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Roll-hexagon transition in an active optical system

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We study the formation of hexagonal patterns and the roll-hexagon transition in an active nonlinear optical system, namely a two-level laser with an injected signal, which displays distinctive and noteworthy features from the most commonly encountered pattern-forming systems with broken inversion symmetry, such as convective hydrodynamic systems and passive nonlinear optical systems. With respect to these systems, we show here that the roll-hexagon transition is accompanied in the Fourier space by a spontaneous breaking of the inversion symmetry $\mathbf{k}\rightarrow-\mathbf{k}$, which is a signature of the different physical mechanisms involved in the elementary process of pattern formation. [S1050-2947(98)50104-7]

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Spontaneous pattern formation in systems driven away from equilibrium is a topic of considerable interest in many areas of physics [1]. Despite the wide diversity of physical systems where patterns may spontaneously occur, the basic mechanisms leading to pattern formation and competition are very general and, in boundary-free systems, they solely reflect the intrinsic symmetry of the problem. An important example of pattern transition is that between hexagons and rolls in two-dimensional isotropic systems with broken inversion symmetry [1–3]. The Rayleigh-Bernard convection with non-Boussinesq fluids [3] has been presented as a paradigm where this transition occurs. In nonlinear optics, hexagonal patterns and roll-hexagon competition have been described in various *passive* nonlinear systems, such as Kerr media with optical feedback [4], optical bistable systems [5], and counterpropagating beams in a nonlinear medium [6]. In these systems the nonlinear dynamics is usually described by a *real* order parameter $u=u(\mathbf{x},t)$, which, close to the instability for pattern formation, may be taken as a linear superposition of N *rolls* with wave vectors $\mathbf{k}_l$ and amplitudes A_l, where $|\mathbf{k}_l|=k_c$ and k_c is the critical wave number fixed by the system parameters. The formation of hexagonal patterns and the roll-hexagon transition arise from the interaction of $N=3$ rolls whose wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ satisfy the resonance condition $\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2=\mathbf{0}$ (see Fig. 1); the corresponding amplitude equations have the canonical form [2,3],

$$\partial_t A_1=\mu A_1+\sigma A_2 A_3^* -(|A_1|^2+\gamma|A_2|^2+\gamma|A_3|^2)A_1, \tag{1}$$

where μ measures the distance from the instability point, $\gamma>1$ is the cross-saturation parameter, and σ is zero when the system has the inversion symmetry $u\rightarrow-u$. Similar equations for A_2 and A_3 are obtained from Eq. (1) by cyclic permutation of the indeces. The bifuration scenario originating from Eqs. (1) is well known, and it explains the subcritical onset of hexagons, the transition between rolls and hexagons, and the existence of two kinds of phase locking leading to either positive or negative hexagons, depending on the sign of σ [2]. It is important to point out that the form of Eqs. (1) is determined not only by the symmetries of the system, but also by the requirement for the order parameter u to be real, so that the elementary patterns are not single *tilted waves* (TW's), but *rolls*, i.e., pairs of oppositely oriented TW's on the critical circle. The real nature of the order parameter is due either to trivial physical constraints or to *conservation laws* that must be satisfied in the elementary process of pattern formation. The former case occurs, for instance, in hydrodynamic systems, where the order parameter is directly related to a physical observable (such as the temperature of the convective fluid). The latter situation is commonplace in *passive* nonlinear optical systems, where the elementary process of pattern formation involves the simultaneous emission of two photons with opposite wave vectors in the transverse plane due to the conservation of the total photon momentum during four-wave mixing in the nonlinear medium [7]. Conversely, it is known that a single TW may be emitted in *active* optical systems, such as two-level and Raman lasers [8], photorefractive oscillators [9], and optical parametric oscillators [10(a)], with the notable exception of degenerate optical parametric oscillators [10(b),(c)]. However, the equations for these systems are usually invariant for inversion, so that the most likely patterns are single TW's or rhomboids [11].

In this Rapid Communication we investigate a type of roll-hexagon competition that occurs in a model of a homogeneously broadened, two-level laser with plane mirrors externally driven by a coherent plane-wave field [laser with

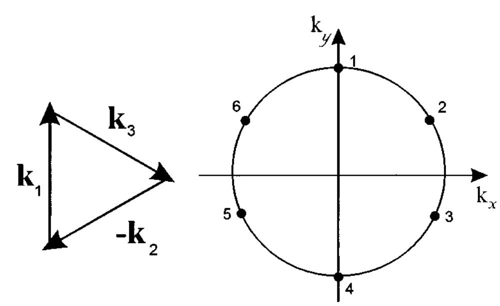

FIG. 1. Geometry for hexagonal patterns in Fourier space.

injected signal (LIS) [12]]. As was recently shown in Ref. [13], the effect of the external field is to break the inversion symmetry of the laser equations, permitting the formation of rolls and hexagons. However, as for two-level lasers without injected signal [8], it turns out that the elementary patterns at the instability point are TW's. This fact profoundly differentiates the pattern-forming properties of the LIS model from other hydrodynamical and optical systems. The case where the injected signal breaks the inversion symmetry and the rotational symmetry of the laser equations as well was investigated in Ref. [13]. Here we concentrate on the case where the external field breaks only the inversion symmetry of the laser equations, leaving the rotational symmetry around the laser cavity axis. In this case we show that the laser dynamics near the onset of instability for pattern formation may be described by a set of *six amplitude equations* for six TW's on the critical circle, located at the vertices of a regular hexagon (see Fig. 1), which represent a generalization of the amplitude equations (1), to the case where the elementary patterns are single TW's. The bifurcation scenario for these equations indicates the formation of regular hexagons and a transition to rolls when the external parameters are varied. However, the patterns selected and the roll-hexagon competition show unusual features that can be summarized as follows: (i) stable rolls are asymmetric, i.e., they are composed by the superposition of two oppositely oriented TW's with different amplitudes, symmetric rolls that are always linearly unstable; (ii) hexagons bifurcate *supercritically* from the trivial zero solution; and (iii) a transition between two kinds of hexagons occurs before rolls are stably selected. In the far field, this transition corresponds to a breaking of the inversion symmetry $\mathbf{k}\rightarrow-\mathbf{k}$ of the hexagonal patterns.

The starting point of the analysis is provided by the Maxwell-Bloch laser equations for a homogeneously broadened, two-level ring laser with an external plane-wave coherent field injected into the laser and propagating parallel to the cavity axis [12,13]. Using the same notations as in Ref. [13], these equations can be written as

$$\partial_T e = ia\nabla^2 e + i\Omega e - \sigma e + \sigma p + E,$$

$$\partial_T p = -p + (r-n)e,$$

$$\partial_T n = -bn + \tfrac{1}{2}(e^* p + e p^*), \tag{2}$$

where e and p are the scaled envelopes for the electric field and polarization of the medium referenced to the atomic transition frequency, n is proportional to the difference of the atomic inversion from its threshold value for lasing action, r is the pump parameter, E is the amplitude of the injected field, whose frequency is assumed to be exactly coincident with the atomic transition frequency, T is the time variable scaled to the decay time of polarization, Ω is the cavity detuning parameter, and σ and b are the decay rates of the electric field and of the population inversion, respectively, scaled to the decay rate of polarization. For a positive detuning parameter ($\Omega>0$) and assuming a weak injected signal, it turns out that the critical modes close to the instability point ($r\simeq 1$) for pattern formation are TW's on the critical circle of radius $k_c=(\Omega/a)^{1/2}$, as occurs for the free-running two-level laser [8]. The competition among these modes may be captured by a weakly nonlinear analysis of the laser Eqs. (2) close to the instability point; with the proper scaling of the laser parameters as discussed in Ref. [13], it turns out that the electric field at leading order may be written as

$$e(\mathbf{x},t)=\sqrt{b}\left(\alpha+\sum_{l=1}^{N} A_l \exp(i\mathbf{k}_l\cdot\mathbf{x})\right), \tag{3}$$

where $|\mathbf{k}_l|=k_c$, $\alpha=iE/\sqrt{b}\Omega$ is proportional to the amplitude of the injected field, N is the number of TW's on the critical circle, and A_l are their complex amplitudes that, neglecting spatial effects, satisfy the following amplitude equations [13]:

$$\begin{aligned}\partial_t A_l = {} & \mu A_l - \alpha^2 A_s^* \delta(\mathbf{k}_s+\mathbf{k}_l)\\ & -2\alpha \sum_{m,s=1}^{N} A_m A_s^* \delta(\mathbf{k}_m-\mathbf{k}_s-\mathbf{k}_l)\\ & -\alpha \sum_{m,s=1}^{N} A_m A_s \delta(\mathbf{k}_m+\mathbf{k}_s-\mathbf{k}_l)\\ & -\sum_{m,r,s=1}^{N} A_m A_r A_s^* \delta(\mathbf{k}_m+\mathbf{k}_r-\mathbf{k}_s-\mathbf{k}_l),\end{aligned} \tag{4}$$

where $\mu=r-1-2\alpha^2$, $t=T\sigma/(1+\sigma)$, and the phase of the external field is chosen in such a way that α is real and positive. The second term on the right-hand side of Eqs. (4) shows that two oppositely oriented TW's are always linearly coupled, whereas the quadratic nonlinear terms are not zero for triadics $\mathbf{k}_m,\mathbf{k}_s,\mathbf{k}_l$ satisfying the resonance conditions $\mathbf{k}_m \pm \mathbf{k}_s-\mathbf{k}_l=0$. Therefore, to fully capture the basic dynamics of TW's on the critical circle, $N=6$ modes located as in Fig. 1 on the vertices of a regular hexagon should be considered. The equations for these modes, labeled 1,2,3, . . . ,6, may be written as

$$\begin{aligned}\partial_t A_l = {} & \mu A_l - \left(2\sum_{j=1}^{6} |A_j|^2 - |A_l|^2\right) A_l - \alpha^2 A_{l+3}^*\\ & -2\alpha A_{l+1}A_{l-1} - 2\alpha A_{l-1}A_{l-2}^* - 2\alpha A_{l+1}A_{l+2}^*,\end{aligned} \tag{5}$$

($l=1,2,\ldots,6$), with the convention that the indices assume cyclically the values 1,2, . . . ,5,6,1,2, ... Note, that if one assumes $A_{l+3}=A_l^*$, Eqs. (5) reduce to the canonical form expressed by Eqs. (1). To investigate the bifurcation properties of the more general amplitude equations (5), let us first observe that they have the gradient form,

$$\partial_t A_l = -\frac{\partial U}{\partial A_l^*}, \tag{6}$$

where the potential $U=U(A_l,A_l^*)$ is given by

$$\begin{aligned}U = {} & -\mu\sum_{l=1}^{6} A_l A_l^* + \sum_{k,l=1}^{6} \gamma_{kl} A_k A_k^* A_l A_l^*\\ & +\tfrac{1}{2}\alpha^2 \sum_{l=1}^{6} (A_l A_{l+3} + A_l^* A_{l+3}^*)\\ & +2\alpha \sum_{l=1}^{6} (A_l^* A_{l-1} A_{l-2}^* + A_l A_{l-1}^* A_{l-2}),\end{aligned} \tag{7}$$

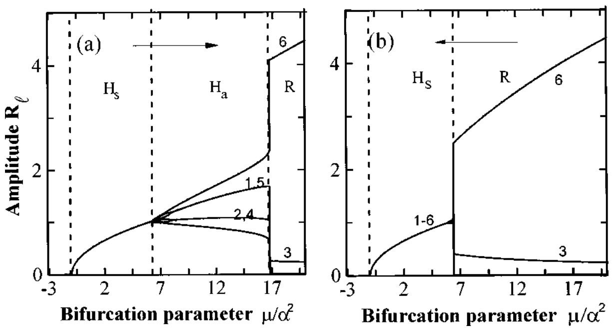

FIG. 2. Bifurcation diagrams of the amplitude equations [Eqs. (5)] obtained (a) when the bifurcation parameter $\eta=\mu/\alpha^2$ is adiabatically increased starting from the trivial solution at $\mu/\alpha^2=-3$, and (b) when the bifurcation parameter is subsequently decreased. The vertical dashed lines separate regions corresponding to different patterns: H_s, H_a are hexagons, R are asymmetric rolls. In the region of H_s hexagons, the six modes have the same amplitudes, whereas in the region of rolls, only two oppositely oriented modes survive (labeled as 3 and 6 in the figure).

and $\gamma_{kl}=(1+\delta_{kl})/2$. Equations (5) imply $dU/dt=-2\Sigma_{l=1}^{6}|\partial U/\partial A_l|^2$, so that the stable solutions of Eqs. (5) are the minima of the potential [Eq. (7)]. It should be noted that, introducing the polar decomposition $A_l=R_l\ \exp(i\phi_l)$, it turns out that the last two sums on the right-hand side of the potential [Eq. (7)] are phase sensitive and, therefore, they tend to favor precise phase-locking conditions. In particular, the third term is minimized for the phase locking $\phi_l+\phi_{l+3}=\pi$, whereas the last term favors a phase locking of the form $\phi_l+\phi_{l-2}-\phi_{l-1}=\pi$, which is exactly the ordinary phase locking for positive hexagonal patterns [3]. These two conditions are, however, not compatible with each other, so that they compete in the dynamics for pattern formation. To discuss the bifurcation properties of Eqs. (5), it is worth noting that, after a suitable rescaling of time t and amplitudes A_l in Eqs. (5), the driving parameters μ and α may enter the equations solely through the ratio $\eta=\mu/\alpha^2$, which we assume, therefore, as the bifurcation parameter of the system. Note that, from a physical viewpoint, $1/\eta$ gives a measure of the phase-sensitivity degree of the system, the limit $\eta=\infty$ corresponding to a complete loss of phase sensitivity (free-running laser). Linear stability analysis of the zero solution $A_l=0$ indicates that the laser emission in the forced mode is stable for $\eta<-1$, and at $\eta=-1$ an instability arises from the growth of independent *symmetric* rolls corresponding to $A_{l+3}=-A_l^*$; however, it turns out that the single-roll solution corresponding, for instance, to $A_1=-A_4^*=[(\mu+\alpha^2)/3]^{1/2}\exp(i\phi)$ and $A_l=0$ for $l\neq 1,4$ (ϕ is an arbitrary phase term) is always unstable with the growth of the other two rolls oriented at an angle $2\pi/3$. The bifurcating solution involves, therefore, all six TW's on the critical circle. The determination of the steady-state solutions of Eqs. (5), involving six modes, is a nontrivial matter to deal with analytically; however, the bifurcation diagram originating from these equations when the driving parameter η is adiabatically increased can easily be determined by numerical integration of Eqs. (5). The bifurcation diagram for the amplitudes R_l obtained by increasing the driving parameter starting from the forced-mode solution ($A_l=0$) is shown in Fig. 2(a). It

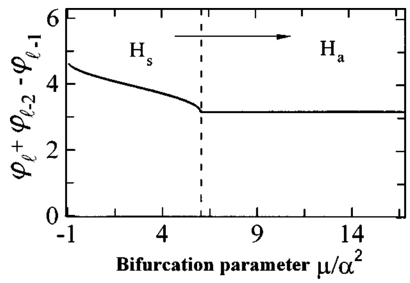

FIG. 3. Phase locking among TW's for H_s and H_a hexagons corresponding to the bifurcation diagram of Fig. 2(a).

turns out that hexagonal patterns bifurcate supercritically from the trivial solution, and for $-1<\eta<\sim 6.4$ the six TW's have the same amplitudes and are phase locked in such a way that $\phi_l+\phi_{l-2}-\phi_{l-1}$ is independent of l ($l=1,2,\ldots,6$), although it varies as η is increased (see Fig. 3). We call these hexagons H_s. The corresponding near-field intensity pattern is shown in Fig. 4(a); it should be noted that, although the phase of modes changes when η is varied, due to phase locking, the resulting intensity pattern is not influenced by these phase changes. At $\eta\simeq 6.4$, a symmetry-breaking bifurcation takes place, resulting in the emission of TW's with different intensities. Although the resulting near-field intensity pattern is very similar to that shown in Fig. 2(a), in the far field this bifurcation breaks the inversion symmetry $\mathbf{k}\rightarrow-\mathbf{k}$ typical of H_s hexagons. We will denote these hexagons by H_a. The phase locking for H_a hexagons is ruled by the condition $\phi_l+\phi_{l-2}-\phi_{l-1}=\pi$ (see Fig. 3), which is characteristic of ordinary hexagonal patterns [3]. However, we stress that in our model this implies asymmetric emission of TW's. We can physically understand this unusual feature on the basis of the energy competition between the two phase-sensitive terms in the potential (7). In fact, as previously discussed, the phase-locking condition $\phi_l+\phi_{l-2}-\phi_{l-1}=\pi$ is energetically favored by the last term of the potential (cubic in the mode amplitudes); however, it also implies $\phi_l+\phi_{l+3}=0$, i.e., an increase of the free energy due to the other phase-sensitive term, which is quadratic and involves the amplitudes of two oppositely oriented TW's. Minimization of this term leads to the spontaneous emission of TW's in opposite directions with different amplitudes, i.e., to H_a hexagons. At $\eta\simeq 16.8$ a third bifurcation takes place that results in the transition from H_a hexagons to *asymmetric* rolls composed by the superposition of two of the six TW's with opposite wave vectors and of different intensities, phase locked according to the rule $\phi_l+\phi_{l+3}=\pi$.

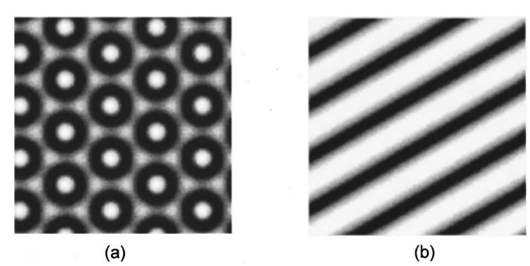

FIG. 4. Near-field intensity patterns obtained for (a) $\mu/\alpha^2=2$ (hexagons) and (b) $\mu/\alpha^2=17$ (rolls).

The amplitudes of the two TW's can be calculated analytically in a closed form and they are given by $R^2_{l,l+3}=[\mu \pm(\mu^2-4\alpha^4)^{1/2}]/2$. The corresponding near-field intensity pattern is shown in Fig. 4(b). Further increasing η, the amplitude of one of the two TW's decays toward zero, which results in the emission of a single tilted wave. Note that the bifurcating scenario so obtained corresponds to a progressive loss of phase sensitivity of the system induced by the external signal, and to the prevalence of the phase-insensitive character of the free-running laser emission. The bifurcation diagram that results when the amplitude equations [Eqs. (5)] are integrated by adiabatically *decreasing* η starting from the asymmetric roll solution is shown in Fig. 2(b). As can be seen, the asymmetric rolls are now stable down to $\eta\simeq6.4$, where a transition to H_s hexagons occurs. This indicates bistability between rolls and H_a hexagons.

In conclusion, we have studied the formation of hexagonal patterns in an active optical system with broken inversion symmetry. Owing to the peculiar mechanism involved in the elementary process of pattern formation, the onset of hexagons and the roll-hexagon transition in this system are deeply different from those most commonly found in other systems with broken inversion symmetry and generally described by the canonical model expressed by Eqs. (1). Although the results presented here have been obtained in a special nonlinear optical system, they can be expected to occur in other pattern-forming systems with broken inversion symmetry where the elementary patterns are tilted waves instead of rolls.

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