Bounds on decoherence and error

L. S. Schulman*
Physics Department, Clarkson University, Potsdam, New York 13699-5820
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When a confined system interacts with its walls (treated quantum mechanically), there is an intertwining of degrees of freedom. We show that *this need not lead to entanglement*, hence decoherence. However, it will generally lead to error. The wave-function optimization required to avoid decoherence is also examined. [S1050-2947(98)05802-8]

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INTRODUCTION

Physical implementation of quantum computing algorithms [1], experimental tests of certain theories [2], as well as other contemporary problems require that for fairly large systems the time evolution be fully described by $\psi \to \exp(-iHt/\hbar)\psi$, with no ''measurement'' or, to be more precise, no decoherence or interaction with the environment. Such an interaction can cause entanglement with environmental degrees of freedom and prevent interference with portions of the wave function that have not experienced the identical interaction. Moreover, those same interactions can induce wave-function errors even within the original system Hilbert space.

For any laboratory system one can expect a degree of entanglement with the environment, simply due to the fact that the system is pinned to the table. In particular, when part of the system rebounds from the walls confining it (even electromagnetic walls) conservation of momentum demands an intertwining of the degrees of freedom. One might expect such confinement to place a fundamental bound on entanglement.

Taking the approach in [3], I begin from this inevitable intertwining and establish the extent to which it leads to entanglement. The measure of entanglement is that given in [4]. There is a surprise in the calculation: for appropriately tailored wave functions, *there need be no decoherence*. This leads us to explore the significance of the tailoring. However, although decoherence is avoidable, we will show that error is not [5].

Whether the decoherence is large or small (for nearly matching wave functions it is of the order of the system to container mass ratio), the resulting amplitude defect must be subtracted from the wave function for *each* collision, allowing for the possibility of physically significant effects.

INTERACTING WITH A WALL

A confined system will, from time to time, interact with its container. Dissipative walls, in the sense that the interaction is an inelastic collision, immediately lead to entanglement; for our bounds we therefore assume that the collision is elastic and involves no degree of freedom beyond that required to contain the system. Our model is therefore the scattering of two point particles: one small (mass m), representing the microscopic system, and one large (mass M), representing the container [6].

Before the collision we assume the wave function to be unentangled, that is, $\Psi_I = \Gamma(X)\Phi(x)$, with position variables X and x corresponding to the large and small masses, respectively. We make several simplifying assumptions: (i) restriction to one dimension, reasonable if the large ''particle'' is in fact a wall; (ii) rapid completion of the scattering; (iii) short-range, infinite repulsion; and (iv) Gaussian wave packets. Assumptions (ii) and (iii) are reasonable and simplify the calculation, and I expect that departures from them will increase decoherence and error. We further comment below on these assumptions.

If the interaction with the wall could be treated as a pure potential interaction with a fixed object, the wave function after the collision would be [7] $\Gamma(X)\Phi(-x)$. On the other hand, the correct form of the final wave function can be seen by going to center of mass coordinates $R=(MX+mx)/\mathcal{M}$, $u=x-X$, with $\mathcal{M}=M+m$. In these coordinates

$$\Psi_I = \Gamma(R-\delta u)\Phi(R+\gamma u),$$

where $\delta = m/\mathcal{M}$ and $\gamma = M/\mathcal{M}$. With the above assumptions, the wave function *after* the collision is

$$\Psi_F = \Gamma(R+\delta u)\Phi(R-\gamma u),$$

i.e., $u \to -u$. To show this, recall that the exact propagator for this problem is

$$G(R'',u'',t;R',u') = g_0^{\mathcal{M}}(R''-R',t)[g_0^{\mu}(u''-u',t) - g_0^{\mu}(u''+u',t)], \tag{1}$$

with

$$g_0^{\nu}(y,t) \equiv \sqrt{\frac{\nu}{2\pi i\hbar t}} \exp\left(\frac{i}{\hbar}\frac{\nu y^2}{2t}\right)$$

the free propagator, and $\mu = mM/\mathcal{M}$. To a good approximation, before the collision the wave function is given by the integral involving $g_0^{\mu}(u''-u',t)$ and after the collision [8] by that involving $g_0^{\mu}(u''+u',t)$. Thus, to get the final wave function, one reverses u.

When reexpressed in terms of x and X,

$$\Psi_F = \Gamma(X(1-2\delta)+2x\delta)\Phi(-x(1-2\delta)+2X\gamma), \tag{2}$$

suggesting that the final wave function has become entangled. For interactions more general than the hard wall

*Electronic address: schulman@polaris.clarkson.edu

there will be more complicated changes in the functions, but since the separate evolution of u and R follows from momentum conservation and the general nature of the two-particle interaction, there is no getting away from the intertwining.

The form we take for the wave function is

$$\Psi_I = \frac{1}{\sqrt{2\pi\sigma\Sigma}} \exp\left(-\frac{X^2}{4\Sigma^2}\right)\exp\left(-\frac{x^2}{4\sigma^2}+ikx\right), \tag{3}$$

with both x and X taking values on the entire real line (the position spreads are $\Delta X=\Sigma$ and $\Delta x=\sigma$, both assumed real). In principle we should use a wave function with $x-x_0$ in place of x above and restrict the relative coordinate to (say) negative values (because the particle is approaching a hard wall from the left). However, because we are able to restrict attention to the reflected wave, setting x_0 to zero only corresponds to assuming a different moment for the initial time at which the system was assumed to be disentangled—this has no effect on our major conclusion (the existence of entanglement-free scattering) and little effect on the other conclusions [10]. Note, by the way, that since this propagator is exact (given the hard-wall assumption), the subsequent time evolution corresponds to to a pair of free particles in the following way. Write down the integral $\int G\Psi_I$, keeping only the $g(u''+u',t)$ term. Now change u' to $-u'$ throughout. The propagator is now the original free-particle propagator, which factors *both* in the center of mass coordinates *and* in the separate x and X coordinates. The result of the transformation $u\rightarrow -u$ is that it is Ψ_I that carries the entanglement. Thus, if (as we show below) the transformed Ψ_I is at any time disentangled with respect to x and X, it will remain that way forever.

We now check error and decoherence. To compute ''error,'' we compare the outgoing wave to the final state, had the wall not been treated dynamically. To compute decoherence we measure the degree of entanglement as defined in [4].

ERROR

We examine the overlap integral of the actual Ψ_F with the wave function that would have resulted from the idealization, $x\rightarrow -x$, namely,

$$\Psi_{\text{test}} \equiv \Gamma(X)\Phi(-x) = \Gamma(R-\delta u)\Phi(-R-\gamma u). \tag{4}$$

Using Eq. (3),

$$\begin{aligned} A &\equiv \int \Psi^*_{\text{test}}\Psi_F = \int dR\, du\, \Gamma^*(R-\delta u)\Phi^*(-R-\gamma u) \\ &\quad \times \Gamma(R+\delta u)\Phi(R-\gamma u) \\ &= \frac{1}{2\pi\sigma\Sigma}\int dR\, du\, \exp\left(-\frac{(R-\delta u)^2}{4\Sigma^2}-\frac{(-R-\gamma u)^2}{4\sigma^2}\right. \\ &\quad \left. -ik(-R-\gamma u)\right)\exp\left(-\frac{(R+\delta u)^2}{4\Sigma^2}-\frac{(R-\gamma u)^2}{4\sigma^2}\right. \\ &\quad \left. +ik(R-\gamma u)\right), \end{aligned}$$

we find

$$A^{-2} = \left[\gamma^2+\delta^2+\gamma^2\lambda+\frac{\delta^2}{\lambda}\right]\exp\left(\frac{4k^2\lambda\sigma^2}{1+\lambda}\right) \quad \text{with } \lambda \equiv \frac{\Sigma^2}{\sigma^2}. \tag{5}$$

To study the extent to which the idealization Eq. (4) can be accurate, we vary σ and Σ so as to minimize the deviation (and maximize A). For $k=0$, A depends only on λ (not the sigmas separately) and is optimized by

$$\lambda_{\max} = \frac{\delta}{\gamma} \approx \frac{m}{M}.$$

Substituting yields $A=1$. There is *no* error. (N.B. This holds *only* for $k=0$ and $\lambda=\lambda_{\max}$.) When $k\neq 0$ we maximize A by optimizing λ for given $k\sigma$. We will see that even at best [11] A falls below unity by $O(\delta)$. For small and large $k\sigma$ analytic forms are

$$\begin{aligned} &\lambda_{\max} \approx \delta/\gamma \quad (\text{as before}), \\ &1-A \approx 2\delta k^2\sigma^2 \quad \text{for small } k\sigma, \\ &\lambda_{\max} \approx \delta/2k\sigma, \quad 1-A\approx 2\delta k\sigma \quad \text{for large } k\sigma. \end{aligned} \tag{6}$$

These behaviors mesh smoothly at $k\sigma\sim 1$. Equation (6) is a lower bound on error. The factor $\delta\approx m/M$ keeps this effect small and is reminiscent of similar factors in measurement theory [12]. It may be appropriate to think of the confinement process as one in which the system's components are constantly bumping up against their container, so that the small δ could pick up a large factor related to the frequency of such interactions.

DECOHERENCE

This is potentially the more damaging effect. A basis independent measure of the degree of entanglement of the particle and wall is given in [4]. It can be shown [13] that this degree of entanglement is 1 minus the largest eigenvalue of $\psi^\dagger\psi$ (or $\psi\psi^\dagger$) considered as a matrix operator with matrix indices the arguments of ψ.

Because we ultimately wish to use the system variable x as if it were unentangled, the wave function is expressed in terms of x and X:

$$\begin{aligned} \Psi_F(x,X) &= \left[\frac{4\Omega\omega}{\pi^2}\right]^{1/4} \exp\{-\Omega[X(1-2\delta)+2\delta x]^2-\omega[x(1 \\ &\quad -2\gamma)+2\gamma X]^2+ik[x(1-2\gamma)+2\gamma X]\}, \end{aligned} \tag{7}$$

with $\Omega\equiv 1/4\Sigma^2$ and $\omega\equiv 1/4\sigma^2$. We can form an operator by integrating either over X or over x. We choose

$$\begin{aligned} F(x',x) &\equiv \int dX\, \Psi^*_F(X,x')\Psi_F(X,x) \\ &= \sqrt{\frac{2\omega\Omega}{\pi D}}\exp\left\{-(x^2+x'^2)\frac{\omega\Omega}{D}-2(x-x')^2\frac{\rho^2}{D}\right. \\ &\quad \left. +ik(1-2\gamma)(x-x')\right\}, \end{aligned} \tag{8}$$

with $D \equiv \Omega(\gamma-\delta)^2+4\omega\gamma^2$ and $\rho \equiv |(\gamma-\delta)(\Omega\delta-\omega\gamma)|$. As indicated, we want the largest eigenvalue of F, now thought of as the integral kernel of an operator. Note that the factor $\exp[ik(1-2\gamma)(x-x')]$ can be dropped because it does not affect the eigenvalue. Next observe that F is almost the same as the kernel of the propagator for the simple harmonic oscillator. Using a standard form for this operator [15], we note the following fact. The operator

$$G(x,y) \equiv \sqrt{\frac{\beta}{\pi \sinh u}} \exp\left[-\frac{\beta}{\sinh u}\left[(x^2+y^2)\cosh u - 2xy\right]\right]$$

has the spectrum $G_n=\exp[-u(n+\frac{1}{2})]$, $n=0,1,2,\ldots$, irrespective of β. (The connection with the oscillator is $\beta = m\omega/2\hbar$ and $\omega t = -iu$.) It is now straightforward to deduce that the spectrum of F is $F_n=(1-e^{-u})e^{-nu}$, with $n = 0,1,\ldots$, and $\sinh u/2 = \sqrt{\omega\Omega}/2\rho$. It follows that the largest eigenvalue of F is

$$F_0 = 1-z^2 \quad \text{with } z=\sqrt{\frac{w^2}{4}+1}-\frac{w}{2}, \quad w \equiv \frac{\sqrt{\omega\Omega}}{\rho}.$$

For small w, $F_0 \sim w$, and for large w, $F_0 \sim 1-1/w^2$.

The first issue is minimizing entanglement, that is, maximizing F_0. Clearly, F_0 reaches its theoretical maximum for $w=\infty$, which requires [16] in turn $\Omega\delta=\omega\gamma$. Recalling the definitions of ω and Ω, this brings us to the same relation, $\Sigma^2/\sigma^2=\delta/\gamma$, that we found when minimizing error [17]. It is interesting that here the entanglement is strictly zero *even when the momentum k is nonzero*—if there is the special matching of wave function spreads. In the absence of matching, the entanglement, hence the decoherence, can be considerable, as indicated by $F_0 \sim w$ for small w.

This decoherence cuts down the *amplitude* of the wave function that can ultimately yield an accurate computational result. By the methods of [4] one can show [13] that the maximum amplitude available in a putative unentangled wave function $\psi(x)$ is $\sqrt{F_0}$ and that for two successive independent collisions it will be the product of two such terms. If F_0 is not extremely close to 1, the effect can build rapidly. Such behavior is to be contrasted with, say, decay, where the initial small deviation is in a phase, so that the effect of many independent such deviations is only quadratic in each of them.

OPTIMAL COHERENCE

The minimization of both error and entanglement have brought to light a matching condition on the spreads of the system and apparatus $\Sigma^2/\sigma^2=m/M$. This may be surprising. Based on the usual idealization of macroscopic objects, one might have thought that there should be no restriction on the *smallness* of ΔX [18]. Aside from considerations of the sort in [2] (and for which $F_0=1$ provides an example of a ''special state''), there is no reason to think that nature would evolve into minimally decohering states [19]. Of course the constructor of a quantum computer may have a strong interest in such minimizing. In any case, it is of interest to consider the possibility that the optimizing condition hold generally. In [3] it was observed that *all* pairs of objects could satisfy the relation above if for each object its mass m (not the same m as before) and its position uncertainty σ_m were related by $\sigma_m^2 \sim 1/m$. Possible justifications were considered in [3], but we here take the relation as a hypothesis and extend it using dimensional analysis [20]. Taking $\hbar=1$ and $c=1$, it is clear that another length (or energy or mass) is needed. For a confined system the quantities that come to mind are an overall length scale for the system and the temperature. The former seems to me ill defined, and in particular an attractive feature of the relation proposed is that it is not vital to distinguish between ''system'' and walls. Using then the temperature (T) and restoring $\hbar$, we find

$$\sigma_m^2 \sim \frac{\hbar^2}{mk_BT}, \tag{9}$$

with k_B the Boltzmann constant. Equation (9) gives a mass m object a packet size that is the geometric mean of its Compton wavelength and $\sim(0.2\text{ cm})/(T\text{ K})$. This does not seem inconsistent with experience. Lower temperature allows larger coherent wave packets, distinguishing this effect from thermal fluctuations [21] where position spread *decreases* with decreasing temperature. If the effective momentum k of the small mass is itself the result of thermal fluctuations, then equipartition relates this to temperature as well. We then have $k^2\sigma^2 \sim (2\hbar^2k^2/2m)/k_BT \sim 1$, independent of temperature [22]. (For $k\sigma=1$, $A\approx 1-1.2\delta$.) This suggests that in a heat bath, $\Delta p \sim \hbar/\Delta x$, since $\langle p\rangle=0$.

LIMITATIONS AND EXTENSIONS

We have shown that confinement need *not* force entanglement, but if the confined objects strike the walls at finite velocity, there must be ''error.'' It must be emphasized that the no-entanglement result depends not only on a particular ratio of spreads for small and large systems, but also on the Gaussian form of the wave packet and on the form of the interaction with the wall. Since my expectation in starting this work was that a nonzero lower bound could be found on entanglement-induced decoherence, it made sense to idealize as much as possible in aiming for the lower bound. It now turns out that zero entanglement is attainable, so that a converse orientation is suitable: which assumptions could be dropped and still maintain zero entanglement? My guess is not many, although given our experience with the absence of momentum-conservation-forced entanglement one should not jump to conclusions. That guess is based on another solvable (nearly solvable, actually) propagator: the harmonic wall. For a wall (in the relative coordinate u) of the form $\theta(u)\frac{1}{2}\mu(\pi/\tau)^2u^2$ and for regions $u<0$, the propagator is the same as that given in Eq. (1), except [23,2] that $g_0^\mu(u''+u',t)$ is replaced by $g_0^\mu(u''+u',t-\tau)$. In other words, the wave function reflects essentially perfectly off the wall, but is delayed by one half period. The difference between this and the free uncoupled propagator (which it would be for $\tau=0$) contains terms of the form $\exp(\text{const}\times i\tau x X)$, which do not seem to me removable, but are small for short reaction times. In any case, this issue remains open.

For applications it is desirable to identify the wall mass M. Even for a vacuum chamber one would not look to the mass of the entire chamber, but only the region affected by the particle's collision, perhaps defined by the wavelength of the appropriate phonon. For ''chambers'' that are magnetic fields (etc.) one can ultimately look to the laboratory equipment that produces these fields.

Finally, there is our decoherence-minimizing relation $\sigma_m^2 \sim 1/m$ or, more ambitiously, $\sigma_m^2 \sim \hbar^2/mk_BT$. Do particles settle into wave packets of this size? Are two-time boundary condition considerations (as in [2]) at work? Or perhaps (not exclusively) arguments of the form in [3] or [19] hold. Yet another question is the form such a relation might take for massless particles. Here too one could ask for decoherence-minimizing scattering.

In conclusion, we have shown that pinning a system to the table does not in itself force entanglement with the degrees of freedom of the container, treating the latter as a fully quantum object. Nevertheless, subject to reasonable assumptions, that pinning will introduce error, in the sense of changed outgoing wave function. Minimizing both decoherence and error are best accomplished when a particular relation exists between the wave function spreads of the system and container. We have also computed the degree of entanglement in situations where the minimum spread condition does not hold.

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[1] Among many references, see, for example, I. L. Chuang, R. Laflamme, P. W. Shor, and W. H. Zurek, Science **270**, 1633 (1995). Note that for quantum computing purposes the entanglement discussed in the present article is relevant only to the extent that different states in the Hilbert space associated with the computation couple differently to the translational degrees of freedom.

[2] L. S. Schulman, *Time's Arrows and Quantum Measurement* (Cambridge University Press, Cambridge, 1997).

[3] L. S. Schulman, Phys. Lett. A **211**, 75 (1996). Note a misprint: ''ϵ'' there should be $2m/\mathcal{M}$ (not $m/\mathcal{M}$).

[4] A. Shimony, Ann. N.Y. Acad. Sci. **755**, 675 (1995).

[5] By ''decoherence'' I mean loss of the primary wave function through entanglement with other degrees of freedom, hence the inability to interfere with portions of the wave function not so entangled. By ''error'' I mean a nonentangled wave function whose value is changed from that associated with idealized potential scattering.

[6] For electromagnetic or other confinement this picture will need extension. However, the primitive underpinning of the derivation, momentum conservation, suggests that such an extension is possible.

[7] A treatment neglecting the dynamical nature of the wall would generally omit the function Γ.

[8] The validity of this assertion depends on the separation of the incoming and outgoing wave packets. To derive a quantitative measure of this I must be more explicit about the propagator in Eq. (1). Because of the hard wall, the coordinate u ($\equiv x-X$) is only defined for negative values (since I take the small particle to be coming from the left). It is more convenient, however, to extend the space to the entire real line and consider the initial wave packet to have consisted of two pieces, one coming from its actual source and one coming from the mirror image. (This is the method of images applied to the path integral [9].) To study the separation of the wave packets I ignore the small effect of the entanglement with the wall [this has been checked with Gaussian wave packets and only changes the outcome by $O(\delta)$]. I thus ascertain whether with the naive calculation (treating the wall as a fixed potential) the incident and reflected wave packets separate. First, if the initial position is $-x_0$, then it must be the case that $|x_0| \gg \sigma$, where σ is the spread of the x-wave function. All that is left to check is that wave-packet spreading during the time it takes for the packets to separate does not overwhelm the effect of the relative velocity of the separating (incident and reflected, or source and image) wave packets. To see this, recall that for a free particle the time-t evolute of a particle with initial wave function $(1/\sigma)\exp\{-[(x-x_0)^2/4\sigma^2]+ik(x-x_0)\}$ is $[1/\sigma(t)]\exp\{-[(x-x_0-v_0t)^2/4\sigma^2(t)]+ik(x-x_0-v_0t)+i\hbar k^2t/2m\}$, with $\sigma^2(t)\equiv\sigma^2+i\hbar t/2m$ and $v_0\equiv\hbar k/m$. Small wave-packet spread implies $\sigma^2 \gg \hbar t/2m$, while having the incoming and outgoing packets separate from one another requires $v_0t>\sigma$. Combining these gives the requirement $\sigma k \gg 1$ in order for our calculational method to be valid.

[9] A. Auerbach and L. S. Schulman, J. Phys. A **30**, 5993 (1997).

[10] Allowing the integration to range over the whole line is another correlate of the method of images and the assumption that the initial wave packet was negligibly small at the wall. To show this I drop the complication of treating the wall dynamically. What one should calculate is $\int_{-\infty}^{0}dx'[g(x-x',t)-g(x+x',t)]\psi_0(x')$. Since ψ_0 vanishes for $x \geq 0$ (so far, I've not set x_0 to zero), the integral over x' can be run over the entire line. For the $g(x+x',t)$ integral we change x' to $-x'$, so that it now looks like a source at $+|x_0|$. *At this point* I set $x_0=0$ and make use of the convenient wave function form given in Eq. (3).

[11] Recall from Ref. [8] that $k\sigma$ should be larger than one for our calculational method to apply.

[12] M. M. Yanase, Phys. Rev. **123**, 666 (1961).

[13] Following Ref. [14] (and in agreement, up to overall constants, with Ref. [4]), one can define the measure of entanglement either as $J\equiv\min_{\rho,\sigma}\int dx\,dy|\psi(x,y)-\rho(x)\sigma(y)|^2$ or as $K\equiv\max_{f,g}|\int dx\,dy\,\psi^*(x,y)f(x)g(y)|$, with $\|f\|=\|g\|=1$ in the second version. One then invokes the following mathematical result. Let A be an arbitrary $n\times n$ matrix and let $\mathcal{L}\equiv\min_{u,v}\Sigma_{i,j}|A_{ij}-u_iv_j^*|^2$. Then if u and v minimize $\mathcal{L}$, they satisfy $A^\dagger u=v\|u\|^2$ and $Av=u\|v\|^2$. From this it follows that $AA^\dagger u=\lambda u$, $A^\dagger Av=\lambda v$, and $\lambda=\|u\|^2\|v\|^2$, the maximum eigenvalue of $A^\dagger A$. Defining $\tilde{u}=u/\|u\|$, $\tilde{v}=v/\|v\|$, and $S_2(B)\equiv\mathrm{Tr}B^\dagger B$ for a matrix B, we can write $\mathcal{L}=S_2(A-\sqrt{\lambda}|\tilde{u}\rangle\langle\tilde{v}|)$. It is also clear that $\langle\tilde{u}|A|\tilde{v}\rangle=\sqrt{\lambda}$, which is real, and finally $\mathcal{L}=1-\lambda$. The equivalence of the definitions K and

J can be seen by noting the correspondence of $\mathcal{L}$ and J, with $\psi(x,y)$ playing the role of A_{ij}. I need to show that functions $\tilde{u}$ and $\tilde{v}$ that minimize J also maximize K. For arbitrary, normalized w and x, let $\tilde{J}\equiv S_2(A-\gamma|w\rangle\langle x|)$, where γ is an arbitrary complex constant. If we adjust $|x\rangle$ to make $\langle x|A|w\rangle$ real, then $\tilde{J}=S_2(A)+|\gamma|^2-\langle x|A|w\rangle(\gamma+\gamma^*)$. Taking γ to be real obviously can only reduce J. Since the foregoing equation holds for any real γ, it is seen that maximizing $\langle x|A|w\rangle$ is the same as minimizing J. It then follows that $\lambda=\gamma^2$, etc.

[14] L. S. Schulman and D. Mozyrsky (unpublished).

[15] L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).

[16] For $m=M$ one can also get $\rho=0$, i.e., no entanglement, but this is not the physical situation emphasized in this article.

[17] Disentanglement with spread matching could have been noted directly from Ψ_F and does not require Ref. [4]. However, the measure of the amplitude defect without matching does require those results.

[18] There is of course Feynman's variation on the two slit experiment [in R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1965)], which relates uncertainties in a macroscopic object to putative measurements of a microscopic one. The corresponding restriction here would be that ΔX $(=\Sigma)$ not be so small that the associated ΔP destroy the small-system interference patterns that we seek. Our optimal Σ is far from such values. We estimate this kinematic effect as follows. Momentum uncertainty ΔP in the big system means uncertainty $\Delta P/M$ in the (velocity) transformation going into the center-of-mass frame. For the small system this velocity uncertainty gives a momentum uncertainty $(\Delta p)'\sim m(\Delta P/M)$ [the prime on $(\Delta p)'$ distinguishes it from the momentum uncertainty in the original wave function, namely, $(\Delta p)_{\rm usual}\sim\hbar/\sigma$]. Taking $\Delta P\sim\hbar/\Sigma$, we find $(\Delta p)'\sim m\hbar/M\Sigma$. Using $\Sigma^2/\sigma^2\approx m/M$ yields $(\Delta p)'/(\Delta p)_{\rm usual}\sim\sqrt{m/M}$.

[19] There *have* been suggestions that nature ought to evolve into coherent states. See W. H. Zurek, S. Habib, and J. P. Paz, Phys. Rev. Lett. **70**, 1187 (1993).

[20] The inverse mass relation was also found in [19]. The energy demanded by dimensional analysis is there an oscillator frequency. It is not clear (to me) whether their $\Delta x^2\sim\hbar/2m\omega$ and my Eq. (9) are related.

[21] H. Grabert, P. Schramm, and G. Ingold, Phys. Rep. **168**, 115 (1988). See in particular Table 2, p. 159.

[22] The condition $k\sigma=1$ in Ref. [8] was not needed for the disentanglement, only for the justification of wave-packet separation and as a limitation on spreading, so as to allow this separation.

[23] L. S. Schulman, Ann. Phys. (N.Y.) **183**, 320 (1988).