Kinematics and hydrodynamics of spinning particles

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In the first part (Secs. I and II) of this paper, starting from the Pauli current, we obtain the decomposition of the nonrelativistic field velocity into two orthogonal parts: (i) the ''classical'' part, that is, the velocity $\mathbf{w}=\mathbf{p}/m$ in the center of mass (c.m.), and (ii) the ''quantum'' part, that is, the velocity $\mathbf{V}$ of the motion of the c.m. frame (namely, the internal ''spin motion'' or *Zitterbewegung*). By inserting such a complete, composite expression of the velocity into the kinetic-energy term of the nonrelativistic classical (i.e., Newtonian) Lagrangian, we straightforwardly get the appearance of the so-called quantum potential associated, as it is known, with the Madelung fluid. This result provides further evidence of the possibility that the quantum behavior of microsystems is a direct consequence of the fundamental existence of spin. In the second part (Secs. III and IV), we fix our attention on the total velocity $\mathbf{v}=\mathbf{w}+\mathbf{V}$, now necessarily considering relativistic (classical) physics. We show that the proper time entering the definition of the four-velocity v^μ for spinning particles has to be the proper time τ of the c.m. frame. Inserting the correct Lorentz factor into the definition of v^μ leads to completely different kinematical properties for v^2. The important constraint $p_\mu v^\mu=m$, identically true for scalar particles but just assumed *a priori* in all previous spinning-particle theories, is herein derived in a self-consistent way. [S1050-2947(98)03701-9]

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I. MADELUNG FLUID: A VARIATIONAL APPROACH

The Lagrangian for a nonrelativistic scalar particle may be assumed to be

$$\mathcal{L}=\frac{i\hbar}{2}\left[\psi^*\partial_t\psi-(\partial_t\psi^*)\psi\right]-\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi-U\psi^*\psi, \tag{1}$$

where U is the external potential energy and the other symbols have the usual meaning. It is known that, by taking the variations of $\mathcal{L}$ with respect to ψ,ψ^*, one can get the Schrödinger equations for ψ^* and ψ, respectively.

In contrast, since a generic scalar wave function $\psi\in\mathbb{C}$ can be written as

$$\psi=\sqrt{\rho}\,\exp[i\varphi/\hbar], \tag{2}$$

with $\rho,\varphi\in\mathbb{R}$, we take the variations of

$$\mathcal{L}=-\left[\partial_t\varphi+\frac{1}{2m}(\nabla\varphi)^2+\frac{\hbar^2}{8m}\left(\frac{\nabla\rho}{\rho}\right)^2+U\right]\rho \tag{3}$$

with respect to ρ and φ. We then obtain [1–3] the two equations for the so-called *Madelung fluid* [4] (which, taken together, are equivalent to the Schrödinger equation):

$$\partial_t\varphi+\frac{1}{2m}(\nabla\varphi)^2+\frac{\hbar^2}{4m}\left[\frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2-\frac{\Delta\rho}{\rho}\right]+U=0 \tag{4}$$

and

$$\partial_t\rho+\nabla\cdot(\rho\nabla\varphi/m)=0, \tag{5}$$

which are the Hamilton-Jacobi and the continuity equations for the ''quantum fluid,'' respectively, where

$$\frac{\hbar^2}{4m}\left[\frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2-\frac{\Delta\rho}{\rho}\right]\equiv-\frac{\hbar^2}{2m}\frac{\Delta|\psi|}{|\psi|} \tag{6}$$

is often called the quantum potential. Such a potential derives from the penultimate term on the right-hand side (rhs) of Eq. (3), that is to say, from the (single) ''nonclassical term''

$$\frac{\hbar^2}{8m}\left(\frac{\nabla\rho}{\rho}\right)^2 \tag{7}$$

entering our Lagrangian $\mathcal{L}$.

Notice that we got the present *hydrodynamical reformulation* of the Schrödinger theory directly from a variational approach [3]. This procedure, as we are going to see, offers us a physical interpretation of the nonclassical terms appearing in Eqs. (3) and (4). On the contrary, Eqs. (4) and (5) are ordinarily obtained by inserting relation (2) into the Schrödinger equation and then separating the real and the imagi-

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nary parts: a rather formal procedure, which does not shed light on the underlying physics.

Let us recall that an early physical interpretation of the so-called quantum potential, that is to say, of term (6) was forwarded by de Broglie's pilot-wave theory [5]; in the 1950s Bohm [6] revisited and completed de Broglie's approach in a systematic way [and sometimes Bohm's theoretical formalism is referred to as the ''Bohm formulation of quantum mechanics,'' alternative and complementary to Heisenberg's (matrices and Hilbert spaces), Schroedinger's (wave functions), and Feynman's (path integrals) theory]. From Bohm's time up to the present, several conjectures about the origin of that mysterious potential have been made by postulating ''subquantal'' forces, the presence of an ether, and so on. Well known are also the derivations of the Madelung fluid within the stochastic mechanics framework [7,2]: In those theories, the origin of the nonclassical term (6) appears as substantially *kinematical*. In the non-Markovian approaches, [2] for instance, after having assumed the existence of the so-called *Zitterbewegung*, a spinning particle appears as an extendedlike object, while the quantum potential is tentatively related to an internal motion.

But we do not need to follow any stochastic approach, even if our philosophical starting point is the *recognition* of the existence [8–12] of a *Zitterbewegung*, diffusive, or *internal* motion [i.e., of a motion observed *in* the center-of-mass (c.m.) frame, which is the one where $\mathbf{p}=\mathbf{0}$ by definition], in addition to the (external, drift, translational, or convective) motion *of* the c.m. In fact, the existence of such an internal motion is denounced not only by the mere presence of spin but by the remarkable fact that in the standard Dirac theory the particle impulse $\mathbf{p}$ is *not* in general parallel to the velocity: $\mathbf{v}\neq\mathbf{p}/m$; moreover, while $[\hat{\mathbf{p}},\hat{H}]=\mathbf{0}$ so that $\mathbf{p}$ is a conserved quantity, the quantity $\mathbf{v}$ is *not* a constant of motion: $[\hat{\mathbf{v}},\hat{H}]\neq\mathbf{0}$ ($\hat{\mathbf{v}}\equiv\boldsymbol{\alpha}\equiv\gamma^0\boldsymbol{\gamma}$ being the usual vector matrix of Dirac theory). Let us explicitly note, moreover, that in dealing with the *Zitterbewegung* it is highly convenient [10,12] to split the motion variables as (the dot meaning derivation with respect to time)

$$\mathbf{x}=\boldsymbol{\xi}+\mathbf{X},\quad \dot{\mathbf{x}}\equiv\mathbf{v}=\mathbf{w}+\mathbf{V}, \tag{8}$$

where $\boldsymbol{\xi}$ and $\mathbf{w}\equiv\dot{\boldsymbol{\xi}}$ describe the motion of the c.m. in the chosen reference frame, while $\mathbf{X}$ and $\mathbf{V}\equiv\dot{\mathbf{X}}$ describe the internal motion with reference to the c.m. frame (c.m.f.). (Notice that what is called the diffusion velocity $\mathbf{v}_{\mathrm{dif}}$ in the stochastic approaches is nothing but our $\mathbf{V}$.) From a dynamical point of view, the conserved electric current is associated with the helical trajectories [8–10] of the electric charge (i.e., with $\mathbf{x}$ and $\mathbf{v}\equiv\dot{\mathbf{x}}$), while the center of the particle Coulombian field is associated with the geometrical center of such trajectories (i.e., with $\boldsymbol{\xi}$ and $\mathbf{w}\equiv\dot{\boldsymbol{\xi}}=\mathbf{p}/m$).

Returning to the Lagrangian (3), it is now possible to attempt an interpretation [3] of the nonclassical term $(\hbar^2/8m)(\boldsymbol{\nabla}\rho/\rho)^2$ appearing therein. So the first term on the rhs of Eq. (3) represents, apart from the sign, the total energy

$$\partial_t\varphi=-E, \tag{9}$$

whereas the second term is recognized to be the kinetic energy $\mathbf{p}^2/2m$ of the c.m. if one assumes that

$$\mathbf{p}=-\boldsymbol{\nabla}\varphi. \tag{10}$$

The third term, which originates the quantum potential, will be shown below to be interpretable as the kinetic energy in the c.m.f., that is, the internal energy due to the *Zitterbewegung* motion. It will soon be realized, therefore, that in the Lagrangian (3) the sum of the two kinetic-energy terms $\mathbf{p}^2/2m$ and $\frac{1}{2}m\mathbf{V}^2$ is nothing but *a mere application of the König theorem*. We are not going to exploit, as is often done, the arrival point, i.e., the Schrödinger equation; in contrast, we are going to exploit a nonrelativistic (NR) analog of the Gordon decomposition [13] of the Dirac current, namely, a suitable decomposition of the *Pauli current* [14]. In so doing, we shall find an interesting relation between *Zitterbewegung* and spin.

II. THE QUANTUM POTENTIAL AS A CONSEQUENCE OF SPIN AND *ZITTERBEWEGUNG*

Over the past 30 years Hestenes [15] systematically employed the Clifford algebra language in the description of the geometrical, kinematical, and hydrodynamical (i.e., *field*) properties of spinning particles, both in relativistic and NR physics, i.e., both for Dirac theory and for Schrödinger-Pauli theory. In the small-velocity limit of the Dirac equation or directly from the Pauli equation, Hestenes obtained the decomposition of the particle velocity

$$\mathbf{v}=\frac{\mathbf{p}-e\mathbf{A}}{m}+\frac{\boldsymbol{\nabla}\times\rho\mathbf{s}}{m\rho}, \tag{11}$$

where the speed of light c is assumed to be equal to 1, the quantity e is the electric charge, $\mathbf{A}$ is the external electromagnetic vector potential, $\mathbf{s}$ is the *spin vector* $\mathbf{s}\equiv\rho^{-1}\psi^\dagger\hat{\mathbf{s}}\psi$, and $\hat{\mathbf{s}}$ is the spin operator, usually represented in terms of Pauli matrices as

$$\hat{\mathbf{s}}\equiv\frac{\hbar}{2}(\sigma_x;\sigma_y;\sigma_z). \tag{12}$$

[Hereafter, every quantity is a *local* or *field* quantity: $\mathbf{v}\equiv\mathbf{v}(\mathbf{x};t)$, $\mathbf{p}\equiv\mathbf{p}(\mathbf{x};t)$, $\mathbf{s}\equiv\mathbf{s}(\mathbf{x};t)$, etc.] As a consequence, the internal (*Zitterbewegung*) velocity reads

$$\mathbf{V}\equiv\frac{\boldsymbol{\nabla}\times\rho\mathbf{s}}{m\rho}. \tag{13}$$

Let us repeat the previous derivation, now by making recourse to the ordinary tensor language, *from* the familiar expression of the Pauli current [14] (i.e., from the Gordon decomposition of the Dirac current in the NR limit):

$$\mathbf{j}=\frac{i\hbar}{2m}[(\boldsymbol{\nabla}\psi^\dagger)\psi-\psi^\dagger\boldsymbol{\nabla}\psi]-\frac{e\mathbf{A}}{m}\psi^\dagger\psi+\frac{1}{m}\boldsymbol{\nabla}\times(\psi^\dagger\hat{\mathbf{s}}\psi). \tag{14}$$

A spinning NR particle can be simply factorized into

$$\psi\equiv\sqrt{\rho}\Phi, \tag{15}$$

Φ being a Pauli two-component spinor, which has to obey the normalization constraint

$$\Phi^\dagger\Phi = 1$$

if we want to have $|\psi|^2 = \rho$.

By definition $\rho\mathbf{s} \equiv \psi^\dagger\hat{\mathbf{s}}\psi \equiv \rho\Phi^\dagger\hat{\mathbf{s}}\Phi$; therefore, introducing the factorization $\psi \equiv \sqrt{\rho}\Phi$ into the above expression (14) for the Pauli current, one obtains [3]

$$\mathbf{j} \equiv \rho\mathbf{v} = \rho\,\frac{\mathbf{p} - e\mathbf{A}}{m} + \frac{\boldsymbol{\nabla}\times\rho\mathbf{s}}{m}, \tag{16}$$

which is nothing but Hestenes's decomposition (11) of $\mathbf{v}$.

The Schrödinger subcase [i.e., the case in which the vector spin field $\mathbf{s} = \mathbf{s}(\mathbf{x},t)$ is constant in time and uniform in space] corresponds to *spin eigenstates*, so we now need a wave function factorizable into the product of a ''nonspin'' part $\sqrt{\rho}e^{i\varphi}$ (*scalar*) and a *spin part* χ (Pauli spinor):

$$\psi \equiv \sqrt{\rho}e^{i\varphi/\hbar}\chi, \tag{17}$$

χ being *constant in time and space*. Therefore, when $\mathbf{s}$ has no precession (and no external field is present: $\mathbf{A} = \mathbf{0}$), we have $\mathbf{s} \equiv \chi^\dagger\hat{\mathbf{s}}\chi = \text{const}$ and

$$\mathbf{V} = \frac{\boldsymbol{\nabla}\rho\times\mathbf{s}}{m\rho} \neq \mathbf{0} \quad (\text{Schrödinger case}). \tag{18}$$

One can notice that, *even in the Schrödinger theoretical framework, the Zitterbewegung does not vanish*, except for plane waves, i.e., for the nonphysical case of $\mathbf{p}$ eigenfunctions, when not only $\mathbf{s}$ but also ρ is constant and uniform, so that $\boldsymbol{\nabla}\rho = \mathbf{0}$. [Notice also that the continuity equation (6), $\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{p}/m) = 0$, can be still rewritten in the ordinary way $\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$. In fact, the quantity $\boldsymbol{\nabla}\cdot\mathbf{V} \equiv \boldsymbol{\nabla}\cdot(\nabla\times\rho\mathbf{s})$ is identically zero, being the divergence of a rotor, so that $\boldsymbol{\nabla}\cdot(\mathbf{p}/m) = \boldsymbol{\nabla}\cdot\mathbf{v}$.]

But let us go on. We may now write

$$\mathbf{V}^2 = \left(\frac{\boldsymbol{\nabla}\rho\times\mathbf{s}}{m\rho}\right)^2 = \frac{(\boldsymbol{\nabla}\rho)^2\mathbf{s}^2 - (\boldsymbol{\nabla}\rho\cdot\mathbf{s})^2}{(m\rho)^2} \tag{19}$$

since in general it holds that

$$(\mathbf{a}\times\mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}\cdot\mathbf{b})^2. \tag{20}$$

Let us observe that, from the smallness of the negative-energy component (the so-called small component) of the Dirac bispinor follows the smallness also of $\boldsymbol{\nabla}\rho\cdot\mathbf{s} \simeq 0$. This was already known from the Clifford algebra approach to Dirac theory, which yielded [15] (β being the Takabayasi angle [16]) $\boldsymbol{\nabla}\cdot\rho\mathbf{s} = -m\rho\,\sin\beta$, which in the NR limit corresponds to $\beta = 0$ (''pure electron'') or $\beta = \pi$ (''pure positron''), so that one gets $\boldsymbol{\nabla}\cdot\rho\mathbf{s} = 0$ and in the Schrödinger case ($\mathbf{s} = \text{const}$ and $\boldsymbol{\nabla}\cdot\mathbf{s} = 0$)

$$\boldsymbol{\nabla}\rho\cdot\mathbf{s} = 0. \tag{21}$$

By putting such a condition into Eq. (19), it assumes the important form

$$\mathbf{V}^2 = \mathbf{s}^2\left(\frac{\boldsymbol{\nabla}\rho}{m\rho}\right)^2, \tag{22}$$

which *finally* allows us to attribute to the so-called nonclassical term (7) of our Lagrangian (3) the simple meaning of kinetic energy of the internal (*Zitterbewegung*) motion [i.e., of kinetic energy associated with the internal (*Zitterbewegung*) velocity $\mathbf{V}$], provided

$$\hbar = 2\mathbf{s}. \tag{23}$$

In agreement with the previously mentioned König theorem, such an internal kinetic energy does appear, in the Lagrangian (3), as correctly added to the (external) kinetic energy $\mathbf{p}^2/2m$ of the c.m. [in addition to the total energy (9) and the external potential energy U].

In contrast, if we assume (within a *Zitterbewegung* philosophy) that $\mathbf{V}$ [Eq. (22)] is the velocity attached to the kinetic-energy term (7), *then we can deduce* Eq. (23), i.e., we deduce that actually

$$|\mathbf{s}| = \tfrac{1}{2}\hbar.$$

Let us mention, by the way, that in the stochastic approaches the (non-classical) stochastic, diffusion velocity is $\mathbf{V} \equiv \mathbf{v}_{\text{dif}} = \nu(\boldsymbol{\nabla}\rho/\rho)$, the quantity ν being the diffusion coefficient of the quantum medium. In those approaches, however, one has to *postulate* that $\nu \equiv \hbar/2m$. In our approach, on the contrary, if we just adopt for the moment the stochastic language, by a comparison of Eqs. (7), (22), and (23) we would immediately *deduce* that $\nu = \hbar/2m$ and therefore the interesting relation

$$\nu = \frac{|\mathbf{s}|}{m}. \tag{24}$$

Let us explicitly remark that, because of Eq. (22), in the Madelung fluid equation (and therefore in the Schrödinger equation) the quantity $\hbar$ is naturally replaced by $2|\mathbf{s}|$, the presence itself of the former quantity no longer being needed. In a way, we might say that it is more appropriate to write $\hbar = 2|\mathbf{s}|$ rather than $|\mathbf{s}| = \hbar/2$.

Let us add, as a last observation, a corollary of our nonrelativistic decomposition of velocity $\mathbf{v}$ into a classical part (depending on φ) plus a part (depending on ρ and) originating the quantum potential. If one requires the latter part (i.e., the *Zitterbewegung* part) of $\mathbf{v}$ to be small $\boldsymbol{\nabla}\rho/\rho \simeq \mathbf{0}$, then one gets immediately the Bohr-Sommerfeld-WKB condition for the Schrödinger equation solutions to be semiclassical: $\boldsymbol{\nabla}\lambda_{\text{dB}} \simeq 0$.

Let us conclude the first part of the present contribution by stressing the following. We first achieved a nonrelativistic, Gordon-like decomposition of the field velocity within the ordinary tensorial language. Second, we derived the quantum potential (without the postulates and assumptions of stochastic quantum mechanics) by simply relating the nonclassical energy term to *Zitterbewegung* and spin. Such results provide further evidence that the quantum behavior of microsystems may be a direct consequence of the existence of spin. In fact, when $\mathbf{s} = \mathbf{0}$, the quantum potential vanishes in the Hamilton-Jacobi equation, which then becomes a totally *classical* and Newtonian equation. We have also seen that the quantity $\hbar$ itself enters the Schrödinger equation owing to the presence of spin. We are easily induced to conjecture that no scalar *quantum* particles exist that are really elementary, but that scalar particles are always constituted by spinning objects endowed with *Zitterbewegung*.

III. THE KINEMATICS OF SPINNING PARTICLES

In the first part of this paper, we addressed ourselves to spin, *Zitterbewegung*, and Madelung fluid in (nonrelativistic) physics. The previous analysis led us to fix our attention in particular on the internal velocity $\mathbf{V}$ of the spinning particle, as well as on its external velocity $\mathbf{w}=\mathbf{p}/m$. In the second part of this article, we want to fix our attention on the *total* velocity $\mathbf{v}=\mathbf{w}+\mathbf{V}$. It is now essential to allow $\mathbf{w}$ to assume any value and therefore to consider *relativistic* physics. In what follows our considerations will be essentially classical, while the quantum side of these last two sections will be studied elsewhere [17].

Before going on, let us make a brief digression by recalling that, since the works of Compton [8], Uhlenbeck and Goudsmit [18], Frenkel [18], and Schrödinger [9] up to the present, many classical theories, often quite different among themselves from a physical and formal point of view, have been advanced for spinning particles (for simplicity, we often write ''spinning particle'' or just ''electron'' instead of the more pertinent expression ''spin-$\frac{1}{2}$ particle''). Following Bunge [19], they can be divided into three classes: (I) strictly *pointlike* particle models, (II) actual extended-type particle models (spheres, tops, gyroscopes, etc.), and (III) mixed models for extendedlike particles, in which the position of the pointlike charge $\mathcal{Q}$ ends up being spatially distinct from the particle c.m.

Notice that in the theoretical approaches of type III, which, being between classes I and II, could answer a dilemma posed by Barut (''If a spinning particle is not quite a point particle, nor a solid three-dimensional top, what can it be?''), the motion of $\mathcal{Q}$ does not coincide with the motion of the particle c.m. This peculiar feature was found to be an actual characteristic [20–22,15,11,10] (called, as we know, the *Zitterbewegung* motion) of spinning particle kinematics. The type-III models, therefore, are *a priori* convenient for describing *Zitterbewegung*, spin, and intrinsic magnetic moment of the electron, while these properties are hardly predicted by making recourse to the pointlike-particle theories of class I. The theories of type III, moreover, are consistent [8–12] with the ordinary quantum theory of the electron (see below). The extendedlike electron models of class III are at present in fashion also because of their possible generalizations to include supersymmetry and superstrings [10(b)]. Finally, the mixed models help bypassing the obvious nonlocality problems involved by a relativistic covariant picture for extended-type (in particular *rigid*) objects of class II. Quite differently, the extendedlike (class III) electron is nonrigid and consequently variable in its shape and in its characteristic size, depending on the considered dynamical situation. This is *a priori* consistent with the appearance in the literature of many different radii of the electron (for instance, in his book [23], McGregor lists on p. 5 seven typical electron radii, from the Compton to the classical and to the magnetic radius). For all these reasons, therefore, the spinning particle we shall have in mind in Sec. IV is to be described by class III theories.

Here we have to rephrase some of the previous considerations in terms of Minkowski (four-dimensional) vectors. For instance, let us recall again that in the ordinary Dirac theory the particle four-impulse p^μ is in general *not* parallel to the four-velocity: $v^\mu \neq p^\mu/m$. Let us repeat that in order to describe the *Zitterbewegung*, in all type-III theories it is very convenient [10–12] to split the motion variables as (the dot now meaning derivation with respect to the *proper* time τ)

$$x^\mu \equiv \xi^\mu + X^\mu, \quad \dot{x}^\mu \equiv v^\mu = w^\mu + V^\mu, \tag{25}$$

where ξ^μ and $w^\mu \equiv \dot{\xi}^\mu$ describe as before the external motion, i.e., the motion of the c.m., while X^μ and $V^\mu \equiv \dot{X}^\mu$ describe the internal motion. From an electrodynamical point of view, as we know, the conserved electric current is associated with the trajectories of $\mathcal{Q}$ (i.e., with x^μ), while the center of the particle Coulomb field, obtained [22], e.g., through a time average over the field generated by the quickly oscillating charge, is associated with the c.m. (i.e., with w^μ, and then, for free particles, with the geometric center of the internal motion). In such a way, it is $\mathcal{Q}$ which follows the (*total*) motion, while the c.m. follows the *mean* motion only. It is worthwhile also to notice that the classical extendedlike electron of type III is totally consistent with the standard Dirac theory; in fact, the above decomposition for the total motion is the classical analog of two well-known quantum-mechanical procedures, i.e., of the *Gordon decomposition* of the Dirac current, and the (operatorial) *decomposition of the Dirac position operator* proposed by Schrödinger in his pioneering works [9]. We shall return to these points below.

The well-known Gordon decomposition of the Dirac current reads [13] (hereafter we shall choose units such that numerically $c=1$)

$$\overline{\psi}\gamma^\mu\psi = \frac{1}{2m}[\overline{\psi}\hat{p}^\mu\psi - (\hat{p}^\mu\overline{\psi})\psi] - \frac{i}{m}\hat{p}_\nu(\overline{\psi}S^{\mu\nu}\psi), \tag{26}$$

$\overline{\psi}$ being the ''adjoint'' spinor of ψ, the quantity $\hat{p}^\mu \equiv i\partial^\mu$ the four-dimensional impulse operator, and $S^{\mu\nu} \equiv (i/4)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ the spin-tensor operator. The ordinary interpretation of Eq. (26) is in total analogy with the decomposition given in Eq. (25). The first term on the rhs ends up being associated with the translational motion of the c.m. (the scalar part of the current, corresponding to the traditional Klein-Gordon current). The second term on the rhs is instead directly connected to the existence of spin and describes the *Zitterbewegung* motion.

In the above-mentioned papers, Schrödinger started from the Heisenberg equation for the time evolution of the acceleration operator in Dirac theory [$\mathbf{v} \equiv \boldsymbol{\alpha}$]

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} = \frac{i}{\hbar}[H,\mathbf{v}] = \frac{2i}{\hbar}(H\mathbf{v} - \mathbf{p}), \tag{27}$$

where H is equal as usual to $\mathbf{v}\cdot\mathbf{p} + \beta m$. Integrating $\mathbf{v}$ this operator equation once over time, after some algebra one can obtain

$$\mathbf{v} = H^{-1}\mathbf{p} - \frac{i}{2}\hbar H^{-1}\mathbf{a}; \tag{28}$$

integrating it a second time, one obtains [14] the spatial part of the decomposition

$$\mathbf{x} \equiv \boldsymbol{\xi} + \mathbf{X}, \tag{29}$$

where (still in the operator formalism)

$$\boldsymbol{\xi} = \mathbf{r} + H^{-1}\mathbf{p}t \tag{30}$$

is related to the motion of the c.m., and

$$\mathbf{X} = \frac{i}{2}\hbar \boldsymbol{\eta} H^{-1} \quad (\boldsymbol{\eta} \equiv \mathbf{v} - H^{-1}\mathbf{p}) \tag{31}$$

is related to the *Zitterbewegung* motion.

IV. KINEMATICAL PROPERTIES OF THE EXTENDEDLIKE PARTICLES

We now want to analyze the formal and conceptual properties of a differed definition for the four-velocity of our extendedlike electron. Such a definition was initially adopted, but without any emphasis, in the papers by Barut and co-workers dealing with a successful model for the relativistic classical electron ([10(a),12]). Let us consider the following. At variance with the procedures followed in the literature from Schrödinger's time up to the present, we have to make recourse not to the proper time of the charge $\mathcal{Q}$, *but rather to the proper time of the center of mass*, i.e., to the time of the c.m.f.[1] As a consequence, the quantity τ in the denominator of the four-velocity definition $v^\mu \equiv dx^\mu / d\tau$ has to be the *latter* proper time. Up to now, with the exception of the above-mentioned papers by Barut and co-workers, in all theoretical frameworks the Lorentz factor has been assumed to be equal to $\sqrt{1-\mathbf{v}^2}$. On the contrary, in the Lorentz factor it has to enter $\mathbf{w}^2$ instead of $\mathbf{v}^2$, the quantity $\mathbf{w} \equiv \mathbf{p}/p^0$ being the three-velocity of the c.m. with respect to the chosen frame ($p^0 \equiv \mathcal{E}$ is the energy). By adopting the correct Lorentz factor, all the formulas containing it are to be rewritten, and get a new physical meaning. In particular, we shall show below that the new definition does actually *imply*[2] the important constraint, which, holding identically for scalar particles, is often just *assumed* for spinning particles:

$$p_\mu v^\mu = m,$$

where m is the physical rest mass of the particle (and not an *ad hoc* masslike quantity M).[3]

Our choice of the proper time τ may be supported by the following considerations.

(i) The *lightlike Zitterbewegung*, when the speed of $\mathcal{Q}$ is constant and equal to the speed of light in vacuum, is certainly the preferred one (among all the *a priori* possible internal motions) in the literature and to many authors it appears to be the most adequate for a meaningful classical picture of the electron. In some special theoretical approaches, the speed of light is even regarded as the quantum-mechanical typical speed for the *Zitterbewegung*. In fact, the Heisenberg principle in the relativistic domain [14] implies (not controllable) particle-antiparticle pair creations when the (c.m.f.) observation involves space distances of the order of a Compton wavelength. Thus $\hbar/m$ is assumed to be the characteristic orbital radius and $2m/\hbar^2$ the (c.m.f.) angular frequency of the *Zitterbewegung*, as first noticed by Schrödinger, and the orbital motion of $\mathcal{Q}$ is expected to be lightlike. Now, if the charge $\mathcal{Q}$ travels at the speed of light, *the proper time of $\mathcal{Q}$ does not exist*, while the proper time of the c.m. (which travels at subluminal speeds) does exist. Adopting as the time the proper time of $\mathcal{Q}$, as is often done in the past literature, automatically *excludes* a lightlike *Zitterbewegung*. In our approach, by contrast, such *Zitterbewegung* motions are not excluded. Analogous considerations may hold for *superluminal Zitterbewegung* speeds, without too much trouble, since the c.m. (which carries the energy impulse and the ''signal'') is always endowed with a subluminal motion.

(ii) The independence between the center-of-charge and the center-of-mass motion becomes evident by our definition. As a consequence, the nonrelativistic limit can be formulated by us in a correct and univocal way. Namely, by assuming the correct Lorentz factor, one can immediately see that the *Zitterbewegung* can go on being a relativistic (in particular light-like) motion even in the nonrelativistic approximation, i.e., when $\mathbf{p} \to \mathbf{0}$ (this is perhaps connected with the nonvanishing of spin in the nonrelativistic limit). In fact, in the nonrelativistic limit, we have to take

$$\mathbf{w}^2 \ll 1$$

and not necessarily

$$\mathbf{v}^2 \ll 1,$$

as was usually assumed in the past literature.

[1]Let us recall once more that the c.m.f. is the frame in which the kinetic impulse vanishes identically, $\mathbf{p}=\mathbf{0}$. For spinning particles, in general, it is *not* the rest frame since the velocity $\mathbf{v}$ is not necessarily zero in the c.m.f.

[2]For all plane-wave solutions ψ of the Dirac equation, we have (labeling by $\langle\ \rangle$ the corresponding *local mean value* or *field density*) $p_\mu \langle \hat{v}^\mu \rangle \equiv p_\mu \psi^\dagger \hat{v}^\mu \psi \equiv p_\mu \psi^\dagger \gamma^0 \gamma^\mu \psi \equiv p_\mu \bar{\psi} \gamma^\mu \psi = m$.

[3]As an example, recall that Pavšič [10(b)] derived, from a Lagrangian containing an *extrinsic curvature*, the classical equation of the motion for a rigid n-dimensional world sheet in a curved background space-time. Classical world sheets describe membranes for $n \geqslant 3$, strings for $n=2$, and point particles for $n=1$. For the special case $n=1$, he found nothing but the traditional Papapetrou equation for a classical spinning particle; also, by quantization of the classical theory, he actually derived the Dirac equation. In Ref. [10(b)], however, M is not the observed electron mass m, and the relation between the two masses reads $m = M + \mu H^2$, the quantity μ being the so-called string rigidity, while H is the second covariant derivative on the world sheet.

(iii) The definition for the four-velocity that we are going to propose [see Eq. (33)] does agree with the natural classical limit of the Dirac current. Actually, it was used in those models that (like Barut and co-workers) define velocity even at the classical level as the bilinear combination $\bar{\psi}\gamma^{\mu}\psi$, via a direct introduction of classical spinors ψ. By the present definition, we shall be able to write the translational term as p^{μ}/m, with the physical mass in the denominator, exactly as in the Gordon decomposition (26). Quite differently, in all the theories adopting as the time the proper time of $\mathcal{Q}$, in the denominator appears an *ad hoc* variable mass M, which depends on the internal *Zitterbewegung* speed V (see below).

(iv) The choice of the c.m. proper time constitutes a natural extension of the ordinary procedure for relativistic scalar particles. In fact, for spinless particles in relativity the four-velocity is known to be univocally defined as the derivative of four-position with respect to the c.m.f. proper time (which is the only one available in that case).

The most valuable reason in support of our definition turns out to be the circumstance that the previous definition

$$v^{\mu}_{\rm std}=(1/\sqrt{1-\mathbf{v}^2};\mathbf{v}/\sqrt{1-\mathbf{v}^2}), \tag{32}$$

where std denotes standard, seems to entail a mass varying with the internal *Zitterbewegung* speed. But let us make explicit our definition for v^{μ}. The symbols that we are going to use possess the ordinary meaning; the difference [24] is that now *the Lorentz factor* $d\tau dt$ *will not be equal to* $\sqrt{1-\mathbf{v}^2}$, *but instead to* $\sqrt{1-\mathbf{w}^2}$. Thus we shall have

$$v^{\mu}\equiv dx^{\mu}/d\tau\equiv(dt/d\tau;d\mathbf{x}/d\tau)\equiv\left(\frac{dt}{d\tau};\frac{d\mathbf{x}}{dt}\frac{dt}{d\tau}\right)$$
$$=(1/\sqrt{1-\mathbf{w}^2};\mathbf{v}/\sqrt{1-\mathbf{w}^2})\quad(\mathbf{v}\equiv d\mathbf{x}/dt). \tag{33}$$

For w^{μ} we can write

$$w^{\mu}\equiv d\xi^{\mu}/d\tau\equiv(dt/d\tau;d\boldsymbol{\xi}/d\tau)\equiv\left(\frac{dt}{\mathrm{d}\tau};\frac{d\boldsymbol{\xi}}{dt}\frac{dt}{d\tau}\right)$$
$$=(1/\sqrt{1-\mathbf{w}^2};\mathbf{w}/\sqrt{1-\mathbf{w}^2})\quad(\mathbf{w}\equiv d\boldsymbol{\xi}/dt) \tag{34}$$

and for the four-impulse

$$p^{\mu}\equiv mw^{\mu}=m(1/\sqrt{1-\mathbf{w}^2};\mathbf{w}/\sqrt{1-\mathbf{w}^2}). \tag{35}$$

[In the presence of an external field such relations remain valid provided one makes the minimal prescription $p\rightarrow p-eA$ (in the c.m.f. we shall have $\mathbf{p}-e\mathbf{A}=\mathbf{0}$ and consequently $\mathbf{w}=\mathbf{0}$, as above).]

Let us now examine the resulting impulse-velocity scalar product $p_{\mu}v^{\mu}$, which has to be a Lorentz invariant, both with our v and with the previous $v_{\rm std}$. With the quantity $p\equiv(p^0;\mathbf{p})$ being the four-impulse and M_1, M_2 two relativistic invariants, we may write

$$p_{\mu}v^{\mu}\equiv M_1\equiv\frac{p^0-\mathbf{p}\cdot\mathbf{v}}{\sqrt{1-\mathbf{w}^2}} \tag{36}$$

or, alternatively,

$$p_{\mu}v^{\mu}_{\rm std}\equiv M_2\equiv\frac{p^0-\mathbf{p}\cdot\mathbf{v}}{\sqrt{1-\mathbf{v}^2}}. \tag{37}$$

If we refer ourselves to the c.m.f. we shall have $\mathbf{p}_{\rm c.m.f.}=\mathbf{w}_{\rm c.m.f.}=\mathbf{0}$ (but $\mathbf{v}_{\rm c.m.f.}\equiv\mathbf{V}_{\rm c.m.f.}\neq\mathbf{0}$) and then

$$M_1=p^0_{\rm c.m.f.} \tag{38}$$

in the first case and

$$p_{\rm c.m.f.}=M_2\sqrt{1-\mathbf{V}^2_{\rm c.m.f.}} \tag{39}$$

in the second case. So we see that the invariant M_1 is actually a constant, which, being nothing but the center-of-mass energy $p^0_{\rm c.m.f}$ can be identified, as we are going to prove, with the physical mass m of the particle. On the contrary, in the second case (the standard one), the center-of-mass energy is variable with the internal motion.

Now, from Eq. (35) we have

$$p_{\mu}v^{\mu}\equiv mw_{\mu}v^{\mu}$$

and, because of Eqs. (33) and (34),

$$p_{\mu}v^{\mu}\equiv m(1-\mathbf{w}\cdot\mathbf{v})/(1-\mathbf{w}^2). \tag{40}$$

Since $\mathbf{w}$ is a vector component of the total three-velocity $\mathbf{v}$ [due to Eqs. (25)] and, moreover, is the orthogonal projection of $\mathbf{v}$ along the $\mathbf{p}$ direction, we can write

$$\mathbf{w}\cdot\mathbf{v}=\mathbf{w}^2,$$

which, introduced into Eq. (40), yields [24] the important relation

$$m=p_{\mu}v^{\mu}. \tag{41}$$

Quite differently, by use of the wrong Lorentz factor, we would have obtained

$$v^{\mu}=(1/\sqrt{1-\mathbf{v}^2};\mathbf{v}/\sqrt{1-\mathbf{v}^2})$$

and consequently

$$p_{\mu}v^{\mu}\equiv m(1-\mathbf{wv})/\sqrt{(1-\mathbf{w}^2)(1-\mathbf{v}^2)}=m\sqrt{1-\mathbf{w}^2}/\sqrt{1-\mathbf{v}^2}$$
$$\neq m.$$

By recourse to the correct Lorentz factor, therefore, we succeeded in showing that the noticeable constraint $m=p_{\mu}v^{\mu}$, trivially valid for scalar particles, holds for spinning particles too. Such a relation (41) would be very useful also for a Hamiltonian formulation of the electron theory [12].

Finally, we want to show that the ordinary kinematical properties of the Lorentz invariant $v^2\equiv v_{\mu}v^{\mu}$ do *not* hold any longer in the case of spinning particles, endowed with *Zitterbewegung*. In fact, it is easy to prove that the ordinary constraint for scalar relativistic particles (the quantity v^2 constant in time and equal to 1) does *not* hold for spinning particles endowed with *Zitterbewegung*. Namely, if we choose as reference frame the c.m.f. in which $\mathbf{w}=\mathbf{0}$, we have [cf. definition (33)]

$$v^{\mu}_{\rm c.m.f.}\equiv(1;\mathbf{V}_{\rm c.m.f.}), \tag{42}$$

wherefrom, with

$$v^2_{\mathrm{c.m.f.}} \equiv 1 - \mathbf{V}^2_{\mathrm{c.m.f.}}, \tag{43}$$

one can deduce [24] the constraints

$$\begin{aligned} &0 < \mathbf{V}^2_{\mathrm{c.m.f}}(\tau) < 1 \Leftrightarrow 0 < v^2_{\mathrm{c.m.f.}}(\tau) < 1 \quad (\text{timelike}),\\ &\mathbf{V}^2_{\mathrm{c.m.f.}}(\tau) = 1 \Leftrightarrow v^2_{\mathrm{c.m.f.}}(\tau) = 0 \quad (\text{lightlike}),\\ &\mathbf{V}^2_{\mathrm{c.m.f.}}(\tau) > 1 \Leftrightarrow v^2_{\mathrm{c.m.f.}}(\tau) < 0 \quad (\text{spacelike}). \end{aligned} \tag{44}$$

Since the square of the total four-velocity is invariant and in particular it is $v^2_{\mathrm{c.m.f}} = v^2$, these constraints for v^2 will be valid in any frame:

$$\begin{aligned} 0 < v^2(\tau) < 1 &\quad (\text{timelike}),\\ v^2(\tau) = 0 &\quad (\text{lightlike}),\\ v^2(\tau) < 0 &\quad (\text{spacelike}). \end{aligned} \tag{45}$$

Note explicitly that the correct application of special relativity to a spinning particle led us, under our hypotheses, to obtain that $v^2 = 0$ in the lightlike case, but $v^2 \neq 1$ in the timelike case and $v^2 \neq -1$ in the spacelike case.

Let us now examine the manifestation and consequences of such constraints in a specific example, namely, the already mentioned theoretical model by Barut and Zanghi [10(a)], which did implicitly adopt as the time the proper time of the c.m.f. In this case, we get that it is in general $v^2 \neq 1$. In fact, one obtains [12] the remarkable relation

$$v^2 = 1 - \frac{\ddot{v}_\mu v^\mu}{4m^2}. \tag{46}$$

In particular [22], in the lightlike case it is $\ddot{v}_\mu v^\mu = 4m^2$ and therefore $v^2 = 0$.

Returning to Eq. (43), note that now the quantity v^2 is no longer related to the external speed $|\mathbf{w}|$ of the c.m. but, on the contrary, to the internal *Zitterbewegung* speed $|\mathbf{V}_{\mathrm{c.m.f}}|$. Note at last that, in general, and at variance with the scalar case, the value of v^2 is not constant in time any longer, but varies with τ (except when $\mathbf{V}^2_{\mathrm{c.m.f.}}$ itself is constant in time).

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