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Josephson effect between trapped Bose-Einstein condensates

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We study the Josephson effect between atomic Bose-Einstein condensates. By drawing on an electrostatic analogy, we derive a semiclassical functional expression for the three-dimensional Josephson coupling energy in terms of the condensate density. Estimates of the capacitive energy and of the Josephson plasma frequency are also given. The effect of dissipation due to the incoherent exchange of normal atoms is analyzed. We conclude that coherent Josephson dynamics may already be observable in current experimental systems. [S1050-2947(98)50501-X]

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Recently [1–3], it has become possible to cool a macroscopic number ($\sim 10^4 - 10^6$) of magnetically confined, spin-polarized atoms down to temperatures on the order of 100 nK while maintaining densities sufficiently high ($10^{11} - 10^{15}$cm^{-3}) to permit the onset of Bose-Einstein condensation (BEC). From the ensuing theoretical work, it has been concluded that these Bose condensed atomic gases behave very differently from the ideal noninteracting gases, which yields the prospect of potentially displaying a rich phenomenology that might include vortex states and the Josephson effect [4–6].

In this paper we present a theoretical study of the Josephson dynamics between two atomic baths that have undergone BEC. The Josephson effect results from a collective mode of two weakly connected systems between which a macroscopic fraction of particles can tunnel with identical probability amplitude. Going beyond previously proposed one-dimensional [7] and dissipation-free [7,8] models, we present here a three-dimensional study of the Josephson effect between Bose condensates and estimate the effect of damping. We calculate the Josephson coupling energy, the capacitive energy (which accounts for quantum fluctuations of the phase), and the frequency of the Josephson plasma oscillation. Our main conclusion is that still lower temperatures than those achieved up to date are needed for a clear realization of the Josephson effect.

The collective dynamics of a Bose condensate at zero temperature is described by its macroscopic wave function $\Psi(\mathbf{r},t)$. If this is factorized as $\Psi=\sqrt{\rho}\,\exp(i\varphi)$, the standard energy functional can be written as

$$H=\int d\mathbf{r}\left[\frac{\hbar^2}{2m}(|\nabla\sqrt{\rho}|^2+\rho|\nabla\varphi|^2)+V_{\text{ext}}\rho+g\rho^2\right] \qquad (1)$$

($g=2\pi\hbar^2 a/m$), and the corresponding Hamilton equations lead to the Gross-Pitaveskii equations [5,9]. Neglecting depletion [10], the normalization can be taken as $\int d\mathbf{r}\ \rho=N$, N being the total number of atoms.

At sufficiently low temperatures the phase within one well can be regarded as uniform. This can be easily seen if one estimates the energy of a one-radian fluctuation of the phase across the condensate near equilibrium in a single spherical harmonic well [see Eq. (1)],

$$\Delta E=\frac{\hbar^2}{2m}\int d\mathbf{r}\ \rho_{\text{eq}}|\nabla\delta\varphi|^2\simeq\frac{\hbar^2}{2m}\frac{N}{4R^2}\simeq 0.7N^{3/5}\frac{\hbar\omega_0}{2}, \qquad (2)$$

where $R=a_0(15aN/a_0)^{1/5}$ is the cloud radius estimated within the Thomas-Fermi approximation [4], ω_0 is the harmonic-oscillator frequency within the well, and $a_0=(\hbar/m\omega_0)^{1/2}$ is the oscillator length. For the last approximate equality we have used typical parameters $a=5$ nm and $a_0=10^{-4}$ cm. The characteristic temperature ($\Delta E/k_B$) of such a fluctuation can be as big as 10 μK for the experiment of Mewes *et al.* [3]. The estimate (2) suggests that, in the interstitial region between two wells, where $\rho(\mathbf{r})$ decreases appreciably, spatial phase variations are less costly and thus easier to create at low temperatures. Here lies the essence of the low-energy Josephson dynamics, which we study below.

Let us consider two weakly connected condensates 1 and 2. We assume that the condensates are confined within spherical harmonic wells of the same frequency ω_0. First we wish to analyze the semiclassical dynamics. It is, of course, well known that the relative number δN and the relative phase χ of the condensate in the two wells may be treated as canonically conjugate variables. Nevertheless, by introducing a sufficiently coarse-grained average n of the number in (say) well 1 we may treat n and χ as simultaneously well-defined, and write for the wave function the ansatz

$$\tilde{\Psi}(\mathbf{r},t)\propto\Psi_1(\mathbf{r};n(t))+e^{i\chi(t)}\Psi_2(\mathbf{r};N-n(t)), \qquad (3)$$

where $\Psi_i(\mathbf{r};n)$ is the (real) equilibrium wave function for the isolated well i containing n bosons. It is straightforward to show that, to lowest order in the overlap integrals $\int\Psi_1\Psi_2$, the energy functional for $\tilde{\Psi}$ takes the form

$$H(\delta N,\chi)\simeq E_B(\delta N)+E_J(\delta N)(1-\cos\chi), \qquad (4)$$

where $E_B(\delta N)$ is the bulk energy of the two isolated wells with $\delta N \equiv N/2 - n$ transferred atoms, and $E_J(\delta N)$ is the Josephson coupling energy. $E_B(\delta N)$ may be expanded as

$$E_B(\delta N) \simeq E_B(0) + \mu'\left(\frac{N}{2}\right)\delta N^2 + \frac{1}{12}\mu'''\left(\frac{N}{2}\right)\delta N^4. \quad (5)$$

Within the Thomas-Fermi approximation, $\mu \sim N^{2/5}$ [4], so that the ratio between the third and second terms in the expansion is $0.32\delta N^2/N^2$, which means that the last term can be neglected in a wide range of situations. To avoid complications stemming from possible resonances between Josephson oscillations (see below) and intrawell excitations, we require $\mu(N/2+\delta N) - \mu(N/2-\delta N) \ll \hbar\omega_0$, where we use the result that the first normal mode of a spherical well lies approximately at $\hbar\omega_0$ above the ground state [5,6]. This condition is realized when $\delta N/N \ll 4.6N^{-2/5}$ for typical parameters. This may seem an important restriction on δN; however, the fraction of transferred atoms can be as big as 10% for the experiment of Anderson *et al.* [1], but it has to be $<2\%$ for the experiment of Mewes *et al.* [3].

The expression for the coupling energy $E_J(\delta N)$ in terms of Ψ_1 and Ψ_2, which can be derived from Eqs. (3) and (4), is rather complicated and difficult to handle. A much simpler expression can be obtained if one notes that the coupling energy must come entirely from the $\nabla\varphi$ term in Eq. (1), which in turn can be approximated as

$$\int_{\text{ext}} d\mathbf{r}\, \frac{\hbar^2}{2m}\rho(\mathbf{r},t)|\nabla\varphi(\mathbf{r},t)|^2 \equiv E[\varphi], \quad (6)$$

where the integration extends over the region exterior to the condensates (the results are quite independent of the precise location of the condensate borders). As argued before, the phase within the condensates can be assumed to be practically uniform. In the limit of very small phase difference, $\chi \ll 1$, the phase term in Eq. (4) can be approximated as $E_J\chi^2/2$. The only way for Eq. (6) to have such a dependence on the total phase difference χ and not on the details of $\nabla\varphi$ is that the condition $\delta E[\varphi]/\delta\varphi(\mathbf{r},t) = 0$ be satisfied. One may note that, except for trivial factors, this is the electrostatic equation for the electric displacement vector $\mathbf{D} \equiv \rho\boldsymbol{\nabla}\varphi$ in a medium with a nonuniform dielectric constant $\rho(\mathbf{r},t)$. Boundary conditions for $\varphi(\mathbf{r},t)$ are given by its value at the borders of each condensate, which act as conductors in this analogy. We have a system of two conductors held at a potential difference χ and a dielectric medium surrounding them. Then $E[\varphi]$ is essentially the energy of this capacitor. Potential theory tells us that $E[\varphi] \simeq (\hbar^2/2m)C[\rho]\chi^2$, which implies

$$E_J = \frac{\hbar^2}{m}C[\rho], \quad (7)$$

where $C[\rho]$ is the mutual capacitance of the two conductors in the presence of such a dielectric [11].

We take the origin of coordinates in the middle point of the double-well configuration, taking the z direction along the line that joins the two minima. Standard variational arguments can be invoked to prove that, for parallel plate boundary conditions,

$$\int \frac{dx\, dy}{\int_1^2 dz\, \rho(x,y,z)^{-1}} \leqslant C[\rho] \leqslant \left[\int_1^2 \frac{dz}{\int dx\, dy \rho(x,y,z)}\right]^{-1}. \quad (8)$$

The lower bound is obtained by removing the positive term $(\partial\varphi/\partial x)^2 + (\partial\varphi/\partial y)^2$ from the energy functional (6), while the upper bound is derived by taking φ independent of x,y. If V_{ext} depends weakly on x,y in the region that controls the capacitance, we can write $\rho(x,y,z) \simeq \rho(0,0,z)$ and then the two bounds become approximately equal to

$$C[\rho] \simeq A\left[\int_1^2 \frac{dz}{\rho(0,0,z)}\right]^{-1}, \quad (9)$$

where A is an effective area.

In order to proceed further, we need an estimate of $\rho = |\Psi|^2$ in the region of interest. For two identical wells in equilibrium, the ground-state wave function is symmetric in z. Neglecting the dependence on x,y, and within the WKB approximation [7], we can write

$$\Psi(z) = \frac{B}{\sqrt{p(z)}}\cosh\left[\frac{1}{\hbar}\int_0^z dz' p(z')\right], \quad (10)$$

where $p(z) = [2m(V_{\text{ext}}(0,0,z) - \mu)]^{1/2}$ and B is a constant to be determined later. Introducing Eq. (10) into Eq. (9) we obtain

$$E_J \simeq \frac{\hbar A|B|^2}{m}\left[2\tanh\left(\frac{S_0}{2}\right)\right]^{-1}, \quad (11)$$

where $S_0 \equiv \int_1^2 p(z)dz/\hbar$. For standard (quartic) barriers, $S_0 \simeq 2\pi\alpha(V_0 - \mu)/\hbar\omega_0$, where V_0 is the barrier height along the line $x = y = 0$, and α is of order unity ($\alpha = 8/3\pi$ for $\mu \ll V_0$ and $\alpha = 1/\sqrt{2}$ for $V_0 - \mu \ll V_0$).

The coefficient B must be calculated by properly matching Eq. (10) with good approximate solutions near the border of each well. If $S_0 \gtrsim 1$, then Eq. (10) can be matched with the solution given in Ref. [7] for the region $R \gg z - R \gg d$, $d = a_0(a_0/2R)^{1/3}$ being the distance from the classical radius R where the Thomas-Fermi approximation begins to fail. The result is $B \simeq ue^{-S_0/2}(\hbar/8\pi a d^3)^{1/2}$, with $u \simeq 0.397$ [7]. To estimate the effective area A, we note [7] that the form of the order parameter has a universal dependence on the position near the boundary of the Thomas-Fermi zone, namely $\Psi(r) \simeq \phi[(r-R)/d]/d\sqrt{8\pi a}$, where ϕ is a universal function (in that it does not depend on the confining potential) that varies appreciably within a scale of unity. Therefore $\rho(x,y,z)$ varies transversally on a scale such that $(\sqrt{x^2+y^2+R^2} - R)/d \sim 1$. These considerations yield an estimate of $A = 2^{2/3}v(a_0/R)^{4/3}\pi R^2$, where $v \sim 1$. The result is

$$E_J \simeq \frac{5.95u^2 v e^{-S_0}}{\tanh(S_0/2)}\left(\frac{N}{2}\right)^{1/3}\left(\frac{15a}{a_0}\right)^{-2/3}\frac{\hbar\omega_0}{2}. \quad (12)$$

We note that, because the prefactor in Eq. (12) has the same $N^{1/3}$ explicit dependence as the critical temperature T_c for a system of free confined bosons (which is approximately equal to the critical temperature of interacting bosons [12]), one could expect a simple relation between the two magnitudes. Knowing that $k_B T_c \simeq N^{1/3}\hbar\omega_0$ [12], then Eq. (12) can be rewritten as (for typical cases)

$$E_J \sim k_B T_c e^{-S_0}. \tag{13}$$

Also within the Thomas-Fermi approximation, a simple expression can be obtained for $E_B(\delta N)$ in Eq. (5), which, up to quadratic order in δN, can be written $E_B(\delta N)-E_B(0)=E_C\delta N^2/2$, where $E_C=2\mu'(N/2)$ is the capacitive energy. From the result of Ref. [4] for $\mu(N/2)$, we obtain

$$E_C \simeq \frac{4}{5}\left(\frac{N}{2}\right)^{-3/5}\left(\frac{15a}{a_0}\right)^{2/5}\frac{\hbar\omega_0}{2}. \tag{14}$$

Collecting terms, the Hamiltonian (4) can be written

$$H \simeq \frac{E_C}{2}\delta N^2 + E_J(1-\cos\chi), \tag{15}$$

giving the well-known equivalence to a pendulum[13]. The frequency of small oscillations (the Josephson plasmon), $\omega_{\mathrm{JP}}=\sqrt{E_J E_C}/\hbar$, turns out to be

$$\omega_{\mathrm{JP}} \simeq 1.54u\sqrt{v}\left(\frac{2a_0}{15aN}\right)^{2/15}\frac{e^{-S_0/2}}{\sqrt{\tanh(S_0/2)}}\omega_0, \tag{16}$$

which is typically a fraction (not necessarily very small) of the confining potential. As usually $\omega_0/2\pi \simeq 10-100$ Hz, we conclude that $\omega_{\mathrm{JP}} \lesssim 10$ Hz.

The ratio E_J/E_C is a good measure of the classical character of the relative phase χ. From Eqs. (12) and (14), we find

$$\frac{E_J}{E_C} \simeq 0.25u^2 v\left(\frac{2a_0}{15aN}\right)^{1/15}\frac{e^{-S_0}}{\tanh(S_0/2)}\frac{Na_0}{a}. \tag{17}$$

By varying S_0, the system can be driven from the classical regime ($E_J \gg E_C$) to the strong quantum limit ($E_J \ll E_C$). However, in the latter case, quantum fluctuations are only important if we operate at ultralow temperatures $k_B T \lesssim E_C$.

The Hamiltonian (15) describes the dynamics of a conservative system. In real life, however, we should expect a certain amount of damping. The most obvious source of such damping is the incoherent exchange of normal atoms, and a quantitative discussion requires a generalization of our results to nonzero temperature. Because the spatial scale of the normal component is quite different, in a harmonic trap, from that of the condensate, this generalization is much less trivial than for the case of a junction linking two homogeneous superconductors or superfluids, and we shall not attempt a quantitative discussion here. However, we shall give two qualitative arguments, based on two very different physical assumptions (corresponding essentially to the high and low barrier limits), to the effect that the damping associated with the normal component will overdamp the Josephson behavior and hence make it in practice unobservable down to temperatures on the order of $\hbar\omega_0/k_B$. In both cases, we find that normal atoms give an Ohmic contribution to the current,

$$I_n = -G\delta\mu, \tag{18}$$

where $\delta\mu$ is the chemical potential difference (for simplicity, we assume equilibrium within each well). As a result, the first Josephson equation is modified to read

$$\frac{d}{dt}\delta N = \frac{E_J}{\hbar}\sin\chi + I_n. \tag{19}$$

In the high barrier limit ($V_0 \gg k_B T$), the basic order-of-magnitude assumption is that any power-law factors occurring are negligible compared to the relevant WKB or Arrhenius-Kramers exponentials. At temperatures $T \lesssim T_c$, the normal component in each well will be appreciable and will be distributed over an energy range $\sim k_B T \gg \hbar\omega_0$, since [10] $k_B T_c/\hbar\omega_0 \sim N^{1/3} \gg 1$. We will assume that $k_B T$ is, nevertheless, small compared to $V_1 \equiv V_0-\mu_0$, with $\mu_0 \equiv \mu(N/2)$. The situation we have is $V_0 \gg k_B T \sim \mu_0 \gg \hbar\omega_0$. Under these conditions the typical spacing of the one-particle energy levels at energies $\sim k_B T$ is small compared to $\hbar\omega_0$, and we can ignore ''level-crossing'' effects [14] and treat the tunneling of uncondensed particles as incoherent. At high T, the total rate of crossing is then given by a standard Arrhenius-Kramers formula, $P_n=(\kappa\omega_0/2\pi)\exp(-V_1/k_B T)$, where usually [15] $\kappa \sim 1$. We have, approximately, $G \simeq P_n N_n/k_B T$ where N_n is the number of normal particles. The degree of damping can be estimated by comparing Eq. (18) with the Josephson supercurrent in a small oscillation of amplitude $\delta\mu$, namely,

$$I_s \sim \frac{E_J\delta\mu}{\hbar^2\omega_{\mathrm{JP}}}. \tag{20}$$

From Eq. (16), we estimate $\omega_{\mathrm{JP}} \sim N_0^{-2/15}e^{-S_0/2}\omega_0$ ($N_0 \sim N$ is the number of condensed particles), valid for typical parameters at $T=0$. Finite T corrections should not change the qualitative conclusions. We find $I_s/I_n \sim 2\pi N_0^{7/15}(\hbar\omega_0/k_B T)^2\exp(V_1/k_B T - S_0/2)$, since $N_n/N \sim (T/T_c)^3$. We conclude that, in the high-barrier limit, $I_s/I_n \ll 1$; hence the motion of the equivalent pendulum is overdamped, at least down to temperatures of the order of $\hbar\omega_0/k_B$. For temperatures below this our estimate fails for several reasons, not least because the density of normal component falls exponentially rather than as T^3.

The estimate presented above is, however, quite irrelevant for today's experiments, because the high-barrier condition requires $S_0/\pi \gg 2\mu_0/\hbar\omega_0 \sim 6-100$ (we take $\alpha \simeq 1$) for typical situations. On the other hand, the prefactor in $E_J/\hbar$ can be estimated as 10^4-10^5 Hz, and thus the critical current itself would be so small as to be completely unobservable. A more relevant regime is that in which the chemical potential lies near the top of the barrier, $V_1 \sim \hbar\omega_0/2$, so that $S_0 \sim 1-5$. In this regime, we can still expect the WKB formula for E_J derived above to yield a reasonable approximation. However, if $k_BT \gg \hbar\omega_0/2 \sim V_1$, the thermal cloud lies mostly above the barrier and a radically different approach is needed to study its transport properties. For simplicity, we introduce the drastic approximation that particles impinging on the barrier with energy E are transmitted with probability 1 if $E > V_0$ and zero if $E < V_0$. Then, the flow of normal atoms due to a fluctuation in $\delta\mu$ is only limited by the ''contact resistance,'' a concept taken from ballistic transport in nanostructures [16]. Adapting standard arguments to the case of bosons, we may write $G \equiv N_{\rm ch}/h$, where $N_{\rm ch}$ is an effective number of available transmissive channels. Within a continuum approximation, and assuming that only transverse channels with a minimum energy between μ_0 and $\mu_0 + k_BT$ are populated, we find (taking $\delta\mu \ll k_BT$) $N_{\rm ch} = mA_nk_BT/2\pi\hbar^2$, where A_n is a mean transverse contact area seen by normal particles [17]. Approximating [10] $A_n \sim 2\pi k_BT/m\omega_0^2$, we find $N_{\rm ch} \sim (k_BT/\hbar\omega_0)^2$, and hence

$$I_s/I_n \sim 2\pi N_0^{7/15}(\hbar\omega_0/k_BT)^2 e^{-S_0/2}, \qquad (21)$$

which, interestingly, is formally equivalent to the high-barrier result (since there $S_0/2 \sim \pi\alpha V_1/\hbar\omega_0 \gg V_1/k_BT$). Taking $k_BT = 10\hbar\omega_0$, we find $I_s/I_n \sim 0.38$ for $N \sim 10^4$, and 3.3 for $N \sim 10^6$, if $S_0 \sim 5$. The equivalent numbers for $S_0 \sim 1$ are 2.8 and 24. We tentatively conclude that effects of coherent Josephson dynamics can be observed in today's atomic Bose condensates if the barrier is low. Underdamped dynamics can be further favored by decreasing T and increasing N. It is important to note, however, that some aspects of the Josephson behavior can be observed in the overdamped regime, provided that thermal fluctuations are unimportant ($k_BT/E_J \lesssim 1$) [18], a not very stringent condition in the low-barrier limit [see Eq. (13)].

In summary, a systematic study of the Josephson effect between two weakly connected atomic Bose-Einstein baths has been presented. We have derived a three-dimensional functional expression for the Josephson coupling energy in terms of the condensate density. The capacitive energy and the frequency of the Josephson plasma oscillation have also been calculated within the WKB approximation. The effect of damping due to the incoherent exchange of normal atoms has been estimated in the limits of high and low potential barrier. Within the low-barrier regime, we find [see Eq. (21)] that weakly damped Josephson dynamics may already be observed in current experimental setups, and that coherence between atomic Bose condensates can be further enhanced by lowering the temperature and the potential barrier, as well as by increasing the number of condensate particles.

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[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995).

[2] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995).

[3] M. O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, C. G. Townsend, and W. Ketterle, Phys. Rev. Lett. **77**, 416 (1996); **77**, 988 (1996).

[4] G. Baym and C. J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).

[5] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).

[6] M. Edwards, P. A. Ruprecht, K. Burnett, R. J. Dodd, and C. W. Clark, Phys. Rev. Lett. **77**, 1671 (1996).

[7] F. Dalfovo, L. Pitaevskii, and S. Stringari, Phys. Rev. A **54**, 4213 (1996).

[8] A. Smerzi *et al.*, Phys. Rev. Lett. (to be published); S. Raghavan *et al.*, e-print cond-mat/99706220.

[9] L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)]; E. P. Gross, Nuovo Cimento **20**, 454 (1961); J. Math. Phys. **4**, 195 (1963).

[10] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. **78**, 3987 (1997).

[11] A more direct argument goes as follows. Rewrite Eq. (6) as $E[\varphi] = (\hbar^2/2m)\int_{\rm ext} d\mathbf{r}\, \rho(\nabla e^{i\varphi}) \cdot (\nabla e^{-i\varphi})$, and let $Z(\mathbf{r})$ be a solution of $\nabla \cdot (\rho \nabla Z) = 0$ with boundary conditions $Z(j) = e^{-i\varphi_j}$ $(j = 1,2)$. Identifying $Z(\mathbf{r}) = e^{-i\varphi(\mathbf{r})}$ and integrating by parts, we obtain $E[\varphi] = (\hbar^2/2m)C[\rho](1 - \cos\chi)$. A drawback of this reasoning is that $|Z| = 1$ is not guaranteed everywhere.

[12] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A **54**, R4633 (1996).

[13] P. W. Andersson, in *Lectures in the Many Body Problem*, edited by E. Caianello (Academic, New York, 1964).

[14] R. Rouse, S. Han, and J. E. Lukens, Phys. Rev. Lett. **75**, 1614 (1995).

[15] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).

[16] Y. Imry, in *Directions in Condensed Matter*, edited by G. Grinstein and E. Mazenko (World Scientific, Singapore, 1986).

[17] A more accurate calculation which exactly accounts for the thermal population of all channels (but yet assumes an energy-independent mean value of A_n) would replace k_BT by $[V_1(e^{V_1/k_BT} - 1)^{-1} - k_BT\ln(1 - e^{-V_1/k_BT})] \simeq k_BT\ln(k_BT/V_1)$.

[18] V. Ambegaokar and B. I. Halperin, Phys. Rev. Lett. **22**, 1364 (1969).