (41) can be written as an asymptotic series [14,15]

$$S(r)=1-\frac{1}{8r^2}-\frac{7}{128r^4}-\frac{75}{1024r^6}-\cdots. \tag{42}$$

Thus, when $r$ is large, $S(r)\to 1$. This is the expected result since a harmonic oscillator in a coherent state with large $r$ is almost classical. Figure 1 shows a plot of the function $S(r)$.

### B. Semiclassical description

The above results will now be compared with a semiclassical treatment similar to the one that was introduced in the preceding section. The second apparatus is always described by classical statistical mechanics. It is prepared in a Liouville distribution $L_2(Q,P)$ and it interacts with the first apparatus, for which we also assume a Liouville distribution. The latter is initially identical to the Wigner function $W(q,p)$ [8] that results from the first stage of the measurement. It is indeed possible to equate these two distributions because the first apparatus is in a coherent state, so that its Wigner function is everywhere positive. (If we had chosen another state, pure or mixed, whose Wigner function had negative regions, it would have been inconsistent to dequantize the first apparatus. This would simply mean that, in such a state, it is not a legitimate measuring apparatus. This point is further discussed in Sec. IV.)

We must now construct an interaction between the two apparatuses in such a way that $Q$ moves to a new value $Q+C$, as in Eq. (17). To respect classical mechanics, this has to be a continuous canonical transformation, generated by a Hamitonian

$$H_{\rm int}=CP/\epsilon, \tag{43}$$

as in Eq. (15). Here

$$C=\cos\theta=q/\sqrt{p^2+q^2}. \tag{44}$$

The reader who is not interested in computational details may skip from here to Eq. (66).

The variable canonically conjugate to $C$ is

$$p_C=(p^2+q^2)^{3/2}/2p=H_{\rm osc}/\sin\theta, \tag{45}$$

as may be checked by computing their Poisson bracket $[C,p_C]=1$. We may also write $p_C$ as

$$p_C=\pm H_{\rm osc}/\sqrt{1-C^2}. \tag{46}$$

Note that $-1\leqslant C\leqslant 1$, but for any given $C$ the domain of $p_C$ extends from $-\infty$ to $\infty$.

With the interaction (43), $C$ and $P$ are constant, while

$$p_C\to p_C'=p_C-P \tag{47}$$

and

$$Q\to Q'=Q+C, \tag{48}$$

as in Eqs. (16) and (17). The joint distribution thus evolves as

$$L_1(C,p_C)L_2(Q,P)\to L_1(C,p_C+P)L_2(Q-C,P). \tag{49}$$

To obtain the probability distribution of $Q$, we have to integrate the right-hand side of Eq. (49) over all the other canonical variables. First, we note that since $p_C$ extends from $-\infty$ to $\infty$, a shift by the parameter $P$ makes no difference in the integral: We can replace in the integrand $L_1(C,p_C+P)$ by $L_1(C,p_C)$. This allows us to return to the original canonical variables

$$L_1(C,p_C)dC\,dp_C=W(q,p)dq\,dp. \tag{50}$$

Once this is done, the integration over $P$ yields

$$\int L_2(Q-C,P)dP=F(Q-C), \tag{51}$$

where $C$ is given by the right-hand side of Eq. (44) and $F(Q)$ is the initial marginal distribution for $Q$.

As in the preceding full quantum treatment, we shall calculate the average final value $\langle Q\rangle$, for a given initial distribution $F(Q)$:

$$\langle Q\rangle=\int_{-\infty}^{\infty}Q\,dQ\int\int dq\,dpW(q,p)F(Q-C). \tag{52}$$

We again shift the origin $Q\to Q+C$ and obtain

$$\langle Q\rangle=\langle Q\rangle_0+\int\int dq\,dp\,W(q,p)\frac{q}{\sqrt{q^2+p^2}}, \tag{53}$$

where we have replaced $C$ by its explicit value (44) and made use of $\int\int W(q,p)dq\,dp=1$ and $\int F(Q)\,dQ=1$.

Explicitly, for the coherent state $|e^{i\mu}r\rangle$ we have (see Ref. [6], pp. 316 and 325)

$$W(q,p)=\pi^{-1}e^{-(q-\langle q\rangle)^2-(p-\langle p\rangle)^2}, \tag{54}$$

where

$$\langle q\rangle=\sqrt{2}r\ \cos\mu, \tag{55}$$

$$\langle p\rangle=\sqrt{2}r\ \sin\mu. \tag{56}$$

We likewise replace $q$ and $p$ by polar coordinates (whose physical meaning is that of action-angle variables [9])

$$q=\sqrt{2}s\ \cos\phi, \tag{57}$$

$$p=\sqrt{2}s\ \sin\phi, \tag{58}$$

so that $dq\ dp=2s\ ds\ d\phi$. The mean displacement of $Q$,

$$\delta Q=\langle Q\rangle-\langle Q\rangle_0, \tag{59}$$

is then found to be, after some rearrangement,

$$\delta Q=\frac{2}{\pi}\int_0^\infty s\ ds\ \oint d\phi\ \cos\phi e^{-2s^2-2r^2+4rs\cos(\phi-\mu)}. \tag{60}$$

Owing to the periodicity of $\phi$, it is possible to shift its origin by $\mu$, so that

$$\delta Q=\frac{2}{\pi}\int_0^\infty s\ ds\ \oint d\phi\ \cos(\phi+\mu)e^{-2s^2-2r^2+4rs\cos\phi}. \tag{61}$$

In the expression $\cos(\mu+\phi)\equiv\cos\mu\cos\phi-\sin\mu\sin\phi$, the second term is odd in $\phi$ and does not contribute to the integral.

Since now only $\cos\phi$ is involved in the integrand, it is convenient to remap the $s\phi$ plane so that $-\infty<s<\infty$ and $0\leqslant\phi\leqslant\pi$. We thus obtain

$$\delta Q=\frac{2}{\pi}\cos\mu e^{-2r^2}\int_{-\infty}^\infty s\ ds\int_0^\pi d\phi\ \cos\phi\ e^{-2s^2+4rs\cos\phi}. \tag{62}$$

The exponent can be written as $-2(s-r\cos\phi)^2+2r^2\cos^2\phi$. We shift the origin of $s$ by $r\cos\phi$ and perform the integration over $s$ explicitly, with the result

$$\delta Q=\frac{2}{\sqrt{\pi}}\cos\mu re^{-2r^2}\int_0^\pi d\phi\ \cos^2\phi\ e^{2r^2\cos^2\phi}. \tag{63}$$

We then substitute $\phi=y/2$ and $\cos^2\phi=(1+\cos y)/2$, and obtain

$$\delta Q=\frac{\cos\mu}{\sqrt{2\pi}}re^{-r^2}\int_0^\pi e^{2r^2\cos y}(1+\cos y)dy. \tag{64}$$

Finally, owing to the identity [16]

$$\int_0^\pi e^{z\cos y}(\cos ny)dy\equiv\pi I_n(z), \tag{65}$$

we have

$$\delta Q=\cos\mu\sqrt{\pi/2}\ re^{-r^2}[I_0(r^2)+I_1(r^2)]. \tag{66}$$

The expression that multiplies $\cos\mu$ (which was the classical result) tends to $r\sqrt{\pi/2}$ when $r$ is small, and to 1 when $r$ is large. It is plotted in Fig. 1.

Why is this result different from the preceding one, in Eq. (41)? The reason is that the two classically equivalent expressions for $C$ in Eq. (44) are not equivalent when these expressions become operators in quantum mechanics. The semiclassical result (66) was obtained by using the Wigner function $W(q,p)$ in Eq. (53) as if it were a classical probability density. This would be justified if the operator $C$, whose expectation value we seek, was of the Weyl-Wigner form [7]. However $C$, which is defined by its spectrum and eigenstates in Eq. (27), is not of that form. It is therefore not surprising that the semiclassical approximation gives a final result that is different from the quantum prediction. (On the other hand, the linear operator $q$ that was used in Sec. II has the Weyl-Wigner form, and therefore the two methods of calculation agree.)

## IV. SUMMARY AND OUTLOOK

The reader who expected to find in this article a solution of the so-called quantum measurement problem may be disappointed. Indeed, that problem is ill defined and it is understood in different ways by various authors [3,4]. Our way of formulating the problem, for which we can indicate a solution, simply is to say that quantum theorists describe the physical world by means of a complex Hilbert space (vectors and operators) that defies any realistic interpretation, while experimenters find plain numbers. The experimenters manipulate measuring instruments made of ordinary matter, for which quantum theory is assumed valid, but the ultimate outcome of the measuring process is essentially classical [1,2]. Therefore, at some stage, a transition has to be made from the quantum formalism to a classical language.

Here the role of decoherence should be clarified. Some authors claim that decoherence provides the solution of the measurement problem (with the particular meaning that they attribute to that problem), but many others dispute this point of view [17]. Yet, decoherence has an essential role in the preceding discussion. We have repeatedly taken the liberty of discarding one link in the von Neumann chain, after moving over to the next link. For example, in Eq. (9) we obtained the probability distribution for the second apparatus by averaging out the dynamical variables of the spin and the first apparatus. This was not a deliberate omission on our part. In the real world, with many interacting degrees of freeedom, correlations between consecutive links are exceedingly difficult to follow [18]. However, ''exceedingly difficult'' is not the same as ''impossible.'' Technological advances occasionally permit us to proceed one more step along the chain and it is important to examine whether this makes any difference in the final result. This is what the present article is about.

We have shown that if the measuring apparatus is suitably chosen (as in Sec. II), the transition from quantum mechanics to classical *statistical* mechanics can proceed in a consistent

way. However, as shown in Sec. III, a ''bad'' choice of apparatus is incompatible with a classical description (more precisely, the semiclassical results do not coincide with those predicted by quantum theory, though they may asymptotically agree for large $r$).

This brings us to the unavoidable fundamental question: What are the properties that are necessary for a physical system to be a legitimate measuring apparatus? Our results indicate that if an apparatus uses as its ''pointer'' a dynamical variable represented by an operator of the Weyl-Wigner form, it is legitimate to dequantize it and to proceed as if its Wigner function were a classical probability density. For other types of operators, the transition from quantum to classical mechanics usually is only an approximation (which may be excellent if the quantum state of the apparatus is quasiclassical).

Furthermore, the replacement of Wigner's function $W(q,p)$ by a Liouville function $L(q,p)$ is consistent only if $W(q,p)\geqslant 0$. We did not check this condition in Sec. II, because we did not need $W(q,p)$: Only the marginal probability distribution for $q$ was required. In general, if Wigner's function is explicitly needed, it has to be non-negative for a semi-classical treatment to proceed. Fortunately, this condition is likely to be fulfilled for any macroscopic apparatus that is not in a pure state, but rather in a mixed state with $\Delta q\Delta p\gg\hbar$ (this inequality is the hallmark of being ''macroscopic'') [19]. All the negative parts of $W$ are completely washed away by the coarseness of the apparatus.

In summary, there is nothing mysterious in the transition from the quantum world to the classical one. There is no need of invoking anthropomorphic concepts, such as consciousness. Plain orthodox quantum mechanics and classical statistical mechanics correctly reproduce all statistical predictions that can be verified in experiments, provided that the measuring apparatus satisfies suitable conditions, such as those discussed above. If enough care is exercised, no inconsistency shall arise.

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