Superposition-preserving photon-number amplifier

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We analyze a pure-state photon-number amplifier (PNA), i.e., a device that performs the action $|\psi_{\text{in}}\rangle = \Sigma_l c_l |l\rangle \rightarrow |\psi_{\text{out}}\rangle = \Sigma_l c_l |Gl\rangle$, where G is a positive integer. We derive expressions for the fidelity of a finite internal state PNA and demonstrate that the amplifier fidelity, i.e., the similarity between the obtained and the desired output state can be made to approach unity. Finally we outline a systematic procedure to find the Hamiltonian for any finite Hilbert-space PNA and compute the Hamiltonian for a simple but nontrivial PNA. [S1050-2947(98)07301-6]

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I. INTRODUCTION

In conjunction with discussions about quantum information processing [1–3], interferometry, and Schrödinger cat-state generation [4], some attention has been devoted to the photon-number amplifier, in the following referred to as a PNA. The concept of the PNA was first brought up by Yuen [5]. However, it should be pointed out that the term has been used to denote two different devices, carrying out somewhat similar, but in this context distinctly different functions. In both classes of devices an input number state $|\psi_{\text{in}}\rangle = |n\rangle$ is transformed to an output state $|\psi_{\text{out}}\rangle = |Gn\rangle$, where G is a positive integer $\geqslant 2$. However, in first class of PNA's, which have been experimentally demonstrated [6], a pure superposition state at the input $|\psi_{\text{in}}\rangle = \Sigma_l c_l |l\rangle$ is transformed to a mixed state at the output with the density matrix $\hat{\rho}_{\text{out}} = \Sigma_l |c_l|^2 |Gl\rangle\langle Gl|$. Hence any phase information in the state is lost rendering the amplifier useless in any scheme where quantum superpositions play a role. In contrast, Yuen [5], and later D'Ariano [7] showed that quantum mechanics allows for a device that leaves the output state in a pure ''transformed'' superposition state, the transformation of such a device being

$$|\psi_{\text{in}}\rangle = \sum_l c_l |l\rangle \rightarrow |\psi_{\text{out}}\rangle = \sum_l c_l |Gl\rangle. \tag{1}$$

This is the defining equation for a superposition-preserving photon-number amplifier, and in the following we will reserve the acronym PNA to describe a device of this type.

While Yuen showed the feasibility for such a PNA, only limited work on specific realizations of PNA's has hitherto been done. D'Ariano derived a unitary transformation and was the first to derive a corresponding Hamiltonian [3,7]. D'Ariano also pointed out that a device with the same general transformation as the PNA, but with G replaced by the inverse of a positive integer, will work as an ideal phase amplifier [3]. In this work we extend the analysis of Yuen and D'Ariano and set out to derive a Hamiltonian (with a quite straightforward physical interpretation) for a PNA within the rotating-wave approximation and with no ''hidden'' classical fields (see below). We assume that the amplifier input mode remains invariant under the amplification process; specifically we assume that the input mode oscillation frequency does not change. In order for the PNA to be representable by a unitary transformation it is necessary to assume that the PNA has an auxiliary internal mode. We shall assume that the frequency of the internal mode coincides with that frequency of the input mode. A consequence of the rotating-wave assumption is that the state amplification will lead to entanglement between the PNA internal mode and the amplified mode. After dissipation of the PNA internal mode energy, the output state will not be pure, as desired, but mixed. Hence the amplified state will only be close to the desired state, but never exactly equal to it.

To justify the rotating-wave approximation we note that in a typical experiment where a PNA might be used, the input mode is narrow band in frequency. If temporal wave-packet modes are used, the pulse length is typically longer than a nanosecond. Hence, the interaction time of the PNA must be of this duration or longer. Assuming that both the input and the amplifier internal mode have frequencies somewhere in, or near, the visible spectrum, every term in a Hamiltonian that creates or annihilates even a single energy quantum too much or too little to conserve energy will have a transition matrix element on the order of 10^{-6} smaller than terms that conserve energy exactly. Since typical single-photon resolution detectors have quantum efficiencies $<90\%$, the energy nonconserving terms can safely be neglected in the description in comparison to typical measurement errors that are $>10^{-1}$.

Conversely, any term necessary for the proper state amplification that does not conserve energy in a PNA Hamiltonian needs to have a nonlinear coupling strength of the order of 10^6 larger than the energy-conserving terms. Since a long-standing problem in the experimental demonstration of even classical nonlinear optical devices is the small nonlinearity of available nonlinear materials, we believe that any proposal of a PNA that relies on energy non-conserving processes will be impossible to implement experimentally.

D'Ariano's Hamiltonian is only within the rotating-wave approximation if either of two conditions are fulfilled. Either the input mode must simultaneously be frequency converted and photon number amplified, or a third classical (and hence undepleted) mode must be invoked. The action of unitary transformation D'Ariano derived, on the state $|n,m\rangle$, is

$$\hat{U}_G |n,m\rangle = |Gn + G\langle m/G\rangle, [m/G] + n\rangle, \tag{2}$$

where the amplification gain G has been assumed to be a positive integer, $[x]$ denotes the integer part of x, and $\langle x\rangle = x-[x]$ is the fractional part of x. If we look at the mapping we see that for most n's and m's energy is not conserved if the frequency of both modes remain invariant throughout the amplification process. If one restricts oneself to the case where $m=0$, then the total energy of the conversion process can still be preserved provided that the amplified mode is frequency converted from ω to $\omega/(G+1)$ by the amplifier. However, in most experiments, and in systems experiments in particular, it is highly desirable that the amplifier input and output mode are equivalent (e.g., same frequency, polarization, and transverse mode). Therefore we restrict our interest to such processes.

A different way of making the mapping (2) energy conserving is to invoke one or more classical nondepleted field(s), which may provide the energy needed to bring the unitary transformation of D'Ariano within the rotating-wave approximation. In the Hamiltonian these fields would be manifested by rapidly oscillating coupling coefficients. This was suggested as a physical realization of a PNA by Yuen in [2]. However, to realize Eq. (2) above even on a rather restricted two-mode Hilbert space would require a large number of classical fields with different frequencies and fixed relative phases (since the input mode superposition should be conserved). This would pose significant difficulties for the experimentalist. In addition, the problem with entanglement between the amplified mode and the mode providing the energy needed for the amplification is ignored in such a treatment. However, our analysis suggests that, practical considerations aside, it is in fact permissible to neglect the entanglement irrespective of the exact state of the mode providing the energy if this state is in a sufficiently large superposition of energy eigenstates.

We note that work related to ours has been performed on quantum state copying [8,9]. These papers delineate how well states can be copied (in general, states cannot be copied perfectly [10]) and in that respect this work is similar to ours. We will restrict our attention to devices where the amplifier input state Hilbert space is limited, and derive the explicit Hamiltonian for one simple, but nontrivial, such device. However, our algorithm for finding the Hamiltonian can be extended to PNA's operating on arbitrary large Hilbert spaces. It may seem to be a limitation to consider PNA's accepting only a limited set of input states. However, if we require that the amplifier internal mode energy, from which the amplifier draws the necessary energy needed to amplify the state, is bounded, it follows that the amplifier cannot properly amplify a highly excited input state. Hence no energy-conserving PNA exist that can operate on an infinite number-state Hilbert space. Additional justification is provided by more mundane considerations. Although we can easily derive the quantum mechanical equations of motion of, e.g., a phase-insensitive linear amplifier and hence exactly obtain the output state for any input state, the ensuing equations of motion are still only valid in a finite Hilbert-space since gain saturation is an inherent limitation of any amplifier for the same reason it is for the PNA.

II. STATE TRANSFORMATIONS AND FIDELITY

We shall consider a PNA with gain G operating on the input Hilbert space spanned by the state vectors (in a number basis) $\{|0\rangle,|1\rangle\cdots|n-1\rangle\}$. To be able to amplify all input states by G, the amplifier internal state must be in a state

$$|\psi_a\rangle = \sum_{k=(n-1)(G-1)}^{(n-1)(G-1)+m-1} d_k|k\rangle, \tag{3}$$

where the upper summation limit is chosen so that the amplifier initial state is in a m-state superposition if all d_k coefficients included in the summation are nonzero. It is quite obvious that the state should be chosen to be pure since the postinteraction input state we are aiming at is pure. The PNA input state is written as

$$|\psi_{\rm in}\rangle = \sum_{l=0}^{n-1} c_l|l\rangle, \tag{4}$$

and according to our desire to make an input-state-independent PNA, the joint state of the input mode and the preinteraction PNA internal mode must be a product state. The output state, in the rotating-wave approximation, will be

$$|\psi_{\rm out}\rangle = \sum_{l=0}^{n-1}\sum_{k=(n-1)(G-1)}^{(n-1)(G-1)+m-1} c_l d_k \exp(-i\theta_{l,k})|Gl,k-l(G-1)\rangle, \tag{5}$$

where we have used the notation $|l\rangle\otimes|k\rangle\equiv|l,k\rangle$, and where $\exp(-i\theta_{l,k})$ is the PNA unitary evolution phase factor. The fact that we have mapped every input state to one, and only one, output state comes from the defining action of the PNA. One immediately sees that the output state becomes entangled with the amplifier internal state due to energy conservation. In addition, it becomes clear that we can assume, without loss of generality, that all the unitary evolution phase factors are zero, and hence all the (complex) d_k coefficients should have the same phase in a polar decomposition. We will take this phase to be zero too, making all d_k's real and positive. In fact, our ''without loss of generality'' statement in the preceding sentence is only true in the operational sense. As we shall see below there exist many unitary transformations that perform the optimal state mapping, and even after a one-to-one state mapping is defined, one can still find infinitely many optimal unitary transformations (with different $\theta_{l,k}$'s), each corresponding to a specific choice of d_k coefficient phases. Nonetheless, all these unitary transformations are of the same ''difficulty'' or ''complexity,'' where we loosely define ''difficulty'' in terms of the number of terms, and the highest power of the creation and annihilation operators in these terms, needed to synthesize the Hamiltonian corresponding to the unitary evolution.

In his original paper Yuen very briefly discussed the necessity and the role of the PNA internal mode, but he made no attempt to include it in the analysis [5]. D'Ariano takes the analysis a step further and includes the PNA internal state in the analysis [7]. However, as pointed out above, D'Ariano's unitary transformation is not per se energy conserving. Energy conservation will entangle the amplified mode with the PNA internal mode, and as one can guess, getting rid of the unwanted entanglement between the modes is the major obstacle in realizing a PNA, both on paper and

TABLE I. A table of the PNA postinteraction state coefficients of a $n=3$, $m=4$, $G=3$ PNA. The preinteraction internal state coefficients are chosen according to Eq. (11). Tracing over the PNA internal state will leave the output state density operator terms pertaining to the internal states $|4\rangle$ and $|6\rangle$ in a (desired) 3-state superposition.

Output	Postinteraction PNA internal state										
state	$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$	$\|4\rangle$	$\|5\rangle$	$\|6\rangle$	$\|7\rangle$	$\|8\rangle$	$\|9\rangle$	$\|10\rangle$
$\|0\rangle$	0	0	0	0	c_0d_4	0	c_0d_6	0	c_0d_8	0	c_0d_{10}
$\|3\rangle$	0	0	c_1d_4	0	c_1d_6	0	c_1d_8	0	c_1d_{10}	0	0
$\|6\rangle$	c_2d_4	0	c_2d_6	0	c_2d_8	0	c_2d_{10}	0	0	0	0

in reality. If the PNA internal state simply is dissipated after the interaction the output state density operator becomes

$$\hat{\rho}_{\text{out}}=\text{Tr}_a\{|\psi_{\text{out}}\rangle\langle\psi_{\text{out}}|\}, \tag{6}$$

where $\text{Tr}_a\{\}$ denotes the partial trace over the amplifier state Hilbert space. It is easy to demonstrate that nothing is gained by letting the postinteraction amplifier mode undergo further interaction. To show this, suppose we prepare some third mode (not necessarily bosonic) in some pure state $|\psi_{\text{aux}}\rangle=\Sigma e_j|\xi_j\rangle$, where $\{|\xi_j\rangle\}$ constitutes a complete set of orthonormal basis states. If we define some arbitrary unitary evolution $\hat{U}_{\text{arb}}$ of the joint state $|\psi_{\text{aux}}\rangle\otimes|\psi_{\text{out}}\rangle$, then we find that

$$\begin{aligned}&\text{Tr}_{a,\text{aux}}\{\hat{U}_{\text{arb}}|\psi_{\text{aux}}\rangle\otimes|\psi_{\text{out}}\rangle\langle\psi_{\text{out}}|\otimes\langle\psi_{\text{aux}}|\hat{U}_{\text{arb}}^{\dagger}\}\\&=\sum\sum\langle n|\otimes\langle\xi|\hat{U}_{\text{arb}}^{\dagger}\hat{U}_{\text{arb}}|\psi_{\text{aux}}\rangle\otimes|\psi_{\text{out}}\rangle\langle\psi_{\text{out}}|\\&\quad\otimes\langle\psi_{\text{aux}}|\hat{U}_{\text{arb}}^{\dagger}\hat{U}_{\text{arb}}|\xi\rangle\otimes|n\rangle=\text{Tr}_a\{|\psi_{\text{out}}\rangle\langle\psi_{\text{out}}|\},\end{aligned} \tag{7}$$

where we have used the fact that since both $|n\rangle$ and $|\xi\rangle$ are complete orthonormal bases, $\hat{U}_{\text{arb}}|\xi\rangle\otimes|n\rangle$ is also a complete orthonormal base on the joint Hilbert space. This proof can be trivially extended to mixed auxiliary states.

After carrying out the trace in Eq. (6) the PNA output state is no longer pure, regardless of the choice of d_k coefficients. One sees that it is impossible to make an ideal PNA if the amplifier internal state is in a finite state superposition. Therefore it is fruitful to define a fidelity measure that gives an indication of how close to the target output state defined by Eq. (1) one can get. Fidelity measures have been discussed in [11,12], and it appears that a suitable fidelity function should be defined as

$$\begin{aligned}f&=\text{Tr}_a\{\sqrt{\hat{\rho}_{\text{target}}}\hat{\rho}_{\text{out}}\sqrt{\hat{\rho}_{\text{target}}}\}=\text{Tr}_a\{\hat{\rho}_{\text{out}}\hat{\rho}_{\text{target}}\}\\&=\langle\hat{\psi}_{\text{target}}|\text{Tr}_a\{\hat{\rho}_{\text{out}}\}|\hat{\psi}_{\text{target}}\rangle,\end{aligned} \tag{8}$$

where the second and third equalities are based on the fact that our target density operator $\hat{\rho}_{\text{target}}\equiv\Sigma_{l=0}^{n-1}\Sigma_{l'=0}^{n-1}c_lc_{l'}^{*}|Gl\rangle\langle Gl'|$ describes a pure state. The fidelity function expresses the quantum-mechanical similarity between two states. In our specific case, where the target state is pure, the fidelity is the diagonal output state density matrix element corresponding to the target state, if a basis is used in which the target state is one of the basis state vectors. Hence the fidelity expresses the probability to find the PNA output state in the target state. The target function f defined above has the properties that $0\leqslant f\leqslant 1$, $f=1$ iff $\hat{\rho}_{\text{out}}\equiv\hat{\rho}_{\text{target}}$ and f is invariant under the permutation between the target and the output mode. Even with this definition it is not possible to get a numerical value of f for given amplifier state coefficients d_k since f is a function of the coefficients c_l of the input state. Since we are interested in a general device that should work for any choice of the coefficients, it is reasonable to average f over all possible input states, i.e. over the $2n$th-dimensional unit radius hypersphere surface. When calculating f for some input state, one finds that one gets terms with two distinct combinations of c_l coefficients, terms with the factor $|c_l|^4$ and terms with the factor $|c_l|^2|c_{l'}|^2$, where $l\neq l'$. After a bit of algebra one finds that [13]

$$\overline{(|c_l|^4)}=\frac{3}{n(n+2)} \tag{9}$$

and

$$\overline{(|c_l|^2|c_{l'}|^2)}=\frac{1}{n(n+2)}, \tag{10}$$

where the overbar signifies the average over all allowed input states (i.e., all possible choices of c_l coefficients). Using this result in the calculation of f one finds that to optimize f for a given amplifier mode m-state superposition, only the d_k coefficients that satisfy $k=k_{\min}+q(G-1)$, where q is a positive integer, $k_{\min}\geqslant(n-1)(G-1)$, and the q's form an unbroken integer series, should be chosen nonzero. This can be seen from Table I. With the choice above all input states can be mapped on the correct output states (i.e., all terms including the state coefficient c_1 will multiply states belonging to the output state energy manifold $1G=3$ in Table I). In addition one sees that one maximizes the number of proper superpositions of the output state, which are entangled with a particular PNA internal state manifold. In the table both internal state manifolds 4 and 6 are connected to superpositions of the correct form, i.e., containing states belonging to all three output state manifolds. As $m\gg n$ only the extreme PNA internal state energy manifolds will not be connected to output state superpositions of the proper form. If these are neglected, leading to a finite but small fidelity penalty, one sees that one can obtain the wanted output state by choosing all d_k coefficients equal.

The minimum energy amplifier state fulfilling the conditions outlined above is the state

$$|\psi_a\rangle=\sum_{k=0}^{m-1} d_{(n-1+k)(G-1)}|(n+k)(G-1)\rangle. \qquad (11)$$

However, all amplifier states with coefficients chosen according to the criteria above will have the same form of the fidelity function f regardless of the gain G and $k_{\min}$.

Here it may be beneficial to pause for a moment and reexamine the arguments leading us to choose the amplifier superposition state in a particular ''optimal'' way. The assumption has been to construct a PNA that is general, i.e., it should accept all possible input states belonging to the proper input state Hilbert space. Since some states are more difficult to photon number amplify than others (the vacuum state, for instance, is trivial to ''amplify'' since it should simply be mapped onto itself) we have assumed that the word ''optimal'' is used in the sense that averaged over all allowed input states, the amplifier is the optimal device under the constraint that the amplifier internal state may not include more than m states in the number-state basis. To quantify this statement, we introduced the fidelity function f, and then it is quite straightforward to derive the optimum values of the coefficients.

One may argue that instead of constraining the number of orthogonal states in the amplifier mode superposition, one should try to minimize the state energy. However, the experimental challenge to prepare the PNA initial state superposition correctly will be much harder than the challenge to supply sufficient energy to create the state. Therefore it seems very reasonable to try to minimize the size of the amplifier state superposition rather than to try to minimize the amplifier state energy.

III. A SIMPLE PNA

To demonstrate the general characteristics of PNA's it is instructive to look at some simple but nontrivial cases. We shall start by looking at the case where $n=3$, $G=2$, and $m=3$. In this case the desired input state transformation is

$$|\psi_{\text{in}}\rangle=c_0|0\rangle+c_1|1\rangle+c_2|2\rangle\rightarrow c_0|0\rangle+c_1|2\rangle+c_2|4\rangle. \qquad (12)$$

If the amplifier initial internal state is chosen according to the criteria (11) above, i.e., $|\psi_a\rangle=d_2|2\rangle+d_3|3\rangle+d_4|4\rangle$, then the fidelity of the PNA is

$$\begin{aligned} f&=\frac{3(|d_2|^2+|d_3|^2+|d_4|^2)}{5} \\ &\quad+\frac{4\text{Re}\{d_2d_3^*\}+4\text{Re}\{d_3d_4^*\}+2\text{Re}\{d_2d_4^*\}}{15} \\ &=\frac{3}{5}+\frac{4\text{Re}\{d_2d_3^*\}+4\text{Re}\{d_3d_4^*\}+2\text{Re}\{d_2d_4^*\}}{15}, \end{aligned} \qquad (13)$$

where $\text{Re}\{x\}$ signifies the real part of x. The expression is invariant with respect to the permutation $d_2\rightleftharpoons d_4$ as expected. Furthermore it is seen that to maximize the expression all the coefficients should have the same phase angle (in a polar coordinate representation), and as stated above, we can, without loss of generality, choose the coefficients real and positive. When we maximize this expression over the d coefficients we find that the maximum average fidelity of such a PNA is $f=(19+33^{1/2})/30\approx 0.82$ for the choice $d_2=d_4=(1+33^{-1/2})^{1/2}/2\approx 0.54$, $d_3=(1/2-33^{-1/2}/2)^{1/2}\approx 0.64$. This should be compared to the fidelity between the untransformed input state and the target state, $f=4/15\approx 0.27$ and the fidelity of a PNA with the preinteraction internal mode in a (sufficiently excited) single number state (rendering the output state density operator $|c_0|^2|0\rangle\langle 0|+|c_1|^2|2\rangle\langle 2|+|c_2|^2|4\rangle\langle 4|$), $f=3/5=0.6$.

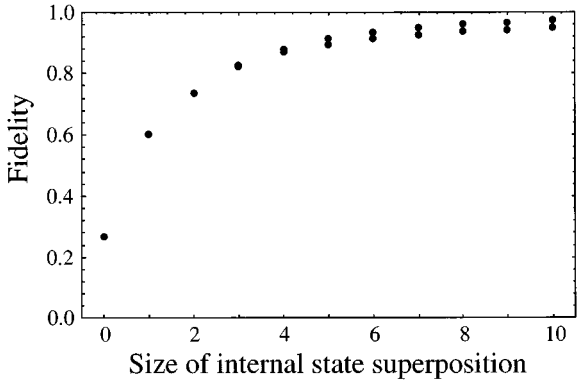

FIG. 1. The average fidelity of a PNA amplifying a 3-state number-state superposition by some integer G, as a function of the number of states in the initial amplifier superposition state. The upper dots are the optimal fidelity, the lower dots are the result if the weights of the amplifier state d_k are all chosen equal. Note that the figure is independent of the choice of amplifier gain G.

In general, for PNA's involving larger quantum superpositions, e.g., larger n's and/or m's, the exact optimum choice of coefficients cannot be solved analytically, but has to be derived by numerical methods. However, we can derive an approximate result for any choice of n, m, and G by noting that if $m\gg n$, the optimum d_k coefficients can be roughly equal. This can be seen directly from Table I. If all $d_k=1/\sqrt{m}$, the PNA state fidelity becomes

$$f=\begin{cases} 1-\dfrac{n^2-1}{3(n+2)m} & \text{for } m\geqslant n \\ \dfrac{3nm+6n+1-m^2}{3n(n+2)} & \text{for } 1\leqslant m<n. \end{cases} \qquad (14)$$

In Fig. 1 we have plotted the fidelity of a PNA accepting 3-state superpositions ($n=3$) as a function of the number of states in amplifier initial state superposition m. Both the exact, numerically derived result (upper dots) and the approximate result (lower dots) are plotted. The point for $m=0$ represents the fidelity between the input state and the target state. As we can see the fidelity approaches unity asymptotically, but the convergence is slow, it goes only as $\propto 1/m$. In Fig. 2 we have plotted the approximate result for a 2-state, 3-state, 4-state, and 5-state PNA as a function of the number of number states in the amplifier initial state superposition. Not surprisingly, the qualitative behavior of all the PNA's are similar. Finally, in Fig. 3, we have plotted the fidelity for an optimized 1, 3, and 10 amplifier internal state superposition PNA accepting a 3-state input superposition. The fidelity is independent of the relative phases between the c_0, c_1, and

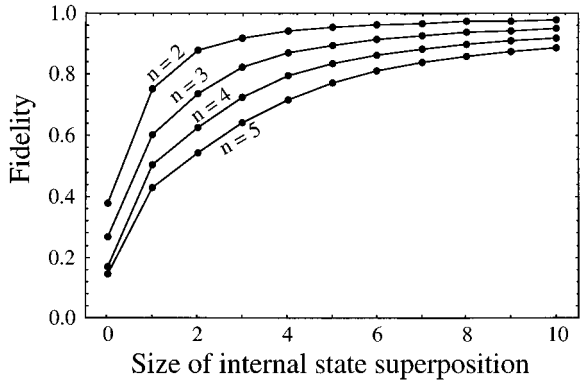

FIG. 2. The average fidelity of PNA's amplifying 2- to 5-state number-state superpositions (top to bottom). The curves were drawn using the approximate formula (13). Optimization will yield slightly higher fidelities.

c_2 coefficients. Therefore we have used $|c_0|$ and $|c_1|$ as our free coordinates and plotted the fidelity along the third axis. We see that as m becomes larger, the figure approaches the unit fidelity plane. One can see that if $m>1$, the input state with lowest fidelity is the state $[\exp(i\theta_0)|0\rangle +\exp(i\theta_2)|2\rangle]/\sqrt{2}$, where θ_0 and θ_2 are arbitrary. This is not surprising since these are the states that require the least and most energy, respectively, in the amplification process. Therefore they leave the most distinctly different ''fingerprints'' on the amplifier internal state and hence are the two states that are the easiest to distinguish using the information left in the postinteraction amplifier state.

IV. PNA HAMILTONIANS

In this section we shall outline a procedure to obtain the Hamiltonian for any specific PNA. We start by noticing that within the rotating-wave approximation energy is preserved; therefore, it is not sufficient that the state transformation is unitary—it must be unitary in each energy manifold. This means that different energy manifolds do not couple and consequently one can solve the manifolds one by one. In any particular energy manifold N with the corresponding energy $\hbar\omega N$ and involving two bosonic modes, there are $N+1$ basis states. In our case it is convenient to take the set $\{|0,N\rangle,|1,N-1\rangle, \ldots ,|N,0\rangle\}$ as the basis states. One then writes down the desired unitary matrix U_0 in this basis. The matrix has the dimension $(N+1)\times(N+1)$. As mentioned above the unitary matrix is not unique. Our choice of the amplifier internal mode coefficients as real and positive constrains us to choosing all the matrix coefficient *responsible for a desired state transformation* to have the same phase. We shall take this phase as zero, which leaves all these coefficients to be unity. The rest of the coefficients can be chosen arbitrarily as long as the matrix is unitary. We seek the Hermitian matrix H_{N0}, which leads to the desired unitary matrix, i.e.,

$$U_{N0}=\exp(-iH_{N0}). \tag{15}$$

To find H_{N0} one finds the eigenbasis set $\{|\phi_1\rangle,|\phi_2\rangle, \ldots ,|\phi_{N+1}\rangle\}$ that diagonalizes U_0, and the corresponding eigenvalues $\{\lambda_1,\lambda_2,\ldots,\lambda_{N+1}\}$. The eigenbasis set can be expressed in the originally chosen basis set by multiplication by the matrix E. In the new basis set, since the corresponding unitary transformation is diagonal, the appropriate generating Hermitian matrix (corresponding to the Hamiltonian) can be diagonal with coefficients h_{eii} such that $\exp(-ih_{eii})=\lambda_i$. Again no unique solution of H_e exists since all the diagonal coefficients can be chosen modulo 2π. We can subsequently transform the obtained Hermitian matrix to the originally chosen base $H_{N0}=(E^{-1})^* H_e (E^{-1})^T$, where $*$ denotes the complex conjugate, and T denotes transpose. In the following we shall refer to this Hermitian matrix as the target Hermitian matrix. We note that a $(N+1)\times(N+1)$ Hermitian matrix can have $(N+1)(N+2)/2$ independent complex coefficients.

What we are really seeking is the Hamiltonian operator whose corresponding matrix in the original basis set is H_{N0}. For this end we construct the normally ordered Hermitian operators

$$(\hat{a}^\dagger)^i(\hat{b}^\dagger)^{N-i}\hat{a}^j\hat{b}^{N-j}+\text{H.c.}, \tag{16}$$

where the annihilation operator $\hat{a}$ operates on the input-output state mode, the corresponding operator $\hat{b}$ operates on the amplifier internal mode, H.c. denotes Hermitian conjugation and $0\leqslant i,\ j\leqslant N$. We note that every such operator is zero when it operates on a state in an energy manifold $<N$. We also note that there are $(N+1)(N+2)/2$ different such operators. The Hermitian matrix in the original state basis has only two nonzero coefficients, namely $h_{i+1,j+1}=h_{j+1,i+1}$. Both of these are real and positive. In addition one can construct another set of Hermitian operators

$$i[(\hat{a}^\dagger)^i(\hat{b}^\dagger)^{N-i}\hat{a}^j\hat{b}^{N-j}-\text{H.c.}]. \tag{17}$$

These operators too are zero when they operate on a state in an energy manifold $<N$. There are $(N+1)(N+2)/2$ different such operators and again the Hermitian matrix in the original state basis has only two nonzero coefficients, namely, $h_{i+1,j+1}=h^*_{j+1,i+1}$. In contrast to the previous case these are both purely imaginary.

Having these $(N+1)(N+2)$ Hermitian operators, their corresponding matrices in the original state basis and the target Hermitian matrix H_{N0} we are ready to find the correct combination of Hermitian operators to describe the needed Hamiltonian (and the needed interaction time). If one is trying to find the operators for the lowest-energy manifold involved a simple inspection of the Hermitian matrices and H_{N0} will do, since every Hermitian matrix corresponds to one, and only one, either real or imaginary part of each coefficient in H_{N0}. For higher manifolds it is not sufficient to simply find the operators corresponding to the target Hermitian matrix, one also has to undo the action of the lower manifold operators on the manifold in question. To this end we compute the Hermitian matrix H_{lN0} corresponding to the sum of these operators in the original basis set. One then forms the matrix $H_{N0}-H_{lN0}$ and proceeds in the same manner outlined above to find the corresponding operators.

To reduce the procedure outlined above to practice we shall show how to find the Hamiltonian for a $n=3$, $m=2$, $G=2$ PNA. The preinteraction PNA state is hence $(c_0|0\rangle +c_1|1\rangle+c_2|2\rangle)\otimes(d_2|2\rangle+d_3|3\rangle)$, where d_2 and d_3 are real

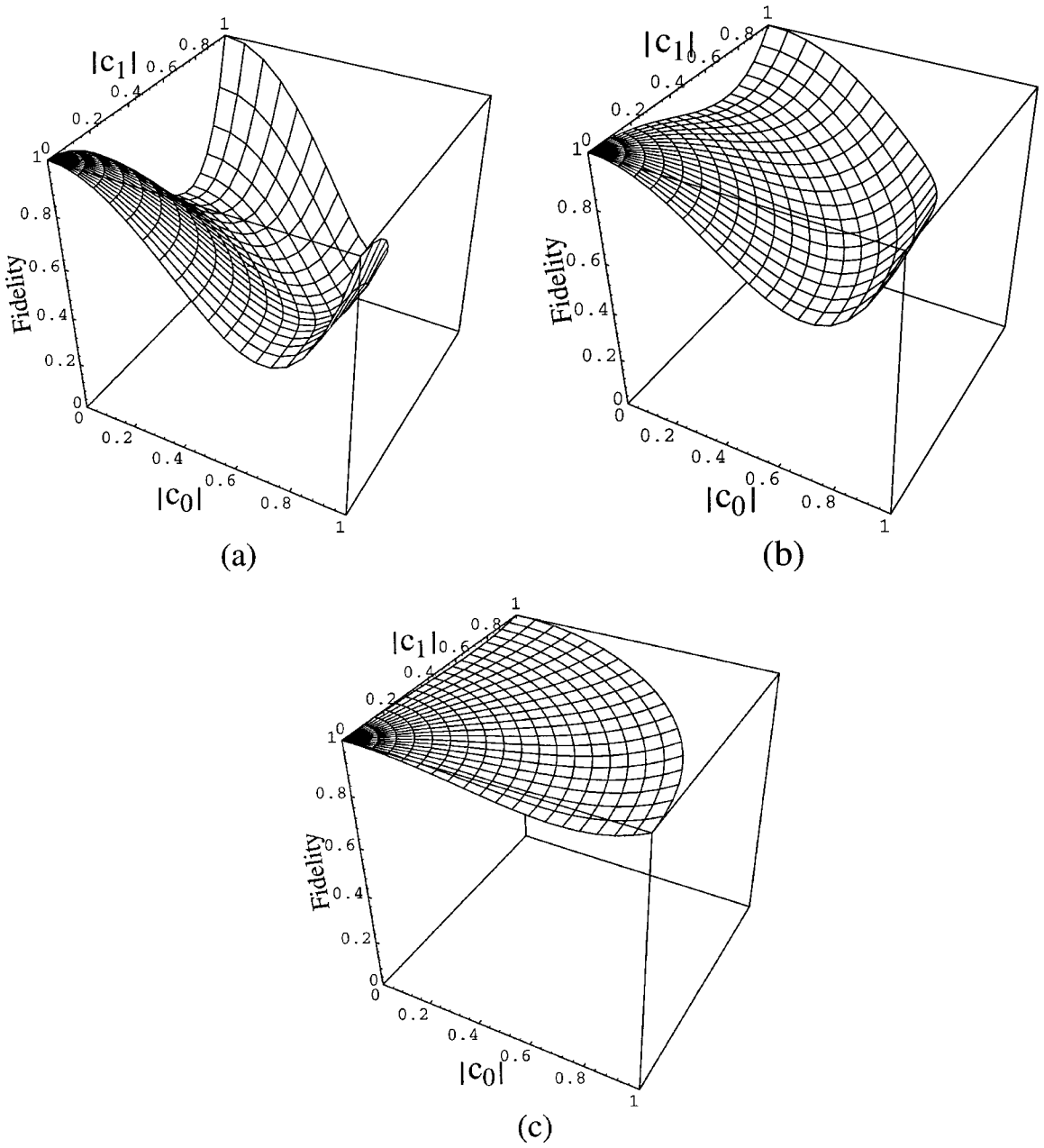

FIG. 3. The fidelity of a $n=3$ PNA for all possible choices of input states. In (a) a $m=1$ PNA has been assumed. In (b) and (c) m is 3 and 10, respectively.

and positive, and the c_i coefficients are arbitrary. The choice $d_2=d_3=2^{-1/2}$ maximizes the fidelity of the transformation, but that is irrelevant for what follows. We begin by noting that the lowest manifold involved is $N=2$, and that a basis set in this manifold is $\{|0,2\rangle,|1,1\rangle,|2,0\rangle\}$. Only the first of these basis states is interesting from the point of view of the PNA, and it shall remain invariant under the PNA amplification process. Hence a possible unitary evolution matrix is

$$U_{2o}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}. \tag{18}$$

Since this manifold (and its unitary evolution) is trivial, we shall forego the prescription above, and by simple inspection we see that an appropriate Hermitian operator is

$$\hat{H}_2\tau/\hbar=\pi(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}). \tag{19}$$

We note in passing that since there are no lower manifolds than the second to worry about, we can (and have) actually use(d) annihilation and creation operators of order 1 although we are dealing with the second manifold.

The third manifold is spanned by the states $\{|0,3\rangle,|1,2\rangle,|2,1\rangle,|3,0\rangle\}$. Of these states the first should remain invariant, and the second should be transformed to $|2,1\rangle$. Hence a permissible unitary transformation is

$$U_{3o}=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}, \tag{20}$$

where we have deliberately chosen a real, positive and diagonal-symmetric unitary transformation. The unitary transformation has the eigenvalues $\{-1,1,1,1\}$, hence a possible corresponding Hermitian matrix in the unitary matrix eigenbase is

$$H_{3e}=\pi\begin{bmatrix}3&0&0&0\\0&4&0&0\\0&0&4&0\\0&0&0&4\end{bmatrix}. \tag{21}$$

Transforming this matrix back to the original basis yields the Hermitian matrix

$$H_{3o}=\pi\begin{bmatrix}4&0&0&0\\0&7/2&1/2&0\\0&1/2&7/2&0\\0&0&0&4\end{bmatrix}. \tag{22}$$

The final Hermitian matrix whose corresponding operator we seek is

$$H_{3o}-H_{l3o}=\pi\left\{\begin{bmatrix}4&0&0&0\\0&7/2&1/2&0\\0&1/2&7/2&0\\0&0&0&4\end{bmatrix}-\begin{bmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\end{bmatrix}\right\}=\pi\begin{bmatrix}1&0&0&0\\0&1/2&1/2&0\\0&1/2&1/2&0\\0&0&0&1\end{bmatrix}, \tag{23}$$

where H_{l3o} is the Hermitian matrix corresponding to the operator (19). By inspection of Eq. (23) we see that the relevant five operators are $(\hat{a}^\dagger)^3\hat{a}^3$, $(\hat{a}^\dagger)^2\hat{b}^\dagger\hat{a}^2\hat{b}$, $\hat{a}^\dagger(\hat{b}^\dagger)^2\hat{a}^2\hat{b}+(\hat{a}^\dagger)^2\hat{b}^\dagger\hat{a}\hat{b}^2$, $\hat{a}^\dagger(\hat{b}^\dagger)^2\hat{a}\hat{b}^2$, and $(\hat{b}^\dagger)^3\hat{b}^3$. The corresponding matrix elements (two for each operator since $h_{ij}=h_{ji}^*$ are $h_{44}=6$, $h_{33}=2$, $h_{23}=2$, $h_{22}=2$, and $h_{11}=6$. Hence, the total operator performing the desired unitary transformation up to the third energy manifold becomes

$$\hat{H}_3\tau/\hbar=\pi\{\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}+[(\hat{a}^\dagger)^3\hat{a}^3+(\hat{b}^\dagger)^3\hat{b}^3]/6+[(\hat{a}^\dagger)^2\hat{b}^\dagger\hat{a}^2\hat{b}+(\hat{a}^\dagger)^2\hat{b}^\dagger\hat{a}\hat{b}^2+\hat{a}^\dagger(\hat{b}^\dagger)^2\hat{a}^2\hat{b}+\hat{a}^\dagger(\hat{b}^\dagger)^2\hat{a}\hat{b}^2]/4\}. \tag{24}$$

As noted in [8], the easier notational way to express such Hamiltonian operators is to use a sum over the outer products of the basis states, i.e., $\hat{H}_3\tau/\hbar=2\pi(|0,2\rangle\langle 0,2|+|1,1\rangle\langle 1,1|+|2,0\rangle\langle 2,0|)+\cdots$. Unfortunately such notation obscures the rather nonlinear interaction needed to obtain the desired unitary evolution. This was specifically pointed out in [8]. D'Ariano's Hamiltonian, on the other hand, used multiphoton annihilation and creation operators [7]. We have chosen not to simplify our Hamiltonians in terms of multiphoton operators but to stick to normally ordered annihilation and creation operators to explicitly bring out the pertinent physics.

Proceeding through the remaining two manifolds for the PNA yields the following interaction Hamiltonian:

$$\begin{aligned}\hat{H}_5\tau/\hbar=\pi\Bigg\{&\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}+\frac{(\hat{a}^\dagger)^3\hat{a}^3+(\hat{b}^\dagger)^3\hat{b}^3}{6}+\frac{(\hat{a}^\dagger)^2\hat{b}^\dagger\hat{a}^2\hat{b}+(\hat{a}^\dagger)^2\hat{b}^\dagger\hat{a}\hat{b}^2+\hat{a}^\dagger(\hat{b}^\dagger)^2\hat{a}^2\hat{b}+\hat{a}^\dagger(\hat{b}^\dagger)^2\hat{a}\hat{b}^2}{4}\\&+\frac{(\hat{a}^\dagger)^3\hat{b}^\dagger\hat{a}^3\hat{b}-(\hat{a}^\dagger)^2(\hat{b}^\dagger)^2\hat{a}^3\hat{b}-(\hat{a}^\dagger)^3\hat{b}^\dagger\hat{a}^2\hat{b}^2}{4}+\frac{(\hat{a}^\dagger)^2(\hat{b}^\dagger)^2\hat{a}^2\hat{b}^2}{8}-\frac{(1+\sqrt{6})[\hat{a}^\dagger(\hat{b}^\dagger)^3\hat{a}^2\hat{b}^2+(\hat{a}^\dagger)^2(\hat{b}^\dagger)^2\hat{a}\hat{b}^3]}{4\sqrt{6}}\\&+\frac{(\hat{a}^\dagger)^5\hat{a}^5+(\hat{b}^\dagger)^5\hat{b}^5}{120}+\frac{(\hat{a}^\dagger)^3(\hat{b}^\dagger)^2\hat{a}^3\hat{b}^2+(5+2\sqrt{6})[(\hat{a}^\dagger)^3(\hat{b}^\dagger)^2\hat{a}^2\hat{b}^3+(\hat{a}^\dagger)^2(\hat{b}^\dagger)^3\hat{a}^3\hat{b}^2]+13(\hat{a}^\dagger)^2(\hat{b}^\dagger)^3\hat{a}^2\hat{b}^3}{48}\\&+\frac{-11(\hat{a}^\dagger)^4\hat{b}^\dagger\hat{a}^4\hat{b}+(12-7\sqrt{2})[(\hat{a}^\dagger)^4\hat{b}^\dagger\hat{a}^3\hat{b}^2+(\hat{a}^\dagger)^3(\hat{b}^\dagger)^2\hat{a}^4\hat{b}]}{96}\\&-\frac{7\sqrt{2}[(\hat{a}^\dagger)^4\hat{b}^\dagger\hat{a}^2\hat{b}^3+(\hat{a}^\dagger)^2(\hat{b}^\dagger)^3\hat{a}^4\hat{b}]+7[(\hat{a}^\dagger)^4\hat{b}^\dagger\hat{a}\hat{b}^4+\hat{a}^\dagger(\hat{b}^\dagger)^4\hat{a}^4\hat{b}]}{96}\\&+\frac{-7\sqrt{2}[(\hat{a}^\dagger)^3(\hat{b}^\dagger)^2\hat{a}\hat{b}^4+\hat{a}^\dagger(\hat{b}^\dagger)^4\hat{a}^3\hat{b}^2]+(12-7\sqrt{2}+4\sqrt{6})[(\hat{a}^\dagger)^2(\hat{b}^\dagger)^3\hat{a}\hat{b}^4+\hat{a}^\dagger(\hat{b}^\dagger)^4\hat{a}^2\hat{b}^3]+13\hat{a}^\dagger(\hat{b}^\dagger)^4\hat{a}\hat{b}^4}{96}\\&+i\frac{(\hat{a}^\dagger)^4\hat{b}^\dagger\hat{a}^3\hat{b}^2-(\hat{a}^\dagger)^3(\hat{b}^\dagger)^2\hat{a}^4\hat{b}-(\hat{a}^\dagger)^4\hat{b}^\dagger\hat{a}^2\hat{b}^3+(\hat{a}^\dagger)^2(\hat{b}^\dagger)^3\hat{a}^4\hat{b}}{48\sqrt{2}}\\&+i\frac{(\hat{a}^\dagger)^3(\hat{b}^\dagger)^2\hat{a}^1\hat{b}^4-\hat{a}^\dagger(\hat{b}^\dagger)^4\hat{a}^3\hat{b}^2-(\hat{a}^\dagger)^2(\hat{b}^\dagger)^3\hat{a}\hat{b}^4+\hat{a}^\dagger(\hat{b}^\dagger)^4\hat{a}^2\hat{b}^3}{48\sqrt{2}}\Bigg\}.\end{aligned} \tag{25}$$

In principle a proper interaction Hamiltonian for any PNA operating on any manifold can be obtained using the procedure outlined above. However, it is clear from this simple, explicit, but nontrivial example that the real worry is not how to calculate the needed Hamiltonian, but how to experimentally realize it. Our explicit expression has the advantage that one sees what is in principle needed, namely, a series of different nonlinear interactions between the two modes. The problem is that the interaction phase shifts are rather large, on the order of unity for field strengths caused by single or few photon states. In reality, for a long time to come, we believe that nonlinear optical materials will lack the needed figure of merit, namely, a sufficiently high ratio between the nonlinear susceptibility and the linear and nonlinear loss.

Before closing this section we shall discuss the simpler $n=2$ PNA's. Such a PNA could find use in, e.g., weak light interferometry and as a Schrödinger cat generator if G is large. Unfortunately this PNA will have Hamiltonians of similar difficulty as the PNA discussed above. Only the $n=2$, $m=2$, $G=2$ PNA will have a less ''difficult'' Hamiltonian than Eq. (25) above. For example, a $n=2$, $m=2$, $G=3$ PNA will have a Hamiltonian consisting of typically more than twenty Hermitian normally ordered operator terms of which about half will be of order five in annihilation and creation. Therefore even PNA's whose input state Hilbert space has the smallest nontrivial dimension will yield complicated interaction Hamiltonians.

V. DISCUSSION

We have demonstrated how to construct a superposition preserving photon-number amplifier if the device is limited to operate on a finite Hilbert space. We argued that the constraint is not as limiting as it seems at first glance, since the same constraint actually is imposed on any real-world linear amplifier. The problem one faces when trying to construct a PNA is that due to the energy transfer between the input-output state and the amplifier internal mode, the joint postinteraction state will be entangled. Therefore the PNA output state will no longer be pure if the amplifier internal state is, e.g., dissipated to reset the amplifier. To classify the fidelity of the output state after tracing over the amplifier mode, we used a fidelity measure introduced in [11]. We then derived a systematic way of preparing the amplifier internal mode and choosing a proper state mapping so that the fidelity of the PNA was maximized when averaged over all allowed input states. We note that the same procedure can be used to construct the unitary transform, in the rotating-wave approximation, for a photon number deamplifier [3]. An additional difficulty with the latter device is that if G is positive but <1 then Gn will in general not be an integer. This was discussed in [3]. However, we assert that the real experimental challenge in constructing either an amplifying or deamplifying device will be the experimental realization of the Hamiltonian needed to optimize the fidelity of the device. We showed that the convergence towards unity fidelity goes as $1/m$, where m is the number of orthogonal states in the initial amplifier internal state superposition. Finally we derived a systematic procedure to derive a Hamiltonian, expressed in annihilation and creation operators that implements the PNA. We noted that no unique such Hamiltonian exists, the same holds for the PNA unitary evolution. However, to solve the ensuing equation systems in higher manifolds, one generally needs most of the degrees of freedom given by the $(N+1)(N+2)$ different Hermitian operators of order N in annihilation and creation. This means that although one Hamiltonian can be said to be ''simpler,'' e.g., by having fewer terms, than some other that performs the same PNA action, the two Hamiltonians will yet have the same ''complexity'' in terms of the order of the nonlinearity needed. The fact that a PNA operating up to manifold N needs operators of order N in annihilation and creation means that it is unrealistic to believe that optimized PNA's can be realized in the near future possibly except for the very simplest devices. However, if one is willing to sacrifice efficiency (where efficiency is defined as the success rate of the desired transformation) it should be possible to construct a PNA with a much simpler Hamiltonian. By, e.g., measurement conditioning it should be possible to identify those experimental runs where one has obtained the desired transformation so that the fidelity is close to unity. Obviously there is going to be a trade-off between the efficiency and the average fidelity of the selected runs. Exactly what this trade-off is will depend on the complexity of the conditioning measurement. This will be the topic of future work.

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[13] Both expressions are independent of the relative polar coordinate phase of c_l and $c_{l'}$. Hence it is actually sufficient to average the expressions over only n coordinates instead of $2n$ coordinates. An easy way to parametrize the problem is to let each possible input state be represented by a point on the unit radius n-dimensional hypersphere. The integration measure is found from the Jacobian of the generalized n-dimensional spherical coordinate system. Equations (8) and (9) can be deduced from the three-dimensional result and an induction proof from n to $n+1$ dimensions.