Collapses and revivals of collective excitations in trapped Bose condensates

R. Graham,* D. F. Walls, and M. J. Collett
Department of Physics, University of Auckland, Auckland, New Zealand

M. Fliesser
Fachbereich Physik, Universität - GH-Essen, D-45117 Essen, Germany

E. M. Wright
Optical Sciences Center, University of Arizona, Tucson, Arizona 85721
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We present a quantum theory of low-lying excitations in a trapped Bose condensate with finite particle numbers. We find that even at zero temperature condensate number fluctuations and/or fluctuations of the excitation frequency due to quantum uncertainties of the mode occupation lead to a collapse of the collective modes due to dephasing. Coherent revivals of the collective excitations are predicted on a much longer time scale. Depletion of collective modes due to second-harmonic generation is discussed. [S1050-2947(97)07312-5]

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I. INTRODUCTION

In order to investigate the properties of the recently created Bose condensates of alkali-metal atoms in magnetic traps [1–4] the excitation of low-lying collective modes by periodic variations of the trap potential has turned out to be a very effective tool. The successful implementation of this idea has produced results for the collective mode spectrum of the trapped condensate [5,6] which are in very good agreement with theoretical predictions based on solutions of the Gross-Pitaevski equation [7–10], i.e., on classical mean field theory at zero temperature. In the experiments, in addition to the mode frequencies, the decay of the collective excitations in real time could be observed. At present there exists no theory which describes the damping of excitations at finite temperature. In this paper we focus solely on the decay of the collective exciations at very low (effectively zero) temperature. In the very low temperature regime it is difficult to imagine any truly dissipative mechanism for the observed decay, as long as the trapped system can be considered closed. The usual dissipative mechanisms in a homogeneous Bose condensate [11] depend on the *continuous* mode spectrum found there.

It has recently been shown [12] that nondissipative interactions give rise to collapses and revivals of the macroscopic wave function for small atomic condensates, analogous to those predicted [13] and observed [14] for a single mode field and a two-level atom. Here we extend this idea to show that very similar mechanisms may lead to the collapse (apparent damping) of collective excitations in a finite Bose condensate with a discrete mode spectrum. For times much longer than the observation time in present experiments, this collapse is predicted to be reversed in ''revivals'' of the collective excitations.

*Permanent address: Fachbereich Physik, Universität - GH Essen, D-45117 Essen, Germany.

We shall investigate two possible mechanisms for the collapse. The first mechanism, discussed in Sec. II, is based on atom-number uncertainty in the condensate. It therefore has the same physical origin as the collapse of the macroscopic wave function [12]. However, it will turn out that this mechanism is much less effective for collective modes. The second mechanism, described in Sec. III, is based on a potential dependence of the mode frequency on the mode occupation. Given such a dependence, quantum uncertainties in the occupation of a mode must lead to its collapse. A two-mode model with this property has been discussed recently by Kuklov *et al.* [15], while Pitaevskii [17] also recently discussed this mechanism in general for nonlinear oscillators and applied it to a special collective mode in trapped condensates. Here we wish to present a theory of the nonlinear self-coupling of the collective hydrodynamic modes in trapped Bose condensates and examine the conditions under which the nonlinear coupling can become effective.

We emphasize that while we propose these collapse-mechanisms as possible explanations for the damping of excitations at zero or very low temperatures, in current experiments there appears to be a substantial finite temperature contribution which is likely to produce the dominant contribution to the currently observed damping times [18].

II. COLLAPSE DUE TO ATOM-NUMBER UNCERTAINTY IN THE CONDENSATE

The mechanisms we present are intrinsically quantum mechanical and, in principle, occur in any *finite* Bose-condensed system, in which the spontaneous symmetry breaking associated with the *infinite* system cannot occur [19]. For this reason the usual Bogoliubov analysis of the low-lying excitations of an infinitely extended Bose gas, which is based on the assumption of a spontaneously broken gauge symmetry, is not directly applicable. It must be modified by eliminating the underlying assumption of broken gauge symmetry before the collapse mechanism proposed here can be described consistently. Here we first present this modification of the Bogoliubov theory, then examine the collapse and revival of the collective excitations.

The grand canonical Hamiltonian of the system is [20]

$$H=\int d^3r\left[\frac{\hbar^2}{2m}\nabla\hat{\psi}^\dagger\cdot\nabla\hat{\psi}+(V(\mathbf{r})+\delta V(\mathbf{r},t)-\mu)\hat{\psi}^\dagger\hat{\psi}+\frac{U_0}{2}\hat{\psi}^{\dagger 2}\hat{\psi}^2\right], \tag{1}$$

where m is the atomic mass, μ the chemical potential, $V(\mathbf{r})$ is the static trap potential, $\delta V(\mathbf{r},t)$ its modulation, and $U_0=4\pi\hbar^2 a/m$ is proportional to the s-wave scattering length a. The Bogoliubov aproach to the low-lying excitations is based on decomposing the boson field annihilation operator $\hat{\psi}(\mathbf{r})$ (and its adjoint) as [21,22]

$$\hat{\psi}(\mathbf{r})=\langle\hat{\psi}(\mathbf{r})\rangle+\delta\hat{\psi}(\mathbf{r}), \tag{2}$$

and then approximating (1) by a quadratic form in the operators $\delta\hat{\psi}(\mathbf{r})$ and $\delta\hat{\psi}^\dagger(\mathbf{r})$. This approach is successful if the expectation value $\langle\hat{\psi}(\mathbf{r})\rangle$ on the right-hand side of Eq. (2) is nonzero, which is the case in an infinitely extended system if the U(1) gauge symmetry is spontaneously broken, or in a finite system if the phase of the condensate is established with respect to a reference, e.g., by a measurement of its phase relative to an infinitely extended Bose-condensed system [16], which itself has a well-defined phase. However, in the experiments on collective excitations in trapped Bose condensates referred to above a phase preparation of the condensate is not made, so that $\langle\hat{\psi}(\mathbf{r})\rangle=0$ on the right-hand side of Eq. (2) and this decomposition becomes useless. In fact, the same kind of problem appears for a laser far above threshold, where Bose condensation of photons in the laser mode occurs, but the phase is not fixed and undergoes a diffusion process [23]. As is done in that case, it is necessary here to replace Eq. (2) by[1]

$$\hat{\psi}(\mathbf{r})=e^{i\hat{\phi}(\mathbf{r})}\sqrt{\rho_0(\mathbf{r})+\delta\hat{\rho}(\mathbf{r})}+\hat{\psi}_1(\mathbf{r}). \tag{3}$$

The decomposition (3) was used by Popov [24] for spatially homogeneous Bose-condensed systems. Here $\rho_0(\mathbf{r})$ is a c number and $\delta\hat{\rho}(\mathbf{r})$ and $\hat{\phi}(\mathbf{r})$ are operators with commutation relations $[\delta\hat{\rho}(\mathbf{r}),e^{i\hat{\phi}(\mathbf{r}')}]=-e^{i\hat{\phi}(\mathbf{r})}\delta^{(3)}(\mathbf{r}-\mathbf{r}')$ and together they provide a quantum mechanical description of the condensate and its collective excitations at long wavelengths [24]. It is convenient to decompose $\delta\hat{\rho}(\mathbf{r})$ further into

$$\delta\hat{\rho}(\mathbf{r})=\delta\hat{\sigma}(\mathbf{r})+\delta\hat{n}/\Omega, \tag{4}$$

where $\delta\hat{n}=\int d^3r\,\delta\hat{\rho}(\mathbf{r},t)$ is canonically conjugate to the spatially averaged phase variable $\hat{\bar{\phi}}=\Omega^{-1}\int d^3r\,\hat{\phi}(\mathbf{r})$ according to the commutaion relation $[\delta\hat{n},\hat{\bar{\phi}}]=i$ and therefore commutes with $\nabla\hat{\phi}$ and has discrete, not necessarily integer eigenvalues with unit spacing. It describes number fluctuations in the condensate. In the following it is useful to introduce the abbreviation $\hat{\rho}(\mathbf{r})=\rho_0(\mathbf{r})+\delta\hat{n}/\Omega$ so that $\rho_0+\delta\hat{\rho}=\hat{\rho}+\delta\hat{\sigma}$. Note that $\hat{\rho}(\mathbf{r})$ also commutes with $\nabla\hat{\phi}$. The operator $\hat{\psi}_1(\mathbf{r})$ appearing in Eq. (3) describes short-wavelength components of $\hat{\psi}(\mathbf{r})$ whose elimination renormalizes the coefficients of the effective long-wavelength theory for $\hat{\phi}(\mathbf{r})$, $\delta\hat{\rho}(\mathbf{r})$ [24], and which provide a particle reservoir for the condensate even in the case $T=0$, to which we confine ourselves here. In the following we shall assume that the elimination of $\hat{\psi}_1$ has already been performed. The ansatz (3) is used in the Hamiltonian (1) to determine the Heisenberg equation of motion for $\delta\hat{\rho}$ and $\hat{\phi}$. We obtain formally (i.e., without paying attention to the fact that operator products at equal points in space are only well defined after choosing an appropriate regularization procedure; for our present purposes this is sufficient, since we are only interested in the quantized fields $\delta\hat{\rho},\delta\hat{\phi}$ either in their free-field regime, as in the present section, or in the case where all modes except one are in their vacuum states, as in the following section),

$$\delta\dot{\hat{\rho}}(\mathbf{r},t)=-(\hbar/m)\nabla\cdot\{[\rho_0(\mathbf{r})+\delta\hat{\rho}(\mathbf{r},t)]^{1/2}\nabla\hat{\phi}(\mathbf{r},t)[\rho_0(\mathbf{r})+\delta\hat{\rho}(\mathbf{r},t)]^{1/2}\}, \tag{5}$$

$$\begin{aligned}\dot{\hat{\phi}}(\mathbf{r},t)=&-\frac{1}{\hbar}\{U_0[\rho_0(\mathbf{r})+\delta\hat{\rho}(\mathbf{r},t)]+V+\delta V(\mathbf{r},t)-\mu\}\\&-\frac{\hbar}{4m}\{[\rho_0(\mathbf{r})+\delta\hat{\rho}(\mathbf{r},t)]^{-1/2}(\nabla\hat{\phi})^2[\rho_0(\mathbf{r})\\&+\delta\hat{\rho}(\mathbf{r},t)]^{1/2}+[\rho_0(\mathbf{r})+\delta\hat{\rho}(\mathbf{r},t)]^{-1/2}(\nabla\hat{\phi})^2[\rho_0(\mathbf{r})\\&+\delta\hat{\rho}(\mathbf{r},t)]^{-1/2}\}+\frac{\hbar}{4m}[\rho_0(\mathbf{r})\\&+\delta\hat{\rho}(\mathbf{r},t)]^{-1/2}(\nabla)^2[\rho_0(\mathbf{r})+\delta\hat{\rho}(\mathbf{r},t)]^{1/2}.\end{aligned} \tag{6}$$

It follows from Eq. (5) that

$$\delta\dot{\hat{n}}=\int d^3r\,\delta\dot{\hat{\rho}}=0, \tag{7}$$

i.e., there is no restoring force on $\delta\hat{n}$, which therefore need not necessarily be small. Therefore we combine this quantity with ρ_0 to extract from Eq. (6) the equation which holds to zeroth-order in the small quantities $\nabla\hat{\phi}$ and $\delta\hat{\sigma}$,

$$\hat{\mu}\hat{\rho}(\mathbf{r})=-\frac{\hbar^2}{4m}\left(\nabla^2\hat{\rho}(\mathbf{r})-\frac{1}{2}\hat{\rho}(\mathbf{r})^{-1}(\nabla\hat{\rho}(\mathbf{r}))^2\right)+V\hat{\rho}(\mathbf{r})+U_0\hat{\rho}(\mathbf{r})^2. \tag{8}$$

This is one component of the Gross-Pitaevski equation and identifies $\hat{\rho}(\mathbf{r})$ as the operator of the density of the condensate, as well as defining the operator of the chemical potential $\hat{\mu}=\mu(N_0+\delta\hat{n})$ as a function of the number operator

[1]The decomposition of ψ and $\psi^\dagger$ into amplitude and phase is beset by well-known problems, which arise due to the existence of the vacuum state satisfying $\psi|0\rangle=0$. However, since $N_0=\int d^3r\,\rho_0(\mathbf{r})\gg 1$ the probability amplitude for the vacuum state $|0\rangle$ is negligibly small.

$N_0+\delta\hat{n}=\int d^3r\,\hat{\rho}(\mathbf{r})$ in the condensate. Next we linearize with respect to the small quantities $\delta\hat{\sigma}(\mathbf{r},t)$ and $\nabla\hat{\phi}(\mathbf{r})$ to obtain

$$\delta\dot{\hat{\sigma}}(\mathbf{r},t)=-\frac{\hbar}{m}\nabla\cdot[\hat{\rho}(\mathbf{r})\nabla\hat{\phi}(\mathbf{r},t)],\tag{9}$$

$$\dot{\hat{\phi}}(\mathbf{r},t)=-\frac{1}{\hbar}U_0\delta\hat{\sigma}(\mathbf{r},t)+\frac{\hbar}{4m}\hat{\rho}(\mathbf{r})^{-1/2}[\nabla^2-\hat{\rho}(\mathbf{r})^{-1/2}(\nabla^2\hat{\rho}(\mathbf{r})^{1/2})]\hat{\rho}(\mathbf{r})^{-1/2}\delta\hat{\sigma}(\mathbf{r},t)-\frac{1}{\hbar}\delta V(\mathbf{r},t).\tag{10}$$

Eliminating $\nabla\hat{\phi}(\mathbf{r})$ from Eqs. (9) and (10) we then get the wave equation for the low-lying excitations,

$$\delta\ddot{\hat{\sigma}}(\mathbf{r},t)=\frac{1}{m}\nabla\cdot\{\hat{\rho}(\mathbf{r})\nabla[U_0\delta\hat{\sigma}(\mathbf{r},t)+\delta V(\mathbf{r},t)]\}-\frac{\hbar^2}{4m^2}\nabla\cdot[\hat{\rho}(\mathbf{r})\nabla\{\hat{\rho}(\mathbf{r})^{-1/2}[\nabla^2-\hat{\rho}(\mathbf{r})^{-1/2}(\nabla^2\hat{\rho}(\mathbf{r})^{1/2})]\hat{\rho}(\mathbf{r})^{-1/2}\delta\hat{\sigma}(\mathbf{r},t)\}].\tag{11}$$

Neglecting the gradient term of fourth order compared to those of second order, and also neglecting $\delta\hat{n}$ compared to ρ_0 and replacing $\delta\hat{\sigma}$ by the classical density fluctuation, we recover Stringari's wave equation [9] for the collective excitations of a trapped Bose condensate. We shall in the following also neglect the higher-order gradient terms, but shall keep the number fluctuation operator $\delta\hat{n}$, which is implicit in $\hat{\rho}$, and examine its consequences in Eq. (11). Since the short-wavelength noncondensate components of the system act as a reservoir even at $T=0$, these number fluctuations may reasonably be taken as Poissonian. We can therefore assume that the initial state of the system is given by a pure state or a mixture with a Gaussian distribution over eigenstates of $\delta\hat{n}$, with zero mean particle fluctuation and mean square deviation given by $\langle(\Delta N_0)^2\rangle=N_0$, the average number of particles in the condensate. This corresponds to a Poissonian distribution of the total particle number in the condensate $\delta\hat{n}+N_0$ around its mean value $N_0\gg1$. We emphasize that due to the entanglement of the condensate with the other states of the system, which have been eliminated, the use of a mixture of $\delta\hat{n}$ eigenstates applies even to a single realization of the experiment, and not only to an ensemble of realizations corresponding to identically prepared experiments. For each eigenvalue δn of $\delta\hat{n}$ we have in principle to determine the solution of Eq. (8) and of Eq. (11) corresponding to a given mode. This gives the normal modes of the density oscillations, after $\delta V(\mathbf{r},t)$ has been switched off, and determines their frequencies ω_ν as a function $\omega_{0\nu}(N_0+\delta\hat{n})$, where $\omega_{0\nu}(N_0)$ is the collective mode frequency determined by the mean-field theory [7–10]. In the quantum ensemble defined by the initial state only eigenvalues δn of $\delta\hat{n}$ which are very small compared to N_0 are contained with appreciable weight. Therefore it is sufficient to expand ω_ν to first order in δn,

$$\omega_\nu=\omega_{0\nu}(1+\gamma_\nu\delta n),\tag{12}$$

with

$$\gamma_\nu=\frac{1}{\omega_{0\nu}}\frac{\partial\omega_{0\nu}}{\partial N_0},\tag{13}$$

a quantity determined by the solutions of the mean-field theory [7–10].

We can now discuss the collapse and revivals of the collective excitations by evaluating the quantum ensemble average of the density oscillation $\langle\delta\hat{\sigma}_\nu(t)\rangle$ for a given mode, assuming that it is coherently excited, e.g., by modulating the trap at the required frequency at times prior to $t=0$, while for $t>0$ the mode is left to evolve freely. We obtain at times $0<t<(\gamma_\nu\omega_{0\nu})^{-1}$

$$\langle\delta\hat{\sigma}_\nu(t)\rangle=\exp\left(-\frac{1}{2}N_0\gamma_\nu^2\omega_{0\nu}^2t^2\right)\left(\langle\delta\hat{\sigma}_\nu(0)\rangle\cos(\omega_{0\nu}t)+\frac{\langle\delta\dot{\hat{\sigma}}_\nu(0)\rangle}{\omega_{0\nu}}\sin(\omega_{0\nu}t)\right).\tag{14}$$

It can be checked that the corrections of order $(\delta n)^2$ to Eq. (12) make a negligible contribution in Eq. (14). Equation (14) shows that the excitation decays by dephasing on a time scale

$$\tau_c=(\sqrt{N_0/2}|\gamma_\nu|\omega_{0\nu})^{-1},\tag{15}$$

in the form of a Gaussian collapse. This collapse occurs even at a temperature $T=0$, and even if the condensate never has a well-defined overall phase. This collapse is therefore different from (and turns out to be less effective than) the collapse of the macroscopic wave function due to number fluctuations [12], which occurs if a well-defined phase of the condensate (with respect to some phase standard, such as another condensate) is prepared, e.g., at $t=0$. On time scales $t\geqslant(\gamma_\nu\omega_{0\nu})^{-1}$ the discreteness of the spectrum of $\delta\hat{n}$ manifests itself and we obtain revivals of $\langle\delta\hat{\rho}_\nu(t)\rangle$ at the times $t=n\pi/\gamma_\nu\omega_{0\nu}$.

To estimate the order of magnitude of the effect and to examine its accessibility to observation we have to obtain an estimate of γ_ν, e.g., for the observed $m=0$ modes in the experimentally realized condensates. We can achieve this goal by using some analytical results due to Stringari [9]. It was shown in Ref. [9] that for $N_0\to\infty$ the mode frequencies become independent of N_0. It was also shown there by using sum-rule arguments that for finite N_0 the $\omega_{0\nu}$ for the low-lying states can be represented in the form $\omega_{0\nu}(N_0)$

$=\omega_{0\nu}(\infty)(1+c_\nu E_{kin}/E_{ho})$, where the c_ν are parameters of order 1 which depend on the trap geometry and the particular mode, and E_{kin}/E_{ho} is the ratio of the kinetic energy and the harmonic trap energy in the ground state. The N_0 dependence of this ratio in the limit of large N_0 of interest here can be estimated by using the Thomas-Fermi approximation [25] as $E_{kin}/E_{ho}=b_\nu(N_0 a/\sqrt{\hbar/m\omega_0})^{-4/5}$ with ω_0 the harmonic trap frequency and a proportionality factor b_ν of order 1 which is asymptotically independent of N_0 but dependent on the trap geometry. For the coefficient γ_ν we obtain finally

$$\gamma_\nu=-\frac{4}{5}b_\nu c_\nu N_0^{-9/5}(a/d_0)^{-4/5}, \tag{16}$$

with $d_0=\sqrt{\hbar/m\omega_0}$, giving a collapse time

$$\tau_c=(5\sqrt{2}/4b_\nu c_\nu)(a/d_0)^{4/5}\frac{N_0^{13/10}}{\omega_{0\nu}}, \tag{17}$$

which is larger than the collapse time for the macroscopic wave function obtained in Ref. [12] roughly by a factor of $\mu/\omega_{0\nu}$.

For the two experiments in which decay times of collective excitations have been measured, the following parameters can be estimated: In the experiment of Ref. [5], the lifetime of the $m=0$ mode (with a frequency of $\nu_{0\nu}=\omega_{0\nu}/2\pi=1.84\nu_r$, where the radial trap frequency has the value $\nu_r=132$ Hz) for a condensate containing $N_0=4500$ rubidium-87 atoms was measured to be 110 ± 25 ms. For this case the following numbers apply: $\omega_{0\nu}/2\pi=187$ Hz, $N_0=4500$, $a=52$ A , $\omega_0=\omega_r(\omega_z/\omega_r)^{1/3}$, and $a/\sqrt{\hbar/m\omega_0}=10^{-2}$. This gives a collapse time of $\tau_c=850$ ms, which is longer than the observed decay time, but this is to be expected, as the latter was measured in a regime where finite temperature effects are not negligible [18].

In the experiment of Ref. [6], the lifetime of the 30 Hz collective excitation of a condensate of $N_0=5\times10^6$ sodium atoms was measured to be 250 ms. Due to the much larger number of atoms used in this experiment the mode frequencies $\omega_{0\nu}$ become insensitive to the dispersion in the number of particles in the condensate [9], and the collapse due to the mechanism under discussion here occurs only on time scales very long compared to the observed damping time.

III. COLLAPSE DUE TO NONLINEARITY OF THE COLLECTIVE MODES

In this section we neglect any uncertainty in the number of particles in the condensate. However, we no longer linearize the Heisenberg equations of motion for $\delta\hat{\rho}$ and $\delta\hat{\phi}$. Instead we consider the case where only a single mode of the collective excitations is appreciably excited and ask for the nonlinear corrections to the dynamics of this mode. This nonlinearity will also give rise to a dephasing and collapse of the collective mode amplitude. This is in the same spirit as the recent work of Kuklov *et al.* [15] and Pitaevskii [17], but differs from the former in that we do not treat a model but rather start from the full microscopic approach, and is more general than the latter because the analysis is not confined to a special symmetric collective mode.

The analysis starts with the nonlinear Heisenberg equations of motion Eqs. (5) and (6) but with the density gradient terms neglected. These can be derived from the Hamiltonian [26]

$$H=\int d^3x\left\{\frac{\hbar^2\hat{\rho}}{2m}(\nabla\hat{\phi})^2+\frac{U_0}{2}(\delta\hat{\sigma})^2+\frac{\hbar^2}{2m}\delta\hat{\sigma}^{1/2}(\boldsymbol{\nabla}\hat{\phi})^2\delta\hat{\sigma}^{1/2}\right\}_{\mathcal{N}}. \tag{18}$$

Questions of operator ordering are not important at this stage. In the end we will choose normal ordering in the mode operators; this is the meaning of the notation $\{\ldots\}_{\mathcal{N}}$. We introduce a mode expansion using the modes of the linearized hydrodynamics,

$$\delta\hat{\sigma}=i{\sum_\nu}'\left(\sqrt{\frac{\hbar\omega_\nu}{2U_0}}F_\nu(\mathbf{r})\alpha_\nu(t)-\text{H.c.}\right), \tag{19}$$

$$\nabla\hat{\phi}={\sum_\nu}'\left(\sqrt{\frac{U_0}{2\hbar\omega_\nu}}\nabla F_\nu(\mathbf{r})\alpha_\nu(t)+\text{H.c.}\right), \tag{20}$$

where Σ' is the mode sum *not* including the spatially and temporally constant part; the constant part is already included in $\hat{\rho}=\rho_0+\delta\hat{n}/\Omega$. Here ω_ν is the frequency associated with the linear mode function $F_\nu(\mathbf{r})$, so that the linear hydrodynamics [9] implies $\alpha_\nu(t)=\alpha_\nu e^{-i\omega_\nu t}$.

The Hamiltonian now becomes

$$H={\sum_\nu}'\hbar\omega_\nu\alpha_\nu^\dagger\alpha_\nu+\hbar{\sum_{\nu\kappa\lambda}}'\left\{\frac{1}{3}C^{(1)}_{(\nu\kappa\lambda)}\alpha_\nu\alpha_\kappa\alpha_\lambda+\frac{1}{2}C^{(2)}_{\nu(\kappa\lambda)}\alpha_\nu^\dagger\alpha_\kappa\alpha_\lambda+\text{H.c.}\right\}, \tag{21}$$

with $C^{(1)}$ and $C^{(2)}$ symmetric in the parenthesized indices and given by

$$C^{(1)}_{(\nu\kappa\lambda)}=\frac{i\hbar}{4m}\sqrt{\frac{U_0}{2\hbar}}(D^{(1)}_{\nu(\kappa\lambda)}+D^{(1)}_{\kappa(\nu\lambda)}+D^{(1)}_{\lambda(\nu\kappa)}), \tag{22}$$

with

$$D^{(1)}_{\nu(\kappa\lambda)}=\sqrt{\frac{\omega_\nu}{\omega_\kappa\omega_\lambda}}\int d^3x F_\nu(\nabla F_\kappa\cdot\nabla F_\lambda), \tag{23}$$

and

$$C^{(2)}_{\nu(\kappa\lambda)}=\frac{i\hbar}{4m}\sqrt{\frac{U_0}{2\hbar}}(2D^{(2)}_{\nu(\kappa\lambda)}+D^{(3)}_{\kappa\nu\lambda}+D^{(3)}_{\kappa\lambda\nu}+D^{(3)}_{\lambda\nu\kappa}+D^{(3)}_{\lambda\kappa\nu}), \tag{24}$$

where

$$D^{(2)}_{\nu(\kappa\lambda)} = \sqrt{\frac{\omega_\nu}{\omega_\kappa \omega_\lambda}} \int d^3x F^*_\nu (\nabla F_\kappa \cdot \nabla F_\lambda), \quad D^{(3)}_{\nu\kappa\lambda} = \sqrt{\frac{\omega_\nu}{\omega_\kappa \omega_\lambda}} \int d^3x F^*_\nu (\nabla F^*_\kappa \cdot \nabla F_\lambda). \tag{25}$$

The resulting equations of motion in the interaction representation are

$$\dot{\alpha}_\nu = -i \sum_{\kappa\lambda}{}' \left\{ C^{(1)*}_{(\nu\kappa\lambda)} e^{i(\omega_\nu+\omega_\kappa+\omega_\lambda)t} \alpha^\dagger_\kappa \alpha^\dagger_\lambda + \frac{1}{2} C^{(2)}_{\nu(\kappa\lambda)} e^{i(\omega_\nu-\omega_\kappa-\omega_\lambda)t} \alpha_\kappa \alpha_\lambda + C^{(2)*}_{\kappa(\nu\lambda)} e^{i(\omega_\nu-\omega_\kappa+\omega_\lambda)t} \alpha^\dagger_\lambda \alpha_\kappa \right\}. \tag{26}$$

We now assume that only a single mode μ is externally excited; all other modes are only excited via their coupling to the mode μ. This implies that α_μ can be considered ''large'' compared to all other mode operators. Using Eq. (26) in this approximation, we have for $\nu=\mu$,

$$\dot{\alpha}_\mu \simeq -i \sum_\kappa{}' \{ 2C^{(1)*}_{(\mu\mu\kappa)} e^{i(2\omega_\mu+\omega_\kappa)t} \alpha^\dagger_\mu \alpha^\dagger_\kappa + C^{(2)}_{\mu(\mu\kappa)} e^{-i\omega_\kappa t} \alpha_\mu \alpha_\kappa + C^{(2)*}_{\mu(\mu\kappa)} e^{i\omega_\kappa t} \alpha^\dagger_\kappa \alpha_\mu + C^{(2)*}_{\kappa(\mu\mu)} e^{i(2\omega_\mu-\omega_\kappa)t} \alpha^\dagger_\mu \alpha_\kappa \}, \tag{27}$$

and for $\nu=\kappa\neq\mu$,

$$\dot{\alpha}_\kappa \simeq -i \left\{ C^{(1)*}_{(\kappa\mu\mu)} e^{i(\omega_\kappa+2\omega_\mu)t} (\alpha^\dagger_\mu)^2 + \frac{1}{2} C^{(2)}_{\kappa(\mu\mu)} e^{i(\omega_\kappa-2\omega_\mu)t} (\alpha_\mu)^2 + C^{(2)*}_{\mu(\kappa\mu)} e^{i\omega_\kappa t} \alpha^\dagger_\mu \alpha_\mu \right\}. \tag{28}$$

In general, the oscillatory terms in this will be changing much more rapidly than α_μ in the interaction picture. The only exception to this is when we have a second harmonic resonance, $\omega_\kappa = 2\omega_\mu$; this case needs separate treatment, and is discussed below. There is no contribution from the apparent resonance when $\omega_\kappa=0$, since this mode is explicitly excluded from the sum; in fact, the coefficient $C^{(2)*}_{\mu(0\mu)}$ would be zero anyway. Thus we may solve Eq. (28) for α_κ treating α_μ as approximately constant, to find

$$\alpha_\kappa \simeq -\left\{ \frac{C^{(1)*}_{(\kappa\mu\mu)}}{\omega_\kappa+2\omega_\mu} e^{i(\omega_\kappa+2\omega_\mu)t} (\alpha^\dagger_\mu)^2 + \frac{1}{2} \frac{C^{(2)}_{\kappa(\mu\mu)}}{\omega_\kappa-2\omega_\mu} e^{i(\omega_\kappa-2\omega_\mu)t} (\alpha_\mu)^2 + \frac{C^{(2)*}_{\mu(\kappa\mu)}}{\omega_\kappa} e^{i\omega_\kappa t} \alpha^\dagger_\mu \alpha_\mu \right\}. \tag{29}$$

We now subsitute this back into Eq. (27) for α_μ, and keep only the nonoscillatory terms, to obtain

$$\dot{\alpha}_\mu = i\kappa \alpha^\dagger_\mu \alpha_\mu \alpha_\mu, \tag{30}$$

where

$$\kappa = 2 \sum_\kappa{}' \left\{ \frac{|C^{(1)}_{(\kappa\mu\mu)}|^2}{\omega_\kappa+2\omega_\mu} + \frac{|C^{(2)}_{\mu(\kappa\mu)}|^2}{\omega_\kappa} + \frac{1}{4} \frac{|C^{(2)}_{\kappa(\mu\mu)}|^2}{\omega_\kappa-2\omega_\mu} \right\}. \tag{31}$$

The form of the matrix elements (22)–(24) implies selection rules for the modes κ contributing to the sum in Eq. (31). For axially symmetric trap potentials with inversion symmetry only modes contribute with positive parity and azimuthal quantum numbers m_κ satisfying $m_\kappa = 2m_\mu$ or $m_\kappa=0$ (via $C^{(2)}_{.(..)} \neq 0$) or $m_\kappa = -2m_\mu$ (via $C^{(1)}_{(\kappa\mu\mu)} \neq 0$).

The effective self-coupling coefficient κ experimentally manifests itself by an energy dependence of the observed mode frequency according to $\omega_\mu(E_\mu) = \omega_\mu - (\kappa/\hbar\omega_\mu)E_\mu$. Having derived Eq. (31), the collapse and revival of the mode μ follow immediately: If at $t=0$ the mode is excited, e.g., in a coherent state with amplitude A, then for $t>0$ its amplitude changes according to

$$\langle A|\alpha_\mu|A\rangle = A\exp[-|A|^2(1-\cos(\kappa t))]\{\cos[|A|^2\sin(\kappa t)] - i\sin[|A|^2\sin(\kappa t)]\}, \tag{32}$$

which for times $|\kappa|t \ll 1$ collapses with the new collapse time $\tau_c = (|A\kappa|)^{-1}$ according to $\langle A|\alpha_\mu|A\rangle \approx A\exp(-\frac{1}{2}(t/t_c)^2$ but is revived at revival times $t_r = 2n\pi\kappa^{-1}$ for integers $n \geq 1$.

We can estimate the order of magnitude of the effective coupling constant κ, in particular its scaling with the system parameters and the number of atoms in the condensate, by using the Thomas-Fermi approximation

$$\kappa \sim \frac{|C|^2}{\omega_0} \sim \frac{\hbar U_0}{m^2\omega_0} \frac{1}{\omega_0} \left(\frac{1}{r_{\rm TF}}\right)^7 \sim \omega_0 N_0^{-7/5} \left(\frac{d_0}{a}\right)^{2/5}, \tag{33}$$

which gives for the collapse time the estimate

$$\tau_c \sim (A\omega_0)^{-1} N_0^{7/5} \left(\frac{a}{d^0}\right)^{2/5}. \tag{34}$$

In this result, derived for an initially excited coherent state, we may replace the coherent amplitude A by the variance of the excited quantum number to generalize it for an arbitrary initially excited state. The scaling of κ with the system parameters is consistent with Pitaevskii [17], who derived the coupling coefficient for a special mode. The scaling of the coefficient with an inverse power of N_0 tends to make it small in most cases. Putting in the numbers for the two experiments, assuming the excited quantum number N_e to be 1% of the total particle number we obtain

$$\text{Rb:}\quad N_e=45,\ A=\sqrt{45},\ \ \tau_c=2.2\ \text{ s},$$

$$\text{Na:}\quad N_e=5\times 10^4,\ \ A=10^2\sqrt{5},\ \ \tau_c=1.47\times 10^3\ \text{ s}, \tag{35}$$

which is longer than the observed decay.

Relatively larger values are obtained for κ (and shorter ones for τ_c) if there is a collective mode near the second harmonic of the externally excited mode. Second-harmonic generation has recently been seen in numerical simulations of the Gross-Pitaevskii equation [27].

We shall briefly discuss this point for the case of an anisotropic axially symmetric trap whose collective mode frequencies and eigenfunctions are known as a function of the anisotropy parameter $\beta=\omega_z/\omega_0$ in the Thomas-Fermi and hydrodynamic limit [9,28], where ω_z and ω_0 are the axial and radial trap frequencies, respectively. The modes are in this case labeled by three quantum numbers (n,j,m) (see Ref. [28]). We are interested in the case where for two different sets of their values we have second-harmonic resonance such that $\omega_\nu=2\omega_\mu$, *and* where the relevant matrix element $C^{(2)}_{\nu(\mu\mu)}$ with $\nu=(\overline{n},\overline{j},\overline{m}),\mu=(n,j,m)$ does not vanish. The latter condition implies certain selection rules, which for the case at hand read: (i) The mode function $F_\nu(\mathbf{r})$ of the second-harmonic mode in cylindrical coordinates ρ,z,φ must be even in z, and (ii) at least one of the two conditions $\overline{m}=2m$ [for $D^{(2)}_{\nu(\mu\mu)}\neq 0, D^{(3)}_{\mu\mu\nu}\neq 0$] or $\overline{m}=0$ [for $D^{\mu\nu\mu}_{(3)}\neq 0$] must be satisfied.

As an example we consider the mode $\mu=(n=0,j=0,m=2)$ which has $\omega_\mu=\sqrt{2}\omega_0, F_\mu(\mathbf{r})=N_\mu\rho^2\exp(2i\varphi)$, where N_μ is a normalization factor, and we shall fix the phases of the mode functions by choosing these normalization factors always positive. Then the two modes $\nu1=(\overline{n}=2,\overline{j}=1,\overline{m}=0)$ with $\omega^2_{\nu1}=[3\beta^2/2+2+\sqrt{(2-3\beta^2/2)^2+2\beta^2}]\omega_0^2$, $F_{\nu1}(\mathbf{r})=N_{\nu1}(-4\rho^2/3-16z^2/3+1)$, and $\nu_2=(\overline{n}=2,\overline{j}=0,\overline{m}=4)$ with $\omega^2_{\nu2}=[3\beta^2/2+10-\sqrt{(6-3\beta^2/2)^2+10\beta^2}]\omega_0^2$, $F_{\nu2}(\mathbf{r})=N_{\nu2}(-16z^2+2\rho^2-1)\rho^4\exp(4i\varphi)$ both satisfy the selection rules. For the special value $\beta^2=(\omega_z/\omega_0)^2=16/7$ it turns out that both modes $\nu1,\nu2$ are degenerate and in second-harmonic resonance with the mode at frequency $\omega_\mu=\sqrt{2}\omega_0$, i.e., $\omega_{\nu1}=\omega_{\nu2}=2\sqrt{2}\omega_0$. (A resonance for this value of β^2 has recently also been noted in Ref. [29]). The relevant matrix elements in this case turn out to be $C^{(2)}_{\nu1(\mu\mu)}=0.562N_0^{-7/10}(d_0/a)^{1/5}\omega_0$ and $C^{(2)}_{\nu2(\mu\mu)}=0.332N_0^{-7/10}(d_0/a)^{1/5}\omega_0$. For such a case of exact second-harmonic resonance, the previous treatment breaks down. The differential equations for the two, or, in the present case even three, resonantly coupled modes must instead be considered together. Neglecting couplings to all other modes, we have for $\omega_{\nu i}=2\omega_\mu$

$$\dot{\alpha}_\mu=-i\sum_i C^{(2)*}_{\nu i(\mu\mu)}\alpha^\dagger_\mu\alpha_{\nu i}, \tag{36}$$

$$\dot{\alpha}_{\nu i}=-\frac{i}{2}C^{(2)}_{\nu i(\mu\mu)}\alpha^2_\mu, \tag{37}$$

which are the coupled mode equations for second-harmonic or subharmonic generation. It can be seen that only the linear combination $\alpha_\nu=\text{const}\Sigma_i C^{(2)*}_{\nu i(\mu\mu)}\alpha_{\nu i}$ of second-harmonic modes couples to the fundamental mode μ, while the linear combination $\overline{\alpha}_\nu=\text{const}[C^{(2)}_{\nu_2(\mu\mu)}\alpha_{\nu1}-C^{(2)}_{\nu1(\mu\mu)}\alpha_{\nu2}]$ is not generated. Choosing the constants $(\text{const}=|C^{(2)}_{\nu(\mu\mu)}|^{-1})$ to normalize these linear combinations, Eqs. (36) and (37) reduce to

$$\dot{\alpha}_\mu=-i|C^{(2)}_{\nu(\mu\mu)}|\alpha^\dagger_\mu\alpha_\nu, \tag{38}$$

$$\dot{\alpha}_\nu=-\frac{i}{2}|C^{(2)}_{\nu(\mu\mu)}|\alpha^2_\mu, \tag{39}$$

where we defined $|C^{(2)}_{\nu(\mu\mu)}|=[|C^{(2)}_{\nu1(\mu\mu)}|^2+|C^{(2)}_{\nu2(\mu\mu)}|^2]^{1/2}$, which takes the value $|C^{(2)}_{\nu(\mu\mu)}|=0.653N_0^{-7/10}(d_0/a)^{1/5}\omega_0$ in our example. These equations have well-known exact solutions [30] in terms of Jacobi elliptic functions, that oscillate with a frequency

$$\Omega_0=|C^{(2)}_{\nu(\mu\mu)}|\sqrt{\langle\alpha^\dagger_\mu\alpha_\mu\rangle}. \tag{40}$$

An order-of-magnitude estimate of this is

$$\Omega_0\sim\sqrt{\frac{n_\mu\hbar U_0}{m^2\omega_0 r^7_{\text{TF}}}}\sim\sqrt{n_\mu}\omega_0 N_0^{-7/10}\left(\frac{d_0}{a}\right)^{1/5}, \tag{41}$$

which is much larger than the nonresonant rate $\kappa\sqrt{n_\mu}$. The initial transfer of energy from the fundamental mode to the second harmonic may be viewed as a collapse with a collapse time $\tau_c\sim\Omega_0$.

The resonance phenomenon found here might help to explain why in the experiments some modes show amplitude dependence of the frequency, while others do not.

IV. CONCLUSIONS

We have investigated two different quantum mechanisms which may lead to a dephasing of collective modes in trapped Bose-Einstein condensates even at zero temperature. Both rely on the possibility that the system may be in a linear superposition of states, each of which has a slightly different frequency for the collective mode. In the first case investigated here the linear superposition combines states with different numbers of atoms in the condensate. This may occur since the noncondensate atoms, which are always present due to many-body interactions [21], act as a particle reservoir for the condensate. This case was earlier shown to give rise to collapses and revivals of the macroscopic wave function [12], once the latter is prepared at a given time with a well-defined (relative) phase, e.g., by a measurement. Here we found that a dephasing effect occurs also for the collective mode but on a different time scale which is typically much longer for large condensates. In principle, the collapse of the macroscopic wave function, if it was initially prepared with a well-defined phase, and of the collective mode amplitude may occur independently and simultaneously, but their simultaneous observation would of course be much more difficult than the observation of each collapse individually. While the dimensionless collapse time of the macroscopic wave function for large N_0 scales like

$$\omega_0\tau_c \sim (d_0/a)^{2/5} N_0^{3/5}/\langle \Delta N_0\rangle \text{ (macroscopic wave function)}, \tag{42}$$

the collapse time of the collective mode based on this mechanism scales as

$$\omega_0\tau_c \sim (d_0/a)^{-4/5} N_0^{9/5}/\langle \Delta N_0\rangle \times \text{ (collective mode due to atom-number uncertainty)}. \tag{43}$$

which corresponds to Eq. (17), which was written for $\langle \Delta N_0\rangle = N_0^{1/2}$.

In the second case investigated by us the linear superposition is one of different quantum number states of the excited collective mode, which will typically be in a coherent state immediately after its excitation from its vacuum state. We have calculated the effective nonlinearity of the excited mode due to its coupling to the other modes. This nonlinearity gives rise to a dispersion of the collective mode frequency within the linear superposition of number states. The collapse time obtained from this mechanism we find [in Eq. (34)] to scale as

$$\omega_0\tau_c \sim (d_0/a)^{-2/5} N_0^{7/5}/\langle \Delta n\rangle \times \text{ (collective mode due to nonlinearity)}. \tag{44}$$

For an initially excited coherent state the variance of the quantum number is related to the average mode energy E_μ by $\langle \Delta n\rangle = \sqrt{E_\mu/\hbar\omega_\mu}$. This scaling is consistent with a result obtained by Pitaevskii [17]. We have also found that the nonlinearity may be strongly enhanced if there is a mode at or close to a resonance with the second harmonic of the excited mode. In the case of such a resonance the transfer of energy from the fundamental mode to the second harmonic for short times looks similar to a collapse of the fundamental mode, which, according to Eq. (41), occurs on a time-scale scaling like

$$\omega_0\tau_c \sim (d_0/a)^{-1/5} N_0^{7/10}/\sqrt{E_\mu/\hbar\omega_\mu} \times (\text{collective mode due to second harmonic}). \tag{45}$$

Due to the scaling of the collapse times with positive powers of the atom number the collapse by one of the mechanisms we have examined could best be observed in comparatively small condensates. At presently achieved temperatures the observed damping of the collective mode occurs at a higher rate than the collapse rates obtained here. However, the observed damping rates were found to be strongly temperature dependent, decreasing rapidly with decreasing temperatures [18]. Therefore the collapse we have investigated here is still masked in present experiments by finite temperature effects, but is predicted to reveal itself when experiments are pushed to smaller temperatures.

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