Reflectionless potentials for the one-dimensional Dirac equation: Pseudoscalar potentials

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For the Dirac equation in one dimension, a method for constructing reflectionless pseudoscalar potentials is presented. This is a relativistic generalization of Kay and Moses's classic method [J. Appl. Phys. **27**, 1503 (1956)] that deals with a similar problem for the Schrödinger equation. This can also be viewed as an inverse scattering problem. For a set of assumed energy levels, the method furnishes a family of potentials, each of which reproduces the energy levels. These potentials are all reflectionless. [S1050-2947(98)02001-0]

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I. INTRODUCTION

In their classic paper of 1956, Kay and Moses gave a method of constructing reflectionless potentials for the Schrödinger equation in one dimension [1]. We are interested in applying the Kay-Moses (KM) method to the one-dimensional Dirac equation. Regarding its behavior under the Lorentz transformation, there are three types of potentials for the Dirac equation: scalar, vector, and pseudoscalar. By the vector type we mean the zeroth component of a Lorentz vector (like the Coulomb potential). We have examined the scalar type potential and have shown how an infinite family of reflectionless potentials can be constructed [2,3]. We also have indicated that reflectionless potentials are impossible if they are of the vector type [2].

The purpose of this paper is to examine the remaining possibility with the pseudoscalar type. We show that the KM method again enables us to construct reflectionless potentials. The problem can also be viewed as an inverse scattering or inverse spectrum problem. For a set of assumed energy levels, we can construct a family of potentials, each of which reproduces the energy levels. These potentials turn out to be all reflectionless. The situation is similar to that of the scalar type, but there are nontrivial differences.

In Sec. II we summarize some general features of the Dirac equation with a pseudoscalar potential. In Sec. III we present a method for constructing a reflectionless potential. Section IV deals with a few examples. Some intriguing aspects of the problem are illustrated. Summary and discussion are given in Sec. V.

II. GENERAL FEATURES OF THE DIRAC EQUATION WITH A PSEUDOSCALAR POTENTIAL

The Dirac equation that we consider is

$$H_D\psi(x)=E\psi(x), \quad H_D=\alpha[p-i\beta f(x)]+\beta m, \tag{1}$$

where $c=\hbar=1$, $p=-id/dx$, and m is the mass of the particle. The wave function ψ has two components. For the Dirac matrices α and β, we use the 2×2 Pauli matrices $\alpha=\sigma_y$ and $\beta=\sigma_z$. The potential in H_D is called the pseudoscalar potential because $\psi^\dagger\alpha\beta\psi$ is a pseudoscalar. Recently, there has been a surge of interest in the ''Dirac oscillator'' that is obtained with the special choice of $f(x)=m\omega x$ [4,5]. In this paper we are interested in the transmission problem. Therefore, we assume that $f(x)$ remains finite as $x\to\pm\infty$. Our goal is to find $f(x)$ such that all waves incident from negative infinity, irrespective of their wave numbers, go to positive infinity without any reflection. In achieving this goal, the following features of the Dirac equation will be useful.

Let the upper (lower) component of ψ be ψ_1 (ψ_2). Then Eq. (1) becomes

$$-\psi_2'+f\psi_2=(E-m)\psi_1, \tag{2}$$

$$\psi_1'+f\psi_1=(E+m)\psi_2, \tag{3}$$

where $\psi_1'=d\psi_1/dx$. These two equations can be manipulated into the supersymmetry (SUSY) pair of Schrödinger equations

$$H_i\psi_i=\left(\frac{p^2}{2m}+U_i\right)\psi_i=\epsilon\psi_i, \quad \epsilon=\frac{E^2-m^2}{2m}, \tag{4}$$

$$U_i(x)=\frac{1}{2m}(f^2\mp f'), \tag{5}$$

where $i=1$ or 2 and the double sign in Eq. (5) is $-$ $(+)$ for $i=1$ (2). In the SUSY terminology, f is the superpotential. Although U_1 and U_2 are different, all eigenvalues, with the possible exception of the smallest one, are shared by H_1 and H_2. For SUSY see, e.g., Refs. [6,7]. It is known that ϵ is non-negative. It is also known that the SUSY pair U_1 and U_2 are phase equivalent, that is, the resulting transmission and reflection coefficients are the same between U_1 and U_2. Therefore, if U_1 is a reflection potential, so is U_2. If U_1 and U_2 are both reflectionless, the underlying pseudoscalar potential is clearly reflectionless.

Let us consider Eq. (4) for $i=1$ with $\epsilon=0$. For this special choice of ϵ, we designate the wave function with ϕ_1,

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$$-\frac{1}{2m}\phi_1''+U_1\phi_1=0. \tag{6}$$

The ϕ_1 and f are related by

$$f(x)=-\frac{\phi_1'}{\phi_1},\quad \phi_1(x)=\exp\left(-\int^x f(x)dx\right). \tag{7}$$

Since $\epsilon=0$ is the minimum possible value of ϵ, we can choose ϕ_1 such that it has no node. Then f is free from singularity. The state that ϕ_1 represents may or may not be a bound state, depending on the asymptotic behavior of f. If f is such that

$$f(\infty)>0,\quad f(-\infty)<0, \tag{8}$$

then the ϕ_1 is normalizable and it represents a bound state. Otherwise ϕ_1 is not normalizable. Conversely, if ϕ_1 is normalizable, f has to conform to Eq. (8). Let us add that if we take Eq. (4) of $i=2$ (instead of $i=1$), we obtain formulas similar to Eqs. (6)–(8) except that f is replaced by $-f$. The H_2 has a bound state with eigenvalue $\epsilon=0$ if and only if f is such that

$$f(\infty)<0,\quad f(-\infty)>0. \tag{9}$$

There is a sort of symmetry between positive- and negative-energy solutions of Eq. (1) in the following sense. If we go to the Foldy-Wouthuysen representation [8,4,5], H_D is transformed to

$$H_{FW}=\beta\sqrt{m^2+p^2+f^2-\beta f'}. \tag{10}$$

With $\beta=1$ (-1) we obtain positive- (negative-) energy states. It is clear that the eigenvalues of H_1 (H_2) are related to positive- (negative-) energy eigenvalues of H_D. The H_1 and H_2 share the same eigenvalues, possibly except the lowest one, and they are phase equivalent. If we find a reflection potential for positive energies, it is also reflectionless for negative energies. We focus on positive-energy states in this paper.

III. HOW TO CONSTRUCT REFLECTIONLESS POTENTIALS

Assume a sequence of N energy eigenvalues of H_D, $E_1<E_2<\cdots<E_N$. The E_i's can take any positive values not smaller than m. Although we use the same index i for H_i, etc., and for E_i, the distinction between the i's should be clear from the context. On the basis of the KM method [1], we can find $f(x)$ such that the Dirac equation has the assumed sequence of energy eigenvalues. It turns out that the pseudoscalar potential with the $f(x)$ so determined is a reflectionless potential. Such $f(x)$ is not unique. We will see that there is a $(N+2)$-parameter family of such potentials.

Before working out the above scenario, let us summarize the gist of the KM method for the nonrelativistic Schrödinger equation [1,9]. Consider the Schrödinger equation

$$\left[\frac{p^2}{2m}+V_{KM}(x)\right]\psi(x)=-\frac{\kappa^2}{2m}\psi(x). \tag{11}$$

Let κ_i $(i=1,2,\cdots,N)$ be N positive constants that are arbitrary but nondegenerate. Let us order them such that $\kappa_1>\kappa_2>\cdots>\kappa_N$. Also let A_i $(i=1,2,\ldots,N)$ be N constants that can be chosen arbitrarily except that they are all positive. The KM potential is defined by

$$V_{KM}(x)=-\frac{1}{m}\frac{d^2}{dx^2}\ln\{\det[I+\hat{A}(x)]\}, \tag{12}$$

where I stands for the $N\times N$ unit matrix and $\hat{A}(x)$ is the $N\times N$ matrix with matrix elements

$$\hat{A}_{ij}(x)=\sqrt{A_iA_j}\frac{e^{(\kappa_i+\kappa_j)x}}{\kappa_i+\kappa_j}. \tag{13}$$

The Schrödinger equation (11) with V_{KM} has N bound states with eigenvalues $-\kappa_i^2/2m$ $(i=1,2,\ldots,N)$. Also V_{KM} is a reflectionless potential. For an arbitrary choice of A_i's, V_{KM} has no symmetry in general. If we choose A_i's as

$$\frac{A_i}{2\kappa_i}=\prod_{j\neq i}\frac{\kappa_i+\kappa_j}{|\kappa_i-\kappa_j|}, \tag{14}$$

then V_{KM} becomes symmetric, i.e., $V_{KM}(x)=V_{KM}(-x)$.

Equation (11) with V_{KM} of Eq. (12) has the scattering state solution

$$\chi_1(k,x)=\left[1+\sum_{i=1}^N\frac{\sqrt{A_i}g_i(x)e^{\kappa_i x}}{ik+\kappa_i}\right]e^{ikx}, \tag{15}$$

where $g_i(x)$'s are defined by the N linear algebraic equations [10]

$$\sum_{j=1}^N[\delta_{ij}+\hat{A}_{ij}(x)]g_j(x)+\sqrt{A_i}e^{\kappa_i x}=0. \tag{16}$$

The $\chi_1(k,x)$ of Eq. (15) is such that $e^{-ikx}\chi(k,x)\to 1$ as $x\to-\infty$, which means that a wave of unit amplitude is incident from negative infinity. There is no reflected wave. Hence V_{KM} is a reflectionless potential. This ends the summary of the KM method.

Returning to the relativistic case, we assume that

$$U_1(x)=\frac{\lambda^2\kappa_1^2}{2m}+V_{KM}(x),\quad \lambda\geq 1. \tag{17}$$

Then Eq. (4) for ψ_1 has N bound states with N eigenvalues

$$\epsilon_i=\frac{E_i^2-m^2}{2m}=\frac{\lambda^2\kappa_1^2-\kappa_i^2}{2m}. \tag{18}$$

There are no other bound states. For the N assumed eigenvalues of H_D, i.e., E_i's, the κ_i's and λ (≥ 1) can be chosen arbitrarily as long as they are related to the E_i's through Eq. (18). Equations (11) and (4) with U_1 of Eq. (17) share the same wave function. Since V_{KM} is reflectionless, so is U_1. Apart from a constant factor, ϕ_1 can be taken as

$$\phi_1(x)=\chi_1(i\lambda\kappa_1,x). \tag{19}$$

Now that the ϕ_1 is known, the potential function f can be determined by Eq. (7). Then we can go on to determine the SUSY partner U_2, which is also reflectionless.

About the symmetry of U_1 and U_2, let us choose A_i's according to Eq. (14) so that U_1 is symmetric. The symmetry of the associated U_2 depends on the choice of λ. If $\lambda=1$, ϕ_1 of Eq. (19) obtains definite parity. Note that this ϕ_1 is the wave function of a bound state. It then follows that $f(x)$ is an odd function of x and U_2 an even function of x.

If $\lambda>1$, the situation can be complicated. The ϕ_1 of Eq. (19) does not represent a bound state. It does not have definite parity, nor the $f(x)$ and U_2 that ensue. Actually, a more general choice of ϕ_1 is possible in this case, that is,

$$\phi_1(x)=\chi_1(i\lambda\kappa_1,x)+\gamma\chi_1(i\lambda\kappa_1,-x), \tag{20}$$

where γ is a constant. We can choose γ arbitrarily as long as ϕ_1 has no node. A similar situation was found in the case of the scalar type potential: see Eq. (3.9) of Ref. [3]. If we take $\gamma=1$, ϕ_1 becomes an even function of x and so does U_2.

Let us sort out how many parameters are involved. The potential contains $2(N+1)$ parameters A_i, κ_i, λ, and γ. If we specify N energies in terms of E_i's, then the potential is left with $N+2$ arbitrary parameters. If we require that U_1 be symmetric, all A_i's are determined by Eq. (14). If we require that U_2 also be symmetric, we have to take $\gamma=1$. If $E_1=m$, then $\lambda=1$ and the value of γ becomes irrelevant.

IV. EXAMPLES

Let us examine a few examples.

Example I. Assume just one positive-energy level E_1 for H_D. The $\kappa_1=\kappa$ and λ are related to E_1 by

$$E_1^2-m^2=(\lambda^2-1)\kappa^2. \tag{21}$$

Otherwise λ and κ can be chosen arbitrarily. We assume that $A_1/2\kappa=1$ so that U_1 becomes symmetric. Then we obtain

$$\chi_1(k,x)=\frac{ik-\kappa\tanh\kappa x}{ik+\kappa}e^{ikx}. \tag{22}$$

Let us start with the simplest case of $E_1=m$. Then $\lambda=1$, but κ is still arbitrary. Let us set $\gamma=0$. Apart from an unimportant constant factor, ϕ_1 is given by

$$\phi_1(x)=\frac{1}{\cosh\kappa x}. \tag{23}$$

Then we find

$$f(x)=\kappa\tanh\kappa x, \tag{24}$$

$$U_1(x)=\frac{\kappa^2}{2m}\left(1-\frac{2}{\cosh^2\kappa x}\right), \tag{25}$$

$$U_2(x)=\frac{\kappa^2}{2m}. \tag{26}$$

The Hamiltonian H_1 has one bound state with $\epsilon=0$, whereas H_2 has no bound state. The two components of the normalized Dirac wave function are given by

$$\psi_1(x)=\sqrt{\frac{\kappa}{2}}\frac{1}{\cosh\kappa x},\quad \psi_2(x)=0. \tag{27}$$

Next let us consider a more general case of $E_1>m$. Then $\lambda>1$. Let us take $\gamma=0$. Apart from a constant factor, ϕ_1 is given by

$$\phi_1(x)=(\lambda+\tanh\kappa x)e^{-\lambda\kappa x}. \tag{28}$$

Then we are led to

$$f(x)=\kappa\left(\tanh\kappa x+\frac{\lambda^2-1}{\lambda+\tanh\kappa x}\right), \tag{29}$$

$$U_1(x)=\frac{\kappa^2}{2m}\left(\lambda^2-\frac{2}{\cosh^2\kappa x}\right), \tag{30}$$

$$U_2(x)=\frac{\kappa^2}{2m}\left[\lambda^2-\frac{2}{\cosh^2\kappa(x+a)}\right], \tag{31}$$

$$e^{2\kappa a}=\frac{\lambda+1}{\lambda-1},\quad \lambda=\coth\kappa a. \tag{32}$$

If $\lambda>1$, f of Eq. (29) does not conform to Eq. (8) nor Eq. (9). Hence there is no bound state with $\epsilon=0$. The H_1 and H_2 both have a bound state with $\epsilon=(\lambda^2-1)(\kappa^2/2m)$. The eigenfunctions of H_1 and H_2 give the two components of the normalized Dirac wave function, which are

$$\psi_1(x)=\frac{1}{2}\sqrt{\frac{\kappa(m+m_\lambda)}{m_\lambda}}\frac{1}{\cosh\kappa x}, \tag{33}$$

$$\psi_2(x)=\frac{\kappa}{2}\sqrt{\frac{\kappa(\lambda^2-1)}{m_\lambda(m+m_\lambda)}}\frac{1}{\cosh\kappa(x+a)}, \tag{34}$$

$$m_\lambda=\sqrt{m^2+\kappa^2(\lambda^2-1)}. \tag{35}$$

Although the two Schrödinger equations (4) are formally decoupled, ψ_1 and ψ_2 are coupled through the Dirac equation (1). Hence their ratio is not arbitrary.

The U_1 obtained above is the Pöschl-Teller potential with one bound state [11]. In additon to the bound state that we obtained above, H_1 has a half-bound state with $\epsilon=(\lambda\kappa)^2/2m$. Its wave function is $\chi(0,x)$, which is not normalizable. If U_1 is made infinitesimally more attractive, the half-bound state becomes a genuine bound state. The U_2 has the same feature. The existence of such a half-bound state is characteristic of all reflectionless potentials [12].

Suppose that we choose $\lambda<1$. Then the eigenvalue of the ground state of H_1 becomes negative and it is no longer possible to have ϕ_1 without a node. This results in a singularity of f. This actually does not matter as far as U_1 is concerned. The singularity of f^2 and that of f' cancel and the resulting U_1 is free from singularity. However, U_2 becomes singular and ψ_2 becomes unnormalizable. This is not acceptable because the Dirac wave function with components ψ_1 and ψ_2 becomes unnormalizable. Let us emphasize that a singular f can be acceptable in the nonrelativistic case, but not in the relativistic case with the Dirac equation.

Example II. We assume one positive-energy level of H_D as in example I, but we take $\gamma=1$. Proceeding in the same way as in example I, we obtain

$$\phi_1(x)=(\lambda+\tanh\kappa x)e^{-\lambda\kappa x}+(\lambda-\tanh\kappa x)e^{\lambda\kappa x}. \quad (36)$$

Then we are led to

$$f(x)=\kappa\left(\tanh\kappa x+\frac{\lambda^2-1}{\tanh\kappa x-\lambda\coth\lambda\kappa x}\right). \quad (37)$$

The U_1 and ψ_1 that follow are the same as those of Eqs. (30) and (33), whereas U_2 and ψ_2 are given by

$$U_2(x)=\frac{\kappa^2}{2m}\left[\lambda^2-\frac{2(\lambda^2-1)(\lambda^2\cosh^2\kappa x+\sinh^2\lambda\kappa x)}{(\lambda\cosh\kappa x\cosh\lambda\kappa x-\sinh\kappa x\sinh\lambda\kappa x)^2}\right], \quad (38)$$

$$\psi_2(x)=\frac{\kappa(\lambda^2-1)}{2}\sqrt{\frac{\kappa}{m_\lambda(m+m_\lambda)}}\frac{\tanh\lambda\kappa x}{\sinh\kappa x\tanh\lambda\kappa x-\lambda\cosh\kappa x}. \quad (39)$$

There is something intriguing. We started by assuming one positive-energy level for H_D and accordingly one bound state for H_1. The H_2, however, obtains two bound states. The ψ_2 of Eq. (39) is an odd function of x and has one node. This ψ_2 therefore represents the first excited state of H_2. Note that, when $\lambda>1$, f of Eq. (37) conforms to Eq. (9). Hence H_2 has a bound state with $\epsilon=0$ [13]. This results in the bound-state solution of the Dirac equation with the two components

$$\psi_1(x)=0,$$

$$\psi_2(x)=\sqrt{\frac{\kappa\lambda(\lambda^2-1)}{2}}\frac{1}{\lambda\cosh\lambda\kappa x-\tanh\kappa x\sinh\lambda\kappa x}. \quad (40)$$

The state that has emerged is the eigenstate of H_D with $E=-m$.

The H_D has only one positive-energy bound state that we initially assumed to exist and two negative-energy bound states. Instead of starting with U_1 as we have done, we can start with U_2 assuming two energy levels with $\kappa_1=\lambda\kappa$ and $\kappa_2=\kappa$. This means that we assume two negative-energy bound states of H_D. We end up with the same results as we have obtained above.

Example III. Assume two positive-energy levels of H_D. Instead of starting with E_i's, let us start with $\kappa_1=2\kappa$ for the lower level and $\kappa_2=\kappa$. Equation (14) leads to $A_1/2\kappa_1=A_2/2\kappa_2=3$. Note that our κ_1 and κ_2 respectively correspond to the κ_2 and κ_1 of example D of Ref. [3]. For simplicity let us set $\lambda=1$ and $\gamma=0$. Then the positive-energy eigenvalues of H_D become $E_1=m$ and $E_2=\sqrt{m^2+3\kappa^2}$. We obtain

$$\chi(k,x)=-\frac{k^2+\kappa^2+3ik\kappa\tanh\kappa x-3\kappa^2\tanh^2\kappa x}{(ik+\kappa)(ik+2\kappa)}e^{ikx}, \quad (41)$$

$$\phi_1(x)=\frac{1}{\cosh^2\kappa x}, \quad (42)$$

$$f(x)=2\kappa\tanh\kappa x, \quad (43)$$

$$U_1(x)=\frac{\kappa^2}{m}\left(2-\frac{3}{\cosh^2\kappa x}\right), \quad (44)$$

$$U_2(x)=\frac{\kappa^2}{m}\left(2-\frac{1}{\cosh^2\kappa x}\right). \quad (45)$$

The U_1 obtained above is the Pöschl-Teller potential with two bound states [11]. In example II, if we take $\lambda=2$ we obtain U_1 and U_2 that are the same as the U_2 and U_1 of the present example, respectively.

Example IV. Let us assume N energy levels with $\kappa_1=N\kappa$, $\kappa_2=(N-1)\kappa$, ... If we choose $\lambda=1$, we obtain

$$\phi_1(x)=\frac{1}{\cosh^N\kappa x}, \quad (46)$$

$$f(x)=N\kappa\tanh\kappa x, \quad (47)$$

$$U_1(x)=\frac{N\kappa^2}{2m}\left(N-\frac{N+1}{\cosh^2\kappa x}\right), \quad (48)$$

$$U_2(x)=\frac{N\kappa^2}{2m}\left(N-\frac{N-1}{\cosh^2\kappa x}\right). \quad (49)$$

V. SUMMARY AND DISCUSSION

We have presented a method for constructing reflectionless pseudoscalar potentials for the one-dimensional Dirac equation. We can view it as an inverse scattering problem. We started with a set of energy levels and constructed potentials, each of which reproduces the assumed energy levels. They are all reflectionless potentials. If the number of the assumed energy levels is N, there is a $(N+2)$-parameter (A_i's, λ, and γ) family of such potentials.

We illustrated the method by a few examples. Example I is similar to example A of Ref. [3] for the scalar-type potential. Example A also has a SUSY pair of H_1 and H_2 with U_1 and U_2, respectively. As far as U_1 is concerned, examples I and A look the same except that the U_1 of the two examples differ by a constant energy shift $\lambda^2\kappa^2/2m$. Note, however, $|E|>m$ in example I, whereas $|E|<m$ in example A. For the U_2, the two examples are very different. Similar remarks apply to the other examples that we examined.

It seems to us that this work, together with the earlier similar work on the scalar-type potential [2,3], covers all conceivable extensions of the method of Kay and Moses to the Dirac equation. There is an interesting relationship between the reflectionless potentials of the scalar type and solutions of the nonlinear Dirac equation with the scalar-type nonlinearity [2]. This is similar to the relation between the KM potential and solutions of the nonlinear Schrödinger equation [14]. We have tried to find such a relationship for

the reflectionless potentials of the pseudoscalar type, but without success.

As an application of the method that we have presented, it would be interesting to construct a pseudoscalar potential such that the energy levels are all equally spaced. Recently, we examined the behavior of a wave packet in the Dirac oscillator potential, which is the pseudoscalar potential with $f(x)=m\omega x$ [5]. We attempted to construct a relativistic analog of Schrödinger's coherent wave packet of the nonrelativistic harmonic oscillator [15]. We found that the wave packet in the Dirac oscillator potential behaves in a rather complicated manner. This is essentially due to the fact that the energy levels of the Dirac oscillator are not equally spaced. If the energy levels are equally spaced, a wave packet will behave much more like Schrödinger's coherent wave packet.

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