Invariant formulation and exact solutions for the relativistic charged Klein-Gordon field in a time-dependent spatially homogeneous electric field

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On the basis of the invariant formulation, we find the quantum and classical exact solutions and corresponding total phase for the relativistic charged Klein-Gordon (KG) field in a time-dependent spatially homogeneous electric field. The total phase includes both the dynamical and geometric phases (Aharonov-Anandan phases). The connection between the quantum and classical solutions is then obtained. From this connection we discuss the condition under which the geometric phase for the KG field can be defined. [S1050-2947(98)03002-9]

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I. INTRODUCTION

Since Berry's discovery of the geometric phase in the quantum adiabatic evolution, there has been a great deal of theoretical and experimental works on this quantum holonomy phenomenon referred to as Berry phase [1]. In a fundamental generalization of Berry's idea, Aharonov and Anandan removed the adiabatic condition and studied the geometric phase for any cyclic evolution [2]. Anandan then pointed out that, in principle, the study of any noncyclic evolution can reduce to the study of cyclic evolution and corresponding phases [3]. The Lewis-Riesenfeld invariant theory [4] was generalized, by introducing a concept of the basic invariants, and used to show explicitly that the study of the exact solutions corresponding to any noncyclic evolution of the driven generalized time-dependent oscillator can reduce to the study of the cyclic evolution and corresponding total phase, including both the geometric phase and dynamical phase [5]. Then, it became more and more recognized that there are actually nothing but different names and attributes given to various parts of the total phase [6] as long as the exact solution of the time-dependent Schrödinger equation with a time-dependent Hamiltonian is concerned. The invariant formulation (representation) in [4,5] for obtaining the exact solutions for systems with time-dependent Hamiltonians is closely related to the study of the phase; it may then be referred to as the phase formulation (representation).

The invariant theory was further extended for investigating the quantum as well as classical systems with non-Hermitian Hamiltonians; for classical ones, the corresponding geometric phase is shown to be the nonadiabatic generalization of the Hannay angle [7]. Recently, it has become evident that the invariant formulation in [4,5] can also be applied to the treatment of more than one-dimensional time-dependent quantum systems (including infinite-dimensional systems: quantum fields [8,9]) if a complete set of invariants can be found.

In this paper the invariant formulation or the phase formulation in [4,5,7,9] is used to investigate both the classical and especially the quantum Klein-Gordon (KG) field. The classical and quantum exact solutions in the phase formulation for a particular case of the KG field are obtained. With the help of the coherent state, we then discuss the classical correspondence and establish the connection between the classical exact solutions and the quantum ones.

In [10] Anandan and Mazur studied the geometric phase for the classical KG field equation that is second order in time and reached a conclusion that the condition under which the geometric phase can be defined is that the (usual) particle creation (annihilation) is absent (equivalent to the absence of the external electric field). It is important to note that, in [10], the condition was obtained without carefully investigating the corresponding quantum theory of the KG field. Apparently, since the creation (annihilation) of KG field-theory-particles is a quantum concept, the quantum KG field theory should be discussed in order to know more about the condition. In the present paper, the quantum KG field theory is studied in detail and the connection between the classical exact solutions and the quantum ones is established. This connection makes it possible to add something to the result obtained in [10].

II. EXACT SOLUTIONS FOR THE CLASSICAL CHARGED KG FIELD

In this section, we study a particular case of the classical KG field in which the homogeneity is assumed for the external time-dependent electric field. The Lagrangian density for a complex scalar KG field interacting with an external electromagnetic field is given by

$$L=(\partial_\mu+ieA_\mu)\phi^*(\partial^\mu-ieA^\mu)\phi-m^2\phi^*\phi, \qquad (2.1)$$

which leads to the following time-dependent Hamiltonian in the Weyl gauge $A^0=0$ [8]:

$$H(t)=\int d^3r[\pi^*\pi+(\vec{\nabla}+ie\vec{A})\phi^*\cdot(\vec{\nabla}-ie\vec{A})\phi+m^2\phi^*\phi], \tag{2.2}$$

where the canonical momentum densities π and π^* are defined as $\pi=\partial L/\partial(\partial_0\phi)$ and $\pi^*=\partial L/\partial(\partial_0\phi^*)$. The canonical Poisson brackets are

$$\{\pi(\vec{r},t),\phi(\vec{r}',t)\}=-\delta^3(\vec{r}-\vec{r}'),$$

$$\{\pi^*(\vec{r},t),\phi^*(\vec{r}',t)\}=-\delta^3(\vec{r}-\vec{r}'),$$

$$\begin{aligned}\{\pi(\vec{r},t),\pi^*(\vec{r}',t)\}&=\{\phi(\vec{r},t),\phi^*(\vec{r}',t)\}\\&=\{\pi(\vec{r},t),\phi^*(\vec{r}',t)\}\\&=\{\pi^*(\vec{r},t),\phi(\vec{r}',t)\}=0.\end{aligned} \tag{2.3}$$

If the homogeneity is assumed for the external time-dependent electric field, we can employ the ''momentum representation'' [8]

$$\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\, e^{i\vec{k}\cdot\vec{r}}F(\vec{k},t),$$

$$\phi^*(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}F^*(\vec{k},t),$$

$$\pi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}P(\vec{k},t),$$

$$\pi^*(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\, e^{i\vec{k}\cdot\vec{r}}P^*(\vec{k},t), \tag{2.4}$$

$$\begin{aligned}H(t)=&\int d^3\vec{k}\{P^*(k,t)P(\vec{k},t)\\&+[(\vec{k}-e\vec{A})^2+m^2]F^*(\vec{k},t)F(\vec{k},t)\\=&\int d^3\vec{k}\, H(\vec{k},t).\end{aligned} \tag{2.5}$$

The corresponding canonical Poisson brackets are

$$\{P(\vec{k},t),F(\vec{k},t)\}=-\delta^3(\vec{k}-\vec{k}'),$$

$$\{P^*(\vec{k},t),F^*(\vec{k},t)\}=-\delta^3(\vec{k}-\vec{k}'),$$

$$\begin{aligned}\{P(\vec{k},t),P^*(\vec{k}',t)\}&=\{F(\vec{k},t),F^*(\vec{k}',t)\}\\&=\{P(\vec{k},t),F^*(\vec{k}',t)\}\\&=\{P^*(\vec{k},t),F(\vec{k}',t)\}=0.\end{aligned} \tag{2.6}$$

It is then easy to get the canonical equations for $F(\vec{k})$, $P^*(\vec{k})$, $F^*(\vec{k})$, and $P(\vec{k})$:

$$\dot{F}(\vec{k},t)=\{F(\vec{k},t),H(t)\}=P^*(\vec{k},t),$$

$$\dot{F}^*(\vec{k},t)=\{F^*(\vec{k},t),H(t)\}=P(\vec{k},t),$$

$$\dot{P}(\vec{k},t)=\{P(\vec{k},t),H(t)\}=-a(\vec{k},t)F^*(\vec{k},t),$$

$$\dot{P}^*(\vec{k},t)=\{P^*(\vec{k},t),H(t)\}=-a(\vec{k},t)F(\vec{k},t), \tag{2.7}$$

where $a(\vec{k},t)=[\vec{k}-e\vec{A}(\vec{k},t)]^2+m^2$. This equation can be rewritten in the matrix form

$$i\frac{\partial}{\partial t}\begin{pmatrix}F(\vec{k},t)\\P^*(\vec{k},t)\end{pmatrix}=\begin{pmatrix}0&i\\-ia&0\end{pmatrix}\begin{pmatrix}F(\vec{k},t)\\P^*(\vec{k},t)\end{pmatrix}, \tag{2.8}$$

$$i\frac{\partial}{\partial t}\begin{pmatrix}F^*(\vec{k},t)\\P(\vec{k},t)\end{pmatrix}=\begin{pmatrix}0&i\\-ia&0\end{pmatrix}\begin{pmatrix}F(\vec{k},t)\\P^*(\vec{k},t)\end{pmatrix}. \tag{2.9}$$

Apparently, we only have to solve Eq. (2.8) since Eq. (2.9) is the complex conjugate to Eq. (2.8). Equation (2.8) is of Schrödinger type with the non-Hermitian Hamiltonian $(\begin{smallmatrix}0&i\\-ia&0\end{smallmatrix})$. The corresponding invariant can be easily obtained [7] (see Appendix B):

$$\hat{I}(\vec{k},t)=\begin{pmatrix}-\rho(\vec{k},t)\dot{\rho}(\vec{k},t)&\rho^2(\vec{k},t)\\-\dfrac{1}{\rho^2(\vec{k},t)}+\rho^2(\vec{k},t)&\rho(\vec{k},t)\dot{\rho}(\vec{k},t)\end{pmatrix}, \tag{2.10}$$

where $\rho(\vec{k},t)$ is the real solution of the auxiliary equation

$$\ddot{\rho}(\vec{k},t)+a(\vec{k},t)\rho(\vec{k},t)=\frac{1}{\rho^3(\vec{k},t)}. \tag{2.11}$$

The eigenkets of $\hat{I}(\vec{k},t)$ are

$$|\pm,\vec{k},t\rangle=\begin{pmatrix}\rho^2(\vec{k},t)\\\pm i[1\mp i\rho(\vec{k},t)\dot{\rho}(\vec{k},t)]\end{pmatrix},$$

$$I(\vec{k},t)|\pm,\vec{k},t\rangle=\mp i|\pm,\vec{k},t\rangle. \tag{2.12}$$

According to the invariant theory in [7] (see Appendix B), the general solution of Eq. (2.7) is

$$\begin{pmatrix} F(\vec{k},t) \\ P^*(\vec{k},t) \end{pmatrix} = \begin{pmatrix} R_1\rho(\vec{k},t)e^{i\theta(\vec{k},t)} + R_2\rho(\vec{k},t)e^{-i\theta(\vec{k}'t)} \\ R_1\left(-\frac{i}{\rho(\vec{k},t)} + \dot{\rho}(\vec{k},t)\right)e^{i\theta(\vec{k},t)} + R_2\left(-\frac{i}{\rho(\vec{k},t)} + \dot{\rho}(\vec{k},t)\right)e^{-i\theta(\vec{k},t)} \end{pmatrix}, \tag{2.13}$$

where R_1, R_2 are arbitrary complex constants and hence both $\rho(\vec{k},t)e^{i\theta(\vec{k},t)}$ and $\rho(\vec{k},t)e^{-i\theta(\vec{k},t)}$ are the particular solutions for $F(\vec{k},t)$ in Eq. (2.7). The expression for the phase $\theta(\vec{k},t)$ is

$$\theta(\vec{k},t) = \int_0^t dt' \frac{1}{\rho^2(\vec{k},t')}. \tag{2.14}$$

It is important to point out that, in accordance with the invariant theory in [5,7], the particular solutions can be made cyclic in a chosen time interval $[0,T]$. The general solution for $F(\vec{k},t)$ is the superposition of the two particular cyclic solutions. This is to say that the study of the general solution for $\phi(\vec{r},t)$ in Eq. (2.4) reduces to the study of the cyclic solutions $\rho(\vec{k},t)e^{i\theta(\vec{k},t)}$ and $\rho(\vec{k},t)e^{-i\theta(\vec{k},t)}$ for arbitrary $\vec{k}$ [with $\pm\theta(\vec{k},t)$ being the phases]. Finally, note that, as $\vec{A}(\vec{k},t)\to 0$, the solutions $\rho(\vec{k},t)e^{i\theta(\vec{k},t)}$ and $\rho(\vec{k},t)e^{-i\theta(\vec{k},t)}$ reduce to the positive-energy and negative-energy solutions in the free KG field theory.

III. QUANTUM MOTION OF THE KG FIELD AND ITS CLASSICAL CORRESPONDENCE

The quantum Hamiltonian for a charged (complex) scalar field with an external electromagnetic field is given by

$$\hat{H}(t) = \int d^3\vec{r}[\hat{\pi}^*\hat{\pi} + (\vec{\nabla} + ie\vec{A})\hat{\phi}^*\cdot(\vec{\nabla} - ie\vec{A})\hat{\phi} + m^2\hat{\phi}^*\hat{\phi}],$$

where $A^0 = 0$. Quantization is performed by imposing the equal-time commutation relations

$$[\hat{\pi}(\vec{r},t), \hat{\phi}(\vec{r}',t)] = -i\delta^3(\vec{r}-\vec{r}'),$$

$$[\hat{\pi}^*(\vec{r},t), \hat{\phi}^*(\vec{r}',t)] = -i\delta^3(\vec{r}-\vec{r}'),$$

$$\begin{aligned}[\hat{\pi}(\vec{r},t), \hat{\pi}^*(\vec{r}',t)] &= [\hat{\phi}(\vec{r},t), \hat{\phi}^*(\vec{r}',t)] \\ &= [\hat{\pi}(\vec{r},t), \hat{\phi}^*(\vec{r}',t)] \\ &= [\hat{\pi}^*(\vec{r},t), \hat{\phi}(\vec{r}',t)] = 0. \end{aligned} \tag{3.1}$$

We choose to work within the functional Schrödinger picture [8] with substitutions

$$\hat{\pi}(\vec{r}) \to \frac{-i\delta}{\delta\phi(\vec{r})}, \quad \hat{\pi}^*(\vec{r}) \to \frac{-i\delta}{\delta\phi^*(\vec{r})}. \tag{3.2}$$

When the homogeneity is assumed for the external electric field, we can employ the momentum representation for the operators

$$\hat{\phi}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}\; e^{i\vec{k}\cdot\vec{r}}\hat{F}(\vec{k}),$$

$$\hat{\phi}^*(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}\; e^{-i\vec{k}\cdot\vec{r}}\hat{F}^*(\vec{k}),$$

$$\frac{\delta}{\delta\phi(\vec{r})} = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}\; e^{-i\vec{k}\cdot\vec{r}} \frac{\delta}{\delta F(\vec{k})},$$

$$\frac{\delta}{\delta\phi^*(\vec{r},t)} = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}\; e^{i\vec{k}\cdot\vec{r}} \frac{\delta}{\delta F^*(\vec{k})}. \tag{3.3}$$

Then we obtain

$$\hat{H}(t) = \int d^3\vec{k}\left\{ -\frac{\delta}{\delta F^*(\vec{k})}\frac{\delta}{\delta F(\vec{k})} + \{[\vec{k} - e\vec{A}(t)]^2 + m^2\}\hat{F}^*(\vec{k})\hat{F}(\vec{k}) \right\}, \tag{3.4}$$

of which the integrand can be shown to be associated with the time-dependent oscillators of two modes. Equation (3.4) leads to the Schrödinger equation for the functional $\Psi[\hat{F},\hat{F}^*;t]$, which describes the quantum motion of the KG field

$$i\frac{\partial}{\partial t}\Psi[\hat{F},\hat{F}^*;t] = \hat{H}(t)\Psi[\hat{F},\hat{F}^*;t]. \tag{3.5}$$

According to the invariant theory [4,5,9], we can find the following invariants for the field:

$$\hat{I}(t) = \int d^3\vec{k}\left[\hat{A}^\dagger(\vec{k},t)\hat{A}(\vec{k},t) + \hat{B}^\dagger(\vec{k},t)\hat{B}(\vec{k},t) + \frac{V}{(2\pi)^3}\right],$$

$$\hat{N}_A(\vec{k},t) = \hat{A}^+(\vec{k},t)\hat{A}(\vec{k},t),$$

$$\hat{N}_B(\vec{k},t) = \hat{B}^+(\vec{k},t)\hat{B}(\vec{k},t), \tag{3.6}$$

where

$$\hat{A}(\vec{k},t)=\frac{1}{\sqrt{2}}\left[\left(\frac{1}{\rho(\vec{k},t)}-i\dot{\rho}(\vec{k},t)\right)\hat{F}(\vec{k})+i\rho(\vec{k},t)\left(\frac{-i\delta}{\delta F^*(\vec{k})}\right)\right],$$

$$\hat{A}^\dagger(\vec{k},t)=\frac{1}{\sqrt{2}}\left[\left(\frac{1}{\rho(\vec{k},t)}+i\dot{\rho}(\vec{k},t)\right)\hat{F}^*(\vec{k})-i\rho(\vec{k},t)\left(\frac{-i\delta}{\delta F(\vec{k})}\right)\right],$$

$$\hat{B}(\vec{k},t)=\frac{1}{\sqrt{2}}\left[\left(\frac{1}{\rho(-\vec{k},t)}-i\dot{\rho}(-\vec{k},t)\right)\hat{F}^*(-\vec{k})+i\rho(-\vec{k},t)\left(\frac{-i\delta}{\delta F(-\vec{k})}\right)\right],$$

$$\hat{B}^\dagger(\vec{k},t)=\frac{1}{\sqrt{2}}\left[\left(\frac{1}{\rho(-\vec{k},t)}+i\dot{\rho}(-\vec{k},t)\right)\hat{F}(-\vec{k})-i\rho(-\vec{k},t)\left(\frac{-i\delta}{\delta F^*(-\vec{k})}\right)\right], \tag{3.7}$$

with $\rho(\vec{k},t)$ being a real solution of the auxiliary equation

$$\ddot{\rho}+\{[\vec{k}-e\vec{A}(t)]^2+m^2\}\rho=\rho^{-3}. \tag{3.8}$$

Note that the invariance of $\hat{I}(t)$ is apparently the consequence of the invariance of $\hat{N}_A$ and $\hat{N}_B$. It is easy to show that the operators $\hat{A}(\vec{k},t),\hat{B}(\vec{k},t)$ satisfy the equal-time commutation relations

$$[\hat{A}(\vec{k},t),\hat{A}^\dagger(\vec{k}',t)]=\delta^3(\vec{k}-\vec{k}'),$$

$$[\hat{B}(\vec{k},t),\hat{B}^\dagger(\vec{k}',t)]=\delta^3(\vec{k}-\vec{k}'). \tag{3.9}$$

$\hat{A}(\vec{k},t),\hat{B}(\vec{k},t)$ may then be referred to as generalized annihilation operators, $\hat{A}^\dagger(\vec{k},t),\hat{B}^\dagger(\vec{k},t)$ as generalized creation operators, and $\hat{N}_A(\vec{k},t),\hat{N}_B(\vec{k},t)$ as generalized particle numbers for the mode-$\vec{k}$ A particle and mode-$\vec{k}$ B particle, respectively. Since $\hat{N}_A(\vec{k},t)$ and $\hat{N}_B(\vec{k},t)$ are both invariants, the generalized particle numbers are conserved.

Now we turn to the problem of exactly solving the functional Schrödinger equation (3.5) by means of the invariant-related unitary transformation method in [9].

We first construct two unitary operators

$$\hat{Q}_A(t)=\exp\Bigg((i/4)\int d^3\vec{k}\bigg\{r(\vec{k},t)\sin\,\alpha(\vec{k},t)\bigg[\hat{F}(\vec{k})\hat{F}^*(\vec{k})-\left(\frac{-i\delta}{\delta F(\vec{k})}\right)\left(\frac{-i\delta}{\delta F^*(\vec{k})}\right)\bigg]+r(\vec{k},t)\cos\,\alpha(\vec{k},t)\times\bigg[\hat{F}^*(\vec{k})\left(\frac{-i\delta}{\delta F(\vec{k})}\right)+\left(\frac{-i\delta}{\delta F^*(\vec{k})}\right)\hat{F}(\vec{k})\bigg]\bigg\}\Bigg),$$

$$\hat{Q}_B(t)=\exp\Bigg((i/4)\int d^3\vec{k}\bigg\{r(-\vec{k},t)\sin\,\alpha(-\vec{k},t)\times\bigg[\hat{F}(-\vec{k})\hat{F}^*(-\vec{k})-\left(\frac{-i\delta}{\delta F(-\vec{k})}\right)\left(\frac{-i\delta}{\delta F^*(-\vec{k})}\right)\bigg]+r(-\vec{k},t)\cos\,\alpha(-\vec{k},t)\bigg[\hat{F}^*(-\vec{k})\left(\frac{-i\delta}{\delta F(-\vec{k})}\right)+\left(\frac{-i\delta}{\delta F^*(-\vec{k})}\right)\hat{F}(-\vec{k})\bigg]\bigg\}\Bigg), \tag{3.10}$$

where $r(\vec{k},t)$ and $\alpha(\vec{k},t)$ are defined by

$$\cosh\, r(\vec{k},t)=\tfrac{1}{2}\{\rho^{-2}(\vec{k},t)+\rho^2(\vec{k},t)+\dot{\rho}^2(\vec{k},t)\},$$

$$\sinh\, r(\vec{k},t)\exp[i\alpha(\vec{k},t)]=\tfrac{1}{2}\{\rho^{-2}(\vec{k},t)-\rho^2(\vec{k},t)+\dot{\rho}^2(\vec{k},t)\}+i\rho(\vec{k},t)\dot{\rho}(\vec{k},t). \tag{3.11}$$

With considerable effort, it can be shown that the unitary operators in Eq. (3.10) transform $\hat{I}(t)$ into $\hat{I}_0$ (by noting that $[\hat{Q}_A(t),\hat{Q}_B(t)]=[\hat{Q}_A(t),\hat{I}_{B0}]=[\hat{Q}_B(t),\hat{I}_{A0}]=0$):

$$\hat{I}_0=Q_A^\dagger(t)\hat{Q}_B^\dagger(t)I(t)\hat{Q}_A(t)\hat{Q}_B(t)=\hat{I}_{A0}+\hat{I}_{B0},$$

$$\hat{I}_{A0}=\hat{Q}_A^\dagger(t)\hat{I}(t)\hat{Q}_A(t)=\int d^3\vec{k}\ \hat{I}_{A0}(\vec{k})=\int d^3\vec{k}\left[\hat{A}_0^\dagger(\vec{k})\hat{A}_0(\vec{k})+\frac{V}{2(2\pi)^3}\right],$$

$$\hat{I}_{B0}=\hat{Q}_B^\dagger(t)\hat{I}(t)\hat{Q}_B(t)=\int d^3\vec{k}\ \hat{I}_{B0}(\vec{k})=\int d^3\vec{k}\left[\hat{B}_0^\dagger(\vec{k})\hat{B}_0(\vec{k})+\frac{V}{2(2\pi)^3}\right], \tag{3.12}$$

where

$$\hat{A}_0^\dagger(\vec{k})=(2^{-1/2})\left\{\hat{F}^*(\vec{k})-i\left[\frac{-i\delta}{\delta F(\vec{k})}\right]\right\},$$

$$\hat{A}_0(\vec{k})=(2^{-1/2})\left\{\hat{F}(\vec{k})+i\left[\frac{-i\delta}{\delta F^*(\vec{k})}\right]\right\},$$

$$\hat{B}_0^\dagger(\vec{k})=(2^{-1/2})\left\{\hat{F}(-\vec{k})-i\left[\frac{-i\delta}{\delta F^*(-\vec{k})}\right]\right\},$$

$$\hat{B}_0(\vec{k})=(2^{-1/2})\left\{\hat{F}^*(-\vec{k})+i\left[\frac{-i\delta}{\delta F(-\vec{k})}\right]\right\}. \tag{3.13}$$

By making use of the unitary operators in Eq. (3.10) and the Backer-Campbell-Hausdorff formula, with lengthy calculations, we obtain $\hat{H}_0(t)$ from $\hat{H}(t)$ (see Appendix A),

$$\hat{H}_0(t)=\hat{Q}_A^\dagger(t)\hat{Q}_B^\dagger(t)\hat{H}(t)\hat{Q}_A(t)\hat{Q}_B(t) - i\hat{Q}_A^\dagger(t)\hat{Q}_B^\dagger(t)\frac{\partial[\hat{Q}_A(t)\hat{Q}_B(t)]}{\partial t}=\int d^3\vec{k}\;\hat{H}_0(\vec{k},t) =\hat{H}_{A0}(t)+\hat{H}_{B0}(t), \tag{3.14}$$

where

$$\hat{H}_{A0}(t)=\int d^3\vec{k}\,\hat{H}_{A0}(\vec{k},t) =\hat{Q}_A^\dagger(t)\hat{H}(t)\hat{Q}_A(t)-i\hat{Q}_A^\dagger(t)\frac{\partial\hat{Q}_A(t)}{\partial t},$$

$$\hat{H}_{B0}(t)=\int d^3\vec{k}\;\hat{H}_{B0}(\vec{k},t) =\hat{Q}_B^\dagger(t)\hat{H}(t)\hat{Q}_B(t)-i\hat{Q}_B^\dagger(t)\frac{\partial\hat{Q}_B(t)}{\partial t}, \tag{3.15a}$$

$$\hat{H}_{A0}(\vec{k},t)=[\rho^{-2}(\vec{k},t)+\dot{\xi}(\vec{k},t)]\hat{I}_{A0}(\vec{k}),$$

$$\hat{H}_{B0}(\vec{k},t)=[\rho^{-2}(\vec{k},t)+\dot{\xi}(\vec{k},t)]\hat{I}_{B0}(\vec{k}), \tag{3.15b}$$

with $\xi(\vec{k},t)=-\tan^{-1}\{\rho(\vec{k},t)\dot{\rho}(\vec{k},t)/[1+\rho^2(\vec{k},t)]\}$. It can be seen from Eq. (3.15b) that (i) $\hat{I}_{A0}(\vec{k})$ and $\hat{I}_{B0}(\vec{k})$ in Eq. (3.12) are time independent and (ii) $\hat{H}_{A0}(\vec{k},t),\hat{H}_{B0}(\vec{k},t)$ differ respectively from $\hat{I}_{A0}(\vec{k}),\hat{I}_{B0}(\vec{k})$ only by multiplying c-number factors. In the discrete notation, $\hat{I}_{A0}$ (or $\hat{I}_{B0}$) in Eq. (3.12) may be regarded as the sum of terms of which each has the form of the Hamiltonian for a simple harmonic oscillator with frequency 1. The solution to the oscillator eigenvalue problem for $k_1,k_2,\ldots$ modes may be characterized by integers $n_{A1},n_{A2},\ldots$ $(n_{A1},n_{A2},\ldots=0,1,2,\ldots)$ and $n_{B1},n_{B2},\ldots$ $(n_{B1},n_{B2},\ldots=0,1,2,\ldots)$. The ground state of $\hat{I}_{A0}(\vec{k})$ and $\hat{I}_{B0}(\vec{k})$ (the state with $n_{A1}=n_{B1}=n_{A2}=n_{B2}=\ldots=0$) is denoted by $|0\rangle$ and satisfies

$$\hat{A}_0(\vec{k})|0\rangle=0,\hat{B}_0(\vec{k})|0\rangle=0.$$

By making use of the ground state $|0\rangle$ and the raising operators $\hat{A}_0^\dagger(\vec{k})$ and $\hat{B}_0^\dagger(\vec{k})$ in Eq. (3.12), we obtain the N_A,N_B particle excited eigenstates of $\hat{I}_{A0}$ and $\hat{I}_{B0}$, respectively:

$$|N_A\rangle\equiv|n_{A1},n_{A2},\ldots(n_{A1}+n_{A2}+\cdots=N_A)\rangle =\left\{[n_{A1}!]^{-1/2}\left[\hat{A}_0^\dagger(\vec{k}_1)\left(\frac{V}{(2\pi)^3}\right)^{-1/2}\right]^{n_{A1}}[n_{A2}!]^{-1/2} \times\left[\hat{A}_0^\dagger(\vec{k}_2)\left(\frac{V}{(2\pi)^3}\right)^{-1/2}\right]^{n_{A2}}\cdots\right\}|0\rangle,$$

$$|N_B\rangle\equiv|n_{B1},n_{B2},\ldots(n_{B1}+n_{B2}+\cdots=N_B)\rangle =\left\{[n_{B1}!]^{-1/2}\left[\hat{B}_0^\dagger(\vec{k}_1)\left[\frac{V}{(2\pi)^3}\right]^{-1/2}\right]^{n_{B1}}[n_{B2}!]^{-1/2} \times\left[\hat{B}_0^\dagger(\vec{k}_2)\left(\frac{V}{(2\pi)^3}\right)^{-1/2}\right]^{n_{B2}}\cdots\right\}|0 \quad (n_{A1},n_{A2},\ldots=0,1,2,\ldots;n_{B1},n_{B2},\ldots=0,1,2,\ldots). \tag{3.16}$$

The eigenstate of $\hat{I}_0=\hat{I}_{A0}+\hat{I}_{B0}$ [see Eq. (3.12)] with particle number N_A+N_B is $|N_A,N_B\rangle_{I_0}=|N_A\rangle|N_B\rangle$. According to the invariant related unitary transformation method in [5,9], from the eigenstates of $\hat{I}_0$, it is easy to obtain the solutions of the Schrödinger equation [associated with $\hat{H}_0(t)$]

$$|N_A,N_B;t\rangle_{S0}=\exp[i\theta_A(t)]\exp[i\theta_B(t)]|N_A,N_B\rangle_{I_0} \quad (N_A,N_B=0,1,2,\ldots),$$

$$\theta_A(t)=-\int_{t_0}^t dt'\langle N_A,N_B;t|H_{A0}(t')|N_A,N_B;t\rangle =\theta_{A0}(t)+n_{A1}\theta_A(\vec{k}_1,t)+n_{A2}\theta_A(\vec{k}_2,t)+\cdots \quad (n_{A1}+n_{A2}+\cdots=N_A),$$

$$\theta_B(t)=\int_{t_0}^t dt'\langle N_A,N_B;t|H_{B0}(t')|N_A,N_B;t\rangle =\theta_{B0}(t)+n_{B1}\theta_B(-\vec{k}_1,t)+n_{B2}\theta_B(-\vec{k}_2,t)+\cdots \quad (n_{B1}+n_{B2}+\cdots=N_B), \tag{3.17}$$

where

$$\theta_A(\vec{k},t)=\int_{t_0}^t dt'\rho^{-2}(\vec{k},t)+\xi(\vec{k},t)-\xi(\vec{k},t_0),$$

$$\theta_B(-\vec{k},t)=\int_{t_0}^t dt'\rho^{-2}(-\vec{k},t)+\xi(-\vec{k},t)-\xi(-\vec{k},t_0),$$

$$\theta_{A0}(t)=\left[-\frac{V}{2(2\pi)}\right]\int d^3\vec{k}\,\theta_A(\vec{k},t),$$

$$\theta_{B0}(t)=\left[-\frac{V}{2(2\pi)}\right]\int d^3\vec{k}\;\theta_B(-\vec{k},t), \tag{3.18}$$

in which $\theta_A(\vec{k},t)$ $[\theta_B(\vec{k},t)]$ is the total phase, including the dynamical phase and geometrical phase, for the mode-$\vec{k}$ A particle [mode-$\vec{k}$ B particle] and $\theta_{A0}(t)$ $[\theta_{B0}(t)]$ is the total phase for the corresponding vacuum. By means of the unitary operators in Eq. (3.10), the particular exact solution of the time-dependent Schrödinger equation (3.5) [associated with $\hat{H}(t)$] can be found to be [5,9]

$$|\psi_{N_AN_B}(t)\rangle_S$$
$$=\hat{Q}_A(t)\hat{Q}_B(t)|N_A,N_B;t\rangle_{S0}$$
$$=\exp[i\theta_A(t)]\exp[i\theta_B(t)]\hat{Q}_A(t)\hat{Q}_B(t)|N_A,N_B\rangle_{I_0}$$
$$=\exp[i\theta_A(t)]\exp[i\theta_B(t)]|N_A,N_B;t\rangle_I, \tag{3.19}$$

where $|N_A,N_B;t\rangle_I$ is the eigenstate of $\hat{I}(t)$ with particle number (N_A+N_B). It is worthwhile to point out that (i) the cyclic property of $|\psi_{N_A,N_B}(t)\rangle_S$ can be discussed as in [5,9] and (ii) the general exact solution of the time-dependent Schrödinger equation (3.5) is a superposition of the particular solutions in Eq. (3.19).

Using the exact solutions of the time-dependent Schrödinger equation (3.5) [associated with $\hat{H}(t)$], we can construct the coherent states

$$|\vec{k},\eta;t\rangle_A=\exp\left(-\frac{|\eta|^2}{2}\right)\sum_n\left(\frac{\eta^n}{\sqrt{n!}}\right)\exp[-in\theta_A(\vec{k},t)]n,t\rangle_{I_A}, \tag{3.20}$$

$$|\vec{k},\eta';t\rangle_B=\exp\left(-\frac{|\eta'|^2}{2}\right)\sum_n\left(\frac{\eta'^n}{\sqrt{n!}}\right)\times\exp[-in\theta_B(-\vec{k},t)]n,t\rangle_{I_B}, \tag{3.21}$$

where

$$|n,t\rangle_{1_A}\equiv|N_A=n,N_B=0;t\rangle_1$$
$$=\hat{Q}_A(t)[n!]^{-1/2}\left[\hat{A}_0^\dagger(\vec{k})\left(\frac{V}{(2\pi)^3}\right)^{-1/2}\right]^n|0\rangle$$
$$=[n!]^{-1/2}\left[\hat{A}^\dagger(\vec{k},t)\left(\frac{V}{(2\pi)^3}\right)^{-1/2}\right]^n|0,t\rangle_{I_A}, \tag{3.22a}$$

$$|n,t\rangle_{1_B}\equiv|N_A=0,N_B=n;t\rangle_1$$
$$=\hat{Q}_B(t)[n!]^{-1/2}\left[\hat{B}_0^\dagger(\vec{k})\left(\frac{V}{(2\pi)^3}\right)^{-1/2}\right]^n|0\rangle$$
$$=[n!]^{-1/2}\left[\hat{B}^\dagger(\vec{k},t)\left(\frac{V}{(2\pi)^3}\right)^{-1/2}\right]^n|0,t\rangle_{I_B}$$
$$(n=0,1,2...), \tag{3.22b}$$

and η,η' are complex constants. It is easy to show that $|\vec{k},\eta;t\rangle_A$ and $|\vec{k},\eta';t\rangle_B$ are the eigenstates of $\hat{A}(\vec{k},t)$ and $\hat{B}(\vec{k},t)$, respectively.

Now we can calculate the expectation values of the operator $\hat{F}(\vec{k})$ for the coherent states in Eqs. (3.20) and (3.21):

$$\langle\hat{F}(\vec{k})\rangle_A\equiv{}_A\langle\vec{k},\eta;t|\hat{F}(\vec{k})|\vec{k},\eta;t\rangle_A$$
$$=\frac{\sqrt{2}}{2}\rho(\vec{k},t)_A\langle\vec{k},\eta;t|[\hat{A}(\vec{k},t)+\hat{B}^\dagger(-\vec{k},t)]|\vec{k},\eta;t\rangle_A$$
$$=\frac{\sqrt{2}}{2}\eta\rho(\vec{k},t)\exp\left\{i\left[\int_0^t dt'\rho^{-2}(\vec{k},t')\right]\right\}, \tag{3.23}$$

$$\langle\hat{F}(\vec{k})\rangle_B\equiv{}_B\langle-k,\eta';t|\hat{F}(\vec{k})|-\vec{k},\eta';t\rangle_B=\frac{\sqrt{2}}{2}\rho(\vec{k},t)_B$$
$$\times\langle-\vec{k},\eta';t|[\hat{A}(\vec{k},t)+\hat{B}^\dagger(-\vec{k},t)]|-\vec{k},\eta';t\rangle_B$$
$$=\frac{\sqrt{2}}{2}\eta'\rho(\vec{k},t)\exp\left\{-i\left[\int_0^t dt'\rho^{-2}(\vec{k},t')\right]\right\}. \tag{3.24}$$

Both $\langle F(\vec{k})\rangle_A$ and $\langle F(\vec{k})\rangle_B$ can be easily shown to be the particular solutions of the classical KG equation (2.7) and are, of course, the same as the solutions $\rho(\vec{k},t)e^{i\theta(\vec{k},t)}$ and $\rho(\vec{k},t)e^{-i\theta(\vec{k},t)}$ obtained in Sec. II up to irrelevant constants. Thus Eqs. (3.23) and (3.24), the classical correspondences of the quantum motion, establish the connection between the classical and quantum exact solutions.

IV. DISCUSSION

In quantum field theory, there are so few systems of physical interest for which the functional Schrödinger equations can be solved exactly that perturbation methods should play an important part in the applications of the theory. We would like to point out that although the case considered in this paper is special, the exact solutions obtained are useful as a starting point for the time-dependent perturbation theory of the scalar field with the additional $(\phi\phi^*)^2$ term in the Hamiltonian in Eq. (2.2) by employing the method in [9].

In Sec. III we indicated two facts: (i) The single generalized particle numbers $\hat{N}_A(\vec{k},t),\hat{N}_B(\vec{k},t)$ are invariants or conserved and defined in terms of generalized creation and annihilation operators and (ii) the total phase $\theta_A(\vec{k},t)$ $[\theta_B(\vec{k},t)]$, including the dynamical phase and the geometrical phase, is for the corresponding single generalized particle. It is clearly seen that these two facts are closely related to each other and it would not be possible to define the total phase $\theta_A(\vec{k},t)$ $[\theta_B(\vec{k},t)]$ for a single generalized particle if the corresponding single generalized particle number were not conserved. Thus we can say that the necessary condition for the total phase $\theta_A(\vec{k},t)$ $[\theta_B(\vec{k},t)]$ and hence the corresponding geometric phase to be defined is that the single generalized particle number should be conserved, namely, the single generalized particle creation (annihilation) should be absent during the time evolution. This condition is essentially in agreement with that obtained by Anandan and Mazur in [10]. However, in the case discussed in this paper, the absence of the generalized particle creation (annihilation) is not equivalent to the absence of the external electric field or to the absence of the usual particle creation (annihilation). In this respect, we have generalized the results obtained in [10], though we have only investigated a special case of the Klein-Gordon field in this paper.

Phase formulation is suitable for the study of the field theory with time-dependent Hamiltonians [9]. It is interesting to use this formulation to investigate the time-dependent Dirac field. Work in this direction is under investigation.

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APPENDIX A: THE INVARIANT-RELATED UNITARY TRANSFORMATION METHOD

In this appendix we first briefly outline the invariant-related unitary transformation method and then use it to deal with the relativistic KG field considered in this paper. We consider a system whose Hamiltonian $\hat{H}(t)$ is time dependent. The invariant $\hat{I}(t)$ for the system satisfies

$$\partial\hat{I}(t)/\partial t - i[\hat{I}(t),\hat{H}(t)]=0. \quad (A1)$$

The eigenvalue equation of $\hat{I}(t)$ can be written as

$$\hat{I}(t)|\lambda_n,t\rangle=\lambda_n|\lambda_n,t\rangle,$$
$$\partial\lambda_n/\partial t=0 \quad (A2)$$

and the time-dependent Schrödinger equation for the system is

$$i\hbar\,\partial|\psi(t)\rangle_s/\partial t=\hat{H}(t)|\psi(t)\rangle_s. \quad (A3)$$

According to the Lewis-Riesenfeld quantum invariant theory, the particular solution of Eq. (A3) is different from the eigenfunction $|\lambda_n,t\rangle$ of the invariant $\hat{I}(t)$ only by a phase factor $\exp[i\gamma_n(t)]$. The general solution of the Schrödinger equation (A3) can be shown to be

$$|\psi(t)\rangle_s=\sum_n C_n \exp[i\gamma_n(t)]|\lambda_n,t\rangle,$$
$$\gamma_n(t)=\int_{t_0}^{t}\langle\lambda_n,t'|i\partial/\partial t'-\hat{H}(t')|\lambda_n,t'\rangle dt',$$
$$C_n=\langle\lambda_n,0|\psi(0)\rangle_s, \quad (A4)$$

On the basis of the Lewis-Riesenfeld quantum invariant theory, the invariant-related unitary transformation method is developed. In some cases of physical interest, it is possible to construct a time-dependent unitary transformation $\hat{Q}(t)$ for a chosen $\hat{I}(t)$ such that $\hat{I}_0=\hat{Q}^\dagger(t)\hat{I}(t)\hat{Q}(t)$ is a time-independent operator with

$$\hat{I}_0|\lambda_n\rangle=\lambda_n|\lambda_n\rangle,$$
$$|\lambda_n\rangle=\hat{Q}^{-1}|\lambda_n,t\rangle, \quad (A5)$$

and the eigenvalue λ_n the same as that in Eq. (A2). By making use of the unitary transformation, we obtain $\hat{H}_0(t)$ from $\hat{H}(t)$,

$$\hat{H}_0(t)=\hat{Q}^\dagger(t)\hat{H}(t)\hat{Q}(t)-i\hat{Q}^\dagger(t)\partial\hat{Q}(t)/\partial t. \quad (A6)$$

This unitary transformation is easily shown to guarantee that the particular solution $|\lambda_n,t\rangle_{s0}$ of the time-dependent Schrödinger equation, associated with $\hat{H}_0(t)$, is different from the eigenfunction $|\lambda_n\rangle$ of the invariant $\hat{I}_0$ only by the same phase factor $\exp[i\lambda_n(t)]$ in Eq. (A4), namely,

$$|\lambda_n,t\rangle_{s0}=\exp[i\gamma_n(t)]|\lambda_n\rangle. \quad (A7)$$

Substitution of $|\lambda_n,t\rangle_{s0}$ into the time-dependent Schrödinger equation [associated with $\hat{H}_0(t)$]

$$i\partial|\lambda_n,t\rangle_{s0}/\partial t=\hat{H}_0(t)|\lambda_n\rangle, \quad (A8)$$

yields

$$-\dot{\gamma}_n(t)|\lambda_n\rangle=\hat{H}_0(t)|\lambda_n\rangle, \quad (A9)$$

which means that $\hat{H}_0(t)$ differs from $\hat{I}_0$ by a c-number factor, depending only on the time t. Thus one is led to the conclusion that if the unitary transformation $\hat{Q}(t)$ is found, the problem of solving the complicated time-dependent Schrödinger equation (A3) reduces to that of solving the much simplified equation (A8). In terms of the solutions of Eq. (A8) and the unitary transformation $\hat{Q}(t)$, the general solution of the time-dependent Schrödinger equation (A3) for the system can be shown to be

$$|\psi(t)\rangle_s=\sum_n C_n \exp[i\gamma_n(t)]\hat{Q}(t)|\lambda_n\rangle,$$
$$\gamma_n(t)=-\int_{t_0}^{t}\langle\lambda_n|\hat{H}_0(t')|\lambda_n\rangle dt'$$
$$=\int_{t_0}^{t}\langle\lambda_n,t'|i\partial/\partial t'-\hat{H}(t')|\lambda_n,t'\rangle dt'. \quad (A10)$$

The statement outlined above is the basic content of the invariant-related unitary transformation method.

In what follows we indicate some steps in using this method to get the exact solution for the KG field. The main steps are the following. (i) Use the quasialgebra [11] associated with the Hamiltonian in Eq. (3.4) to find the unitary transformation in Eq. (3.10) [9]. (ii) Calculate $\hat{I}_0$ in Eq. (3.12). The correctness of the unitary transformation is verified if $\hat{I}_0$ is time independent. Actually, the unitary transformation is found from the calculation of $\hat{I}_0$. (iii) Calculate $H_0(t)=Q_A^\dagger Q_B^\dagger H(t)Q_AQ_B-iQ_A^\dagger Q_B^\dagger[\partial(Q_AQ_B)/\partial t]$. (iv) Find the eigenstates of $\hat{I}_0$. (v) Calculate the corresponding phase factor to obtain the solution of the Schrödinger equation (3.5).

The method for calculating $\hat{I}_0$ is the same as that for $\hat{H}_0$. Here we only present the calculation of $\hat{H}_0$ in some detail since it is more complicated. $\hat{H}_0$ is defined in Eq. (3.14) as

$$\hat{H}_0(t)=\hat{Q}_A^\dagger(t)\hat{Q}_B^\dagger(t)\hat{H}(t)\hat{Q}_A(t)\hat{Q}_B(t) -i\hat{Q}_A^\dagger(t)\hat{Q}_B^\dagger(t)\frac{\partial[\hat{Q}_A(t)\hat{Q}_B(t)]}{\partial t} =\int d^3\vec{k}\,\hat{H}_0(\vec{k},t)=\hat{H}_{A0}(t)+\hat{H}_{B0}(t), \quad \text{(A11)}$$

$$\hat{H}_{A0}(t)=\int d^3\vec{k}\,\hat{H}_{A0}(\vec{k},t) =\hat{Q}_A^\dagger(t)\hat{H}(t)\hat{Q}_A(t)-i\hat{Q}_A^\dagger(t)\frac{\partial\hat{Q}_A(t)}{\partial t},$$

$$\hat{H}_{B0}(t)=\int d^3\vec{k}\,\hat{H}_{B0}(\vec{k},t) =\hat{Q}_B^\dagger(t)\hat{H}(t)\hat{Q}_B(t)-i\hat{Q}_B^\dagger(t)\frac{\partial\hat{Q}_B(t)}{\partial t}. \quad \text{(A12)}$$

By means of the Baker-Campbell-Hausdorff formula, we calculate the first term in Eq. (A12),

$$\hat{Q}_A^\dagger\hat{H}(t)\hat{Q}_A=\frac{1}{2}\int d^3\vec{k}\Big\{\{\omega^2[\cosh(r/2)-\sinh(r/2)\cos\alpha]^2 +\sinh^2(r/2)\sin^2\alpha\}\hat{F}(k)\hat{F}^*(k) +\{[\cosh(r/2)+\sinh(r/2)\cos\alpha]^2 +\omega^2\sinh^2(r/2)\sin^2\alpha\}\frac{-i\delta}{\delta F(k)}\frac{-i\delta}{\delta F^*(k)} +\sinh(r/2)\sin\alpha\{[\cosh(r/2)+\sinh(r/2)\cos\alpha] +\omega^2[\cosh(r/2)-\sinh(r/2)\cos\alpha]\} \times\left[\hat{F}^*(k)\frac{-i\delta}{\delta F^*(k)}+\frac{-i\delta}{\delta F(k)}\hat{F}(k)\right]\Big\}, \quad \text{(A13)}$$

$$\hat{Q}_B^\dagger\hat{H}(t)\hat{Q}_B=\frac{1}{2}\int d^3\vec{k}\Big\{\{\omega^2[\cosh(r/2)-\sinh(r/2)\cos\alpha]^2 +\sinh^2(r/2)\sin^2\alpha\}\hat{F}(-k)\hat{F}^*(-k) +\{[\cosh(r/2)+\sinh(r/2)\cos\alpha]^2 +\omega^2\sinh^2(r/2)\sin^2\alpha\}\frac{-i\delta}{\delta F(-k)}\frac{-i\delta}{\delta F^*(-k)} +\sinh(r/2)\sin\alpha\{[\cosh(r/2)+\sinh(r/2)\cos\alpha] +\omega^2[\cosh(r/2)-\sinh(r/2)\cos\alpha]\} \times\left[\hat{F}^*(-k)\frac{-i\delta}{\delta F^*(-k)}+\frac{-i\delta}{\delta F(-k)}\hat{F}(-k)\right]\Big\}. \quad \text{(A14)}$$

By means of the same formula, with length calculations, we get the second term in Eq. (A12)

$$-i\hat{Q}_A^\dagger\frac{\partial\hat{Q}_A}{\partial t}=\frac{1}{4}\int d^3\vec{k}\Big\{[r\sin\alpha-\alpha(\cosh r-1)(\cos\alpha)\alpha] \times[\hat{F}(\vec{k})\hat{F}^*(\vec{k})][-r\sin\alpha-\alpha(\cosh r-1) -\sinh r(\cos\alpha)\alpha]\left[\frac{-i\delta}{\delta F(\vec{k})}\frac{-i\delta}{\delta F^*(\vec{k})}\right] \times[r\cos\alpha-(\sinh r)(\sin\alpha)\alpha] \times\left[\hat{F}^*(\vec{k})\frac{-i\delta}{\delta F^*(\vec{k})}+\frac{-i\delta}{\delta F(\vec{k})}\hat{F}(\vec{k})\right]\Big\}, \quad \text{(A15)}$$

$$-i\hat{Q}_B^\dagger\frac{\partial\hat{Q}_B}{\partial t}=\frac{1}{4}\int d^3\vec{k}\Big\{[r\sin\alpha-\alpha(\cosh r-1)(\cos\alpha)\alpha] \times[\hat{F}(-\vec{k})\hat{F}^*(-\vec{k})] \times[-r\sin\alpha-\alpha(\cosh r-1) -\sinh r(\cos\alpha)\alpha]\left[\frac{-i\delta}{\delta F(-\vec{k})}\frac{-i\delta}{\delta F^*(-\vec{k})}\right] \times[r\cos\alpha-(\sinh r)(\sin\alpha)\alpha] \times\left[\hat{F}^*(-\vec{k})\frac{-i\delta}{\delta F^*(-\vec{k})}+\frac{-i\delta}{\delta F(-\vec{k})}\hat{F}(-\vec{k})\right]\Big\}. \quad \text{(A16)}$$

Finally, using Eqs. (A11)–(A16) and the auxiliary equation (3.8), we obtain

$$\hat{H}_{A0}(\vec{k},t)=[\rho^{-2}(\vec{k},t)+\dot{\xi}(\vec{k},t)]\hat{I}_{A0}(\vec{k}),$$

$$\hat{H}_{B0}(\vec{k},t)=[\rho^{-2}(-\vec{k},t)+\dot{\xi}(-\vec{k},t)]\hat{I}_{B0}(\vec{k}), \quad \text{(A17)}$$

where $\xi(\vec{k},t)=-\tan^{-1}\{\rho(\vec{k},t)\dot{\rho}(\vec{k},t)\}/[1+\rho^2(\vec{k},t)]$ and $\rho(\vec{k},t)$ is the solution of the auxiliary equation (3.8).

APPENDIX B: THE INVARIANT METHOD FOR THE SYSTEM WITH A NON-HERMITIAN TIME-DEPENDENT HAMILTONIAN

In Ref. [7] the Lewis-Riesenfeld invariant theory (LRIT) for Hermitian Hamiltonians was generalized and used to treat the system with a non-Hermitian Hamiltonian in a finite-dimensional Hilbert space. For this system, we can define the non-Hermitian invariant with the equation

$$\frac{d\hat{I}}{dt}\equiv\frac{\partial\hat{I}}{\partial t}-i[\hat{I},\hat{H}]=0, \quad \text{(B1)}$$

where $\hat{I}(t)$ has complete biorthonormal set of the eigenstates $|\tilde{\psi}_\lambda(t)\rangle$ and $|\psi_\lambda(t)\rangle$ ($\lambda=1,2,\ldots,N$) satisfying

$$\hat{I}(t)|\psi_\lambda(t)\rangle = I_\lambda|\psi_\lambda(t)\rangle,$$

$$\hat{I}^\dagger(t)|\tilde{\psi}_\lambda(t)\rangle = I_\lambda^*|\tilde{\psi}_\lambda(t)\rangle,$$

$$\langle\tilde{\psi}_\mu(t)|\psi_v(t)\rangle = \delta_{\mu v},$$

$$\sum_\lambda |\tilde{\psi}_\lambda(t)\rangle\langle\psi_\lambda(t)| = 1. \quad \text{(B2)}$$

As in the LRIT, it can be shown that

$$|\Psi_\lambda(t)\rangle_S = \exp[i\alpha_\lambda(t)]|\psi_\lambda(t)\rangle,$$

$$|\tilde{\Psi}_\mu(t)\rangle_S = \exp[i\alpha^*{}_\mu(t)]|\tilde{\psi}_\mu(t)\rangle, \quad \text{(B3)}$$

with

$$\alpha_\lambda(t) = \int_0^t \left\langle \tilde{\psi}_\lambda(t') \middle| i\frac{\partial}{\partial t'} - \hat{H}(t') \middle| \tilde{\psi}_\lambda(t') \right\rangle dt', \quad \text{(B4)}$$

are the particular solutions, respectively, of the Schrödinger equations

$$i\partial_t|\Psi(t)\rangle_S = \hat{H}(t)|\Psi(t)\rangle_S,$$

$$i\partial_t|\tilde{\Psi}(t)\rangle_S = \hat{H}^\dagger(t)|\tilde{\Psi}(t)\rangle_S. \quad \text{(B5)}$$

Then the general solutions of the Schrödinger equations are

$$|\Phi(t)\rangle_S = \sum_\lambda C_\lambda \exp[i\alpha_\lambda(t)]|\psi_\lambda(t)\rangle,$$

$$_S\langle\Phi(t)| = \sum_\mu C_\mu \exp[-i\alpha_\mu(t)]\langle\psi_\mu(t)|, \quad \text{(B6)}$$

respectively.

The statement outlined above is the basic content of the generalization of the LRIT in [7]. In [7] a system was studied with the non-Hermitian Hamiltonian of the form [see Eq. (30) in Ref. [7]]

$$\hat{H}(t) = \begin{pmatrix} iy(t) & iz(t) \\ -ix(t) & -iy(t) \end{pmatrix}. \quad \text{(B7)}$$

The corresponding Schrödinger equations is

$$i\frac{\partial}{\partial t}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} iy(t) & iz(t) \\ -ix(t) & -iy(t) \end{pmatrix}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}, \quad \text{(B8)}$$

which is of the same form as Eqs. (2.8) and (2.9) in the present paper. Thus the solutions obtained in Ref. [7] can be used to get the solutions of Eqs. (2.8) and (2.9) in the present paper by noting that the solutions there in Ref. [7] are required to be real, while in the present paper there is no such requirement.

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