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Multiply connected Bose-Einstein-condensed alkali-metal gases: Current-carrying states and their decay

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The ability to support metastable current-carrying states in multiply connected settings is one of the prime signatures of superfluidity. Such states are investigated theoretically for the case of trapped Bose condensed alkali-metal gases, particularly with regard to the rate at which they decay via thermal fluctuations. The lifetimes of metastable currents can be either longer or shorter than experimental time scales. A scheme for the experimental detection of metastable states is sketched. [S1050-2947(98)50303-4]

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Multiply connected superfluid and superconducting systems can support states in which a persistent macroscopic particle current flows. While not truly eternal, these states can have extraordinarily long lifetimes, their decay requiring the occurrence of certain relatively infrequent but nevertheless topologically accessible (quantum or thermal) collective fluctuations [1–4]. With the many considerable successes and rapid progress in the experimental exploration of Bose-Einstein-condensed (BEC) alkali-metal gas systems [5], it seems reasonable to anticipate that multiply connected settings for BEC will soon become available, thus allowing superfluid properties such as persistent currents to be sought. The purpose of the present paper is to address, theoretically, the ability of BEC alkali-metal gas systems in multiply connected settings to support metastable current-carrying states, and to address the stability and decay of such states via thermal fluctuations. Complementary work by Rokhsar [6] addresses related questions regarding the creation of these states and their stability.

We adopt a phenomenological description in which we characterize the state of the BEC system by a macroscopic wave function $\Psi(\mathbf{r})$, in terms of which the condensate density n and current density $\mathbf{j}$ are given by

$$n(\mathbf{r})=|\Psi(\mathbf{r})|^2, \tag{1a}$$

$$\mathbf{j}(\mathbf{r})=\frac{\hbar}{2im}(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*). \tag{1b}$$

The free energy $\mathcal{F}$ of the state is given by the Gross-Pitaevskii form

$$\mathcal{F}=\int d^3r\left\{\frac{\hbar^2}{2m}|\nabla\Psi|^2+[V(\mathbf{r})-\mu]|\Psi|^2+\frac{g}{2}|\Psi|^4\right\}, \tag{2}$$

where m is the mass of an individual atom, $V(\mathbf{r})$ is an effective external potential describing the magnetic and optical confinement of the atoms, and g $(\equiv 4\pi\hbar^2 a/m)$ represents the interatomic interaction, with a being an effective scattering length. For the sake of simplicity we neglect any possible effects due to spin. This description is appropriate for analyzing the behavior of a BEC system at chemical potential μ and temperature T [7]. In order that the condensate be able to undergo the free-energy (and angular-momentum) changing fluctuation necessary for current dissipation, the condensate must not be isolated. Therefore we restrict the system to be at temperatures not far below the critical temperature T_c, in which case the noncondensed atoms serve to provide an energy and angular-momentum reservoir.

As our aim is to address multiply connected systems, we consider trap potentials $V(\mathbf{r})$ that confine the gas to a cylindrically symmetric toroidal region (Fig. 1). Hence, V depends only on r and z, where $\{r,\phi,z\}$ are the usual cylindrical polar coordinates. Moreover, we restrict our attention to systems in which the circumference of the torus L $(=2\pi\overline{r})$ is considerably greater than the condensate healing length ξ $[\approx(\hbar^2/mg\overline{n})^{1/2}$, where $\overline{n}$ is related to the maximum particle density], and the thickness of the torus R is comparable to or smaller than ξ [8]. This corresponds to a regime of low condensate density; hence, the Thomas-Fermi approximation [9] is not applicable. Traps having these gross features should be achievable by the use of magnetic and optical forces [10]. There are two main reasons for considering this setting: (i) there would be no locally stable current-carrying states if L were comparable to or smaller than ξ; (ii) for

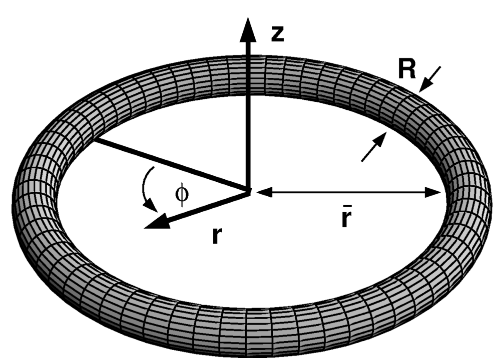

FIG. 1. Envisaged geometry of a trap supporting metastable current-carrying BEC states. The condensate healing length ξ is regarded as being small, compared with the circumference of the torus L $(=2\pi\overline{r})$, but larger than its thickness R.

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thicker samples (i.e., $R>\xi$), the relevant dissipative processes by which the persistent current decays become significantly more complicated (ultimately involving the nucleation of vortex rings) [11].

We shall be concerned with events in which the system decays from some metastable current-carrying state $\Psi_{\rm m}$ (which is a local minimum of $\mathcal{F}$) to a lower-energy (and typically more stable) state via a thermal fluctuation [12]. The current decays through a dissipative process during which the condensate density shrinks in magnitude over a region whose length is comparable to ξ. Dynamically, one can envisage this process as occurring via the passage of a vortex across the sample: a free-energy barrier must be overcome for this event to occur. The height of this barrier δF is given by the difference between the free energy of (the metastable state) $\Psi_{\rm m}$ and that of the transition state $\Psi_{\rm t}$, i.e., the lowest possible free-energy high point en route through configuration space between the initial and final metastable states. This thermally activated process should occur at a rate $\omega_0 e^{-\delta\mathcal{F}/kT}$, where, as we shall discuss later, the attempt frequency ω_0 does not contribute significantly to the temperature dependence of the rate.

In order to calculate the barrier heights, we first identify the collection of metastable current-carrying states $\{\Psi_{\rm m}\}$ and the relevant states $\{\Psi_{\rm t}\}$ for transitions between them. Both families of states are stationary points of $\mathcal{F}$, and therefore satisfy the time-independent Gross-Pitaevskii equation

$$\frac{\delta\mathcal{F}}{\delta\Psi^*}=-\frac{\hbar^2}{2m}\nabla^2\Psi+[V(\mathbf{r})-\mu]\Psi+g|\Psi|^2\Psi=0, \quad (3)$$

subject to periodic boundary conditions in the coordinate ϕ.

To address Eq. (3) we introduce a complete orthonormal set of ''transverse'' eigenfunctions $H_\nu(r,z)$ and the associated energy eigenvalues λ_ν, labeled by the ''channel'' index ν, which solve the eigenproblem

$$-\frac{\hbar^2}{2m}(r^{-1}\partial_r r\partial_r+\partial_z^2)H_\nu+V(r,z)H_\nu=\lambda_\nu H_\nu. \quad (4)$$

We then expand Ψ in terms of these eigenfunctions:

$$\Psi(r,\phi,z)=\sum_\nu F_\nu(\phi)H_\nu(r,z). \quad (5)$$

By inserting this expansion into Eq. (3) and making use of the orthogonality conditions for H_ν, we arrive at the following set of nonlinear coupled ordinary differential equations for the ''longitudinal'' wave functions $F_\nu(\phi)$:

$$F''_\nu+\alpha_\nu F_\nu-\beta\sum_{\nu_1\nu_2\nu_3}\Gamma^{\nu\nu_1}_{\nu_2\nu_3}F^*_{\nu_1}F_{\nu_2}F_{\nu_3}=C_\nu, \quad (6)$$

where primes denote derivatives with respect to ϕ, and the coefficients are defined by

$$\alpha_\nu\equiv 2m\overline{r^2}(\mu-\lambda_\nu)/\hbar^2, \quad (7a)$$

$$\beta\equiv 2m\overline{r^2}g/\hbar^2, \quad (7b)$$

$$\Gamma^{\nu\nu_1}_{\nu_2\nu_3}\equiv\int dz\int r\,dr\,H_\nu H_{\nu_1}H_{\nu_2}H_{\nu_3}. \quad (7c)$$

The terms C_ν, given in footnote [13], are functions of the $\{F_\nu\}$ and are negligible when the transverse extent of the condensate R is small compared with the circumference of the torus L, scaling as $(R/L)^2$.

The physical condition imposed above, viz., that ξ be much larger than the torus thickness R, enforces the condition $\lambda_0<\mu<\lambda_{\nu\neq 0}$. Hence, except for $\nu=0$, we have $\alpha_\nu<0$. In practice, α_ν is expected to be quite large, scaling as $(L/R)^2$.

In the low-density limit the $\nu=0$ channel should dominate, all other channels being occupied weakly. We incorporate this notion by introducing a book-keeping parameter Λ into Eq. (6): for $\nu\neq 0$ we make the replacement $\beta\rightarrow\Lambda\beta$. It can be verified that the nonleading terms in Λ may be neglected for the present purposes. (Incorporating their effects is straightforward, if tedious.) Similarly, one can also incorporate the effects of the $C_\nu[F]$. Hence, we find that the relevant states $\{\Psi_{\rm m}\}$ and $\{\Psi_{\rm t}\}$ can adequately be described in terms of F_0.

Within the approximation scheme just outlined, the uniform current-carrying states have the form $\Psi_{\rm m}=f_{\rm m}e^{iS_{\rm m}}H_0$, with

$$f_{\rm m}^2=N_{\rm m}/2\pi=(\alpha-n_{\rm m}^2)/\beta\Gamma, \quad (8a)$$

$$S_{\rm m}=n_{\rm m}\phi, \quad (8b)$$

for integral $n_{\rm m}$. For the sake of brevity, we now write α and Γ in place of α_0 and Γ^{00}_{00}. At the stated level of approximation, $N_{\rm m}$ is the number of condensed particles in the metastable state and Γ^{-1} is R^2L, i.e., the volume occupied by the condensate. By considering the second variation of $\mathcal{F}$ it can be readily shown that these states are local minima (and hence metastable), provided $n\zeta\leqslant 1$, where $\zeta_{\rm m}\equiv(4\pi/\beta\Gamma N_{\rm m})^{1/2}\approx 2\pi\xi/L$ is the dimensionless coherence length. (This limit on the maximum stable value of n_m is the same as one would find using Landau's criterion for the critical velocity.)

The transition states $\Psi_{\rm t}=f_{\rm t}e^{iS_{\rm t}}H_0$ are given by

$$f_{\rm t}^2=(N_{\rm t}/2\pi)[1-\Delta^2\,\mathrm{sech}^2(\Delta\phi/\zeta_{\rm t})], \quad (9a)$$

$$f_{\rm t}^2\partial_\phi S_{\rm t}=(N_{\rm t}/2\pi)n_{\rm t}. \quad (9b)$$

Far from a region of length ξ, the amplitude $f_{\rm t}$ is constant $(f_{\rm t}^2\sim N_{\rm t}/2\pi)$ and the phase $S_{\rm t}$ winds uniformly $(S_{\rm t}\sim n_{\rm t}\phi)$. The coefficients in Eq. (9) appear simplest when expressed in terms of the dimensionless coherence length $\zeta_{\rm t}\equiv(4\pi/\beta\Gamma N_{\rm t})^{1/2}$:

$$N_{\rm t}/2\pi=(\alpha-n_{\rm t}^2)/\beta\Gamma, \quad (10a)$$

$$n_{\rm t}=n-\pi^{-1}\cos^{-1}(n_{\rm t}\zeta_{\rm t}), \quad (10b)$$

$$\Delta^2=1-(n_{\rm t}\zeta_{\rm t})^2. \quad (10c)$$

As $N_{\rm m}$ and $N_{\rm t}$ differ only by quantities of order ξ/L, either of them may be used to characterize the number of condensed particles. The transition states must have the property that they are saddle points of $\mathcal{F}$ with only one direction of negative curvature. (This unstable direction is the relevant reaction coordinate.) It can readily be shown that the states in Eq. (9) satisfy this condition as long as $\xi > R$, or equivalently $\mu < \lambda_{\nu\neq 0}$. Thus, our approximation scheme for a BEC in a three-dimensional trap reduces the problem precisely to the one-dimensional problem addressed by Little [1], Langer and Ambegaokar [2], and McCumber and Halperin [3]. A useful by-product of the present approach is that it provides a scheme for determining the intrinsic resistance of superconducting wires clad by normal-state materials (and thus having proximity-induced superconductivity).

Having found the relevant states, we now calculate the free-energy barrier for dissipative fluctuations. It can be shown that states Ψ satisfying Eq. (3) have a free energy

$$\mathcal{F} = -\frac{g}{2}\sum_{\nu_0\nu_1\nu_2\nu_3}\Gamma^{\nu_0\nu_1}_{\nu_2\nu_3}\int d\phi F^*_{\nu_0}F^*_{\nu_1}F_{\nu_2}F_{\nu_3}. \tag{11}$$

Using this expression, along with Eqs. (8) and (9), we find that

$$\delta\mathcal{F} = \frac{1}{2}\,\delta\mathcal{F}_0[\Delta(2+(n_t\zeta_t)^2) - 3n_t\zeta_t\cos^{-1}(n_t\zeta_t)], \tag{12}$$

where $\delta\mathcal{F}_0$ is the long wavelength (i.e., $n_t \to 0$) value of $\delta\mathcal{F}$; i.e.,

$$\delta\mathcal{F}_0 = \frac{\hbar^2}{m}\left(\frac{32N_t^3 a}{9R^2L^3}\right)^{1/2}. \tag{13}$$

We now develop order-of-magnitude estimates for the decay rates of metastable states via thermal fluctuations. Let us consider ^{87}Rb, for which the scattering length a is 5.8 nm. We take a harmonic trapping potential $V(\mathbf{r}) = (1/2)m\omega^2[(r-\overline{r})^2+z^2]$, whose ground-state width $\sqrt{\hbar/m\omega}$ can be identified with the width of the condensate R. To estimate T_c we consider N noninteracting atoms in the potential $V(\mathbf{r})$. By virtue of the geometry (i.e., $R \ll L$) we can ignore the curvature of the torus, giving us a density of states $\rho(E) = (4/3)(1/\hbar\omega)^2(mL^2/2\pi^2\hbar^2)^{1/2}E^{3/2}$. Integrating this with the Bose occupation factor reveals that $T_c \approx 1.28(\hbar^2/m)(N/R^4L)^{2/5}$. For example, if we assume that $N \approx 10^6$, $N_t \approx 2.5\times 10^4$, $R \approx 1$ μm, and $L \approx 100$ μm, then we find $\delta\mathcal{F}_0/k_B = 3.2$ μK, and $T_c = 0.28$ μK. The barrier height is sensitive to changes in N_t, and can therefore be manipulated by heating or cooling the sample.

The Arrhenius formula for the decay rate in terms of the barrier height is $\Gamma \approx \omega_0 e^{-\delta\mathcal{F}/kT}$. The attempt frequency ω_0 can be estimated by using the value of the microscopic relaxation time τ, together with the assumption that each coherence volume in the sample fluctuates independently [1]. A realistic estimate for τ is the classical collision time for a dilute gas [i.e., $\tau^{-1} \sim \sigma n v \sim a^2(N/V)(k_B T/m)^{1/2} \sim 5\times 10^4$ Hz], giving lifetimes for the metastable states on the order of seconds. Even beyond the limits of validity of our calculation, one expects $\delta\mathcal{F}$ to be a monotonically increasing function of the density. Hence, the barriers can be extremely large at low temperatures, allowing a continuous tuning of the metastable state lifetime from microseconds to times longer than the lifetime of the condensate.

We now discuss two of the issues necessary for the experimental testing of the predictions presented in this paper. We have been considering the decay of metastable states, but have not yet addressed the issue of how to create them. Various approaches to creating a metastable current-carrying state that rely on the superfluid properties of the condensate have been discussed in detail by Rokhsar [6]. Another technique, which does not rely on the superfluid nature of the condensate, takes advantage of the spatial separation of the condensed and noncondensed atoms. One imagines starting the whole system rotating (for instance, by using a rotating nonaxisymmetric field) then applying a localized perturbation (such as from a sharply focused laser) that stops the thermal atoms but leaves the condensate rotating.

The second experimental matter to be addressed is the detection of metastable current-carrying states. Perhaps the least difficult scheme would make use of present phonon imaging techniques [14]. The experimental configuration could be as follows: A pulse of laser light generates a local rarefaction of the condensate, which then travels as two waves, one moving clockwise, the other counterclockwise. By nondestructive imaging techniques one might then observe where the two waves meet, which gives the velocity of the metastable supercurrent. This is only feasible if the speed of sound c is comparable to the velocity v with which the condensate moves around the annulus. Linearizing Eq. (6) gives $c = (g\Gamma N_m/2\pi m)^{1/2} \approx 1.2$ mm/s, which is only 30 times greater than $v = \hbar/m\overline{r} \approx 46$ μm/s.

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[1] W. A. Little, Phys. Rev. **156**, 396 (1967).
[2] J. Langer and V. Ambegaokar, Phys. Rev. **164**, 498 (1967).
[3] D. McCumber and B. Halperin, Phys. Rev. B **1**, 1054 (1970).
[4] See also M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1975), Chap. 7.
[5] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995); C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995); K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *ibid.* **75**, 3969 (1995).
[6] D. S. Rokhsar, Phys. Rev. Lett. **79**, 2164 (1997); D. S. Rokhsar, e-print cond-mat/9709212.
[7] Such a description is closer in spirit to the phenomenological Ginzburg-Landau one than to the zero-temperature Gross-Pitaevskii description. *A priori*, the effective coefficients in Eq.

(2) differ from the zero-temperature values. However, this difference is not essential, in that we express results in terms of experimental quantities, such as the number of condensed particles N and their wave vector k. In our numerical estimates we do need the effective scattering length a, which we approximate by its zero-temperature value. The absence of a vector potential in Eq. (2) indicates that the ''normal'' fluid of thermal excitations is at rest.

[8] For the case of superconducting wires, the intermediate regime in which L does not greatly exceed ξ has been analyzed by M. B. Tarlie, E. Shimshoni, and P. M. Goldbart, Phys. Rev. B **49**, 494 (1994); M. B. Tarlie, Ph.D. thesis, University of Illinois at Urbana-Champaign, 1995 (unpublished).

[9] G. Baym and C. J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).

[10] For example, one can envisage creating a multiply connected sample via the modification of an existing pancake-shaped magnetic trap [5], by puncturing it axially with a blue-detuned (and thus repulsive) laser beam.

[11] J. S. Langer and M. E. Fisher, Phys. Rev. Lett. **19**, 560 (1967); for a discussion see, e.g., R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, England, 1991), Chap. 8.

[12] In this paper we need not consider decay processes associated with quantum fluctuations, since in the temperature region in question $(T \sim T_c)$, thermal fluctuations dominate.

[13] These terms are given by $C_\nu = \Sigma_{\nu_1} \alpha_{\nu_1} \Delta_{\nu\nu_1} F_{\nu_1} + \beta \Sigma_{\nu_1\nu_2\nu_3} F^*_{\nu_1} F_{\nu_2} F_{\nu_3} D^{\nu_2\nu_3}_{\nu\nu_1}$, in which the quadratic coefficients $\Delta_{\nu\nu_1}$ and the quartic coefficients $D^{\nu_2\nu_3}_{\nu\nu_1}$ are given by $\Delta_{\nu\nu_1} = \overline{r^{-2}} \int d\sigma (r^2 - \overline{r^2}) H_\nu H_{\nu_1}$, and $D^{\nu_2\nu_3}_{\nu\nu_1} = \overline{r^{-2}} \int d\sigma (r^2 - \overline{r^2}) H_\nu H_{\nu_1} H_{\nu_2} H_{\nu_3}$. In these integrals the measure $d\sigma$ is $r dr dz$, with r ranging from 0 to ∞ and z ranging over the whole real line.

[14] M. R. Andrew, K. M. Kurn, H.-J. Miesner, D. S. Durfee, C. G. Townsend, S. Inouye, and W. Ketterle, Phys. Rev. Lett. **79**, 553 (1997).