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State reconstruction for a collection of two-level systems

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A method for reconstructing the quantum state of a system of atoms or spins is proposed. The Q distribution is derived from the measured population in the ground state of a system that has interacted with an external field. Using multipole operators the Q function is inverted to derive the density matrix. [S1050-2947(97)07012-1]

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INTRODUCTION

There have been a number of proposals for the reconstruction of the density matrix for a single mode of the radiation field [1–5]. The Vogel-Risken [1] method as well as the displacement method [3,4] have been implemented and the reconstructions of several states of the radiation field have been reported [6–8]. Some of the above proposals have been generalized to two modes of the radiation field [9]. However, almost all the literature has been exclusively devoted to the reconstruction of the state of the harmonic oscillator systems, which include radiation field, the motional state of the trapped atom and ion [8], and the vibrational state of the molecular wave packet in harmonic [10(a)] and anharmonic potentials [10(b)]. Very little has been done on other systems. Here we report reconstruction of the state of a spin system with arbitrary spin value. This study is relevant to a number of systems in quantum optics and in other fields, for example, a collection of two-level atoms can be modeled [11] in terms of a spin system with spin value $N/2$. The polarization of light can also be described by spin, e.g., the well-known Stokes parameters [12] can be written in terms of the expectation values of spin operators. The fluctuations in spin operators will lead to fluctuations in Stokes parameters. Such fluctuations have in fact been used to introduce the concept of polarization squeezing [13]. One thus needs to have a reconstruction procedure for a spin system with spin value S [14]. In what follows we present a method for the reconstruction of the spin density matrix. The present method in combination with the known methods for the radiation field will enable us to reconstruct the complete density matrix of the interacting atom-field system.

RECONSTRUCTION METHOD

Let ρ be the density matrix for a spin system with spin value S that we need to reconstruct. Let $|\theta,\varphi\rangle$ be the atomic coherent state [15] defined by

$$|\theta,\varphi\rangle = D(\theta,\varphi)|S,-S\rangle \equiv \sum_{m=-S}^{S} \binom{2S}{S+m}^{1/2} \left(\cos\frac{\theta}{2}\right)^{S-m} \left(\sin\frac{\theta}{2}\right)^{S+m} \times e^{i(S+m)\varphi}|S,m\rangle, \tag{1}$$

$$D(\theta,\varphi) \equiv \exp\left\{\frac{\theta}{2}(S^{+}e^{i\varphi} - S^{-}e^{-i\varphi})\right\}. \tag{2}$$

A useful quasidistribution for spin system is [15–17]

$$Q(\theta,\varphi) = \frac{2S+1}{4\pi}\langle\theta,\varphi|\rho|\theta,\varphi\rangle \tag{3}$$

such that

$$\int Q(\theta,\varphi)\sin\theta \, d\theta \, d\varphi = 1. \tag{4}$$

On using Eqs. (1) and (3) we can write Q as

$$Q(\theta,\varphi) = \frac{2S+1}{4\pi}\langle S,-S|\tilde{\rho}|S,-S\rangle, \tag{5}$$

where

$$\tilde{\rho} = D^{+}(\theta,\varphi)\rho D(\theta,\varphi). \tag{6}$$

Note that $\tilde{\rho}$ is obtained from ρ if the system is allowed to interact with an external field Hamiltonian proportional to $i(S^{+}e^{i\varphi} - S^{-}e^{-i\varphi})$. Then $Q(\theta,\varphi)$ is the probability of detecting the system in the ground state $|S,-S\rangle$ after it has interacted with external field. This then leads to the determination of the quasidistribution $Q(\theta,\varphi)$. We next need to have an inversion formula for ρ in terms of $Q(\theta,\varphi)$.

Note that from the definition (3) and (1) we have

$$Q(\theta,\varphi) \equiv \frac{2S+1}{4\pi}\sum\sum \rho_{mm'}\left(\cos\frac{\theta}{2}\right)^{2S-m-m'}\left(\sin\frac{\theta}{2}\right)^{2S+m+m'} e^{-i(m-m')\varphi}\binom{2S}{S+m}^{1/2}\binom{2S}{S+m'}^{1/2}. \tag{7}$$

This is an expansion in terms of the powers of $\cos\theta/2$ and $\sin\theta/2$, etc. However, for an inversion we need to expand $Q(\theta,\varphi)$ in terms of orthogonal polynomials. Then the inversion is easily done. For the present problem the appropriate set of polynomials are the associated Legendre polynomials (in reality the spherical harmonics).

This last objective can be achieved by using multipole operators T_{kq} defined by [18]

$$T_{kq}=\sum_{mm'}(-1)^{S-m}(2k+1)^{1/2}\times\begin{pmatrix} S & k & -S \\ -m & q & m' \end{pmatrix}|S,m\rangle\langle S,m'|. \tag{8}$$

These operators form a complete set and have the properties

$$\mathrm{Tr}(T^{+}_{k_1q_1}T_{k_2q_2})=\delta_{k_1k_2}\delta_{q_1q_2},\quad T^{+}_{kq}=(-1)^{q}T_{k,-q}. \tag{9}$$

Thus we have the important relation [16]

$$\rho=\sum_{kq}\rho_{kq}T_{kq},\qquad \rho_{kq}=Tr(T^{+}_{kq}\rho). \tag{10}$$

We have previously shown that [16]

$$Q(\theta,\varphi)=\frac{2S+1}{4\pi}\sum_{kq}\rho_{kq}Y_{kq}(\theta,\varphi)\frac{(-1)^{k-q}\sqrt{4\pi}\;2S!}{\sqrt{(2S-k)!(2S+k+1)!}} \tag{11}$$

and hence

$$\rho_{kq}\equiv\frac{\sqrt{4\pi}}{(2S+1)!}(-1)^{k-q}\sqrt{(2S-k)!(2S+k+1)!}\int Q(\theta,\varphi)\times Y_{kq}(\theta,\varphi)\sin\theta\,d\theta\,d\varphi. \tag{12}$$

On substituting Eq. (12) in Eq. (10) we get the full density matrix. We therefore have the following reconstruction algorithm—determine the probability of finding the system characterized by $\tilde{\rho}$ in the ground state; this determines the Q- function which should be inverted to find ρ_{kq} [Eq. (12)]; these ρ_{kq}'s yield full density matrix via Eq. (10). We also note that for both spin systems and two-level systems $\tilde{\rho}$ is obtained from ρ by the application of external field. For states of light polarization $\tilde{\rho}$ is obtained from ρ by rotations on Poincàre sphere [19]—such rotations are easily performed by various optical devices.

We finally also note that the Wigner function $W(\theta,\varphi)$ defined by [16,20]

$$W(\theta,\varphi)=\sqrt{\frac{2S+1}{4\pi}}\sum_{kq}\rho_{kq}Y_{kq}(\theta,\varphi) \tag{13}$$

can obviously be constructed by using Eq. (12). Other quasidistributions for spin systems can be similarly constructed.

It is clear that the method of zero counts in the system characterized by the transformed density matrix $\tilde{\rho}$ is applicable rather universally for now we understand it in the context of radiation field and two-level systems. Clearly it could be developed [21,22] along similar lines, for example in the context of a system of N three-level and multilevel atoms [21] [SU(3) group]. Finally we note that with the current reconstruction procedure along with the methods of Refs. [1–5], we can reconstruct the density matrix of a correlated radiation-matter system.

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