Generation of phase-coherent states

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An interaction scheme involving nonlinear $\chi^{(2)}$ media is suggested for the generation of phase-coherent states (PCSs). The setup is based on parametric amplification of the vacuum followed by up-conversion of the resulting twin beam. The involved nonlinear interactions are studied by the exact numerical diagonalization. An experimentally achievable working regime to approximate PCSs with a high conversion rate is given and the validity of the parametric approximation is discussed.
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I. INTRODUCTION

Optical processes taking place in $\chi^{(2)}$ media involve three light waves, yielding to a considerably rich variety of nonlinear phenomena, both in the semiclassical [1] and in the quantum domain [2]. The quantum statistical properties of the radiation coming from such interactions have attracted much attention (see, for example, Refs. [3,4]). Squeezing, antibunching, and entanglement have been predicted and subsequently observed in a series of fascinating experiments [5,6]. Most of the theoretical approaches to a quantum theory of three-wave devices have been carried out using the so-called parametric approximation [7]. In this framework one of the field modes is in a strong semiclassical coherent state, so that its depletion as well as its quantum fluctuations can be neglected. A fully analytical treatment of the quantum dynamics is not available, whereas numerical methods have been developed in the cases of photon number [8] or coherent input states [9]. In this paper the three-wave dynamics is evaluated without an approximation for arbitrary input states, resorting to the numerical block diagonalization of the Hamiltonian in invariant subspaces of the constants of motion.

In the rotating-wave approximation the nondegenerate three-wave interactions are described by the Hamiltonian

$$\hat{H} \propto \chi^{(2)}[abc^{\dagger} + a^{\dagger}b^{\dagger}c], \tag{1}$$

where a, b, and c are the annihilation operators of the three relevant modes, whose frequencies satisfy the relation $\omega_c = \omega_a + \omega_b$. Depending on the input state of the field, the Hamiltonian (1) describes phase-insensitive amplification or frequency up- or down-conversion. The first kind of process occurs in situations with small a and b and large coherent c, so that the pumping mode can be considered as undepleted and treated as a c number, in the so-called parametric approximation. On the other hand, when all three modes participate in the quantum dynamics we are in the presence of frequency up- and down-conversion processes.

In the present paper we are interested in the situation depicted in Fig. 1, where the interaction Hamiltonian (1) is applied twice: in the first step as a parametric (spontaneous) down-conversion of the vacuum state, generating a twin beam on modes a and b, and in the second step as the up-conversion of a twin beam into mode c.

This scheme is of interest because, as shown in the following, the outgoing quantum state of radiation closely resembles the phase-coherent state (PCS)

$$|\lambda\rangle = \sqrt{1-|\lambda|^2}\sum_{n=0}^{\infty} \lambda^n|n\rangle, \tag{2}$$

which has been introduced by Shapiro *et al.* in Ref. [10]. The phase-coherent states are interesting because they are optimum phase states for both the Süssmann and the reciprocal peak likelihood [10] measure of phase uncertainty [11,12]. On the other hand, they also could serve as *seed* state [13–15] in sampling canonical phase distribution by unconventional heterodyne detection [16]. Moreover, one should mention that the PCSs maintain phase coherence under phase amplification [17], such that they are privileged states for phase-based communication channels.

In suggesting the present scheme we have been inspired by Ref. [18], where an ideal scheme using a photon number duplicator (PND) was suggested for PCS synthesis from a twin beam. As a matter of fact, in such photon recombination process the PND is well approximated by the up-conversion from Hamiltonian (1) with mode c initially in the vacuum
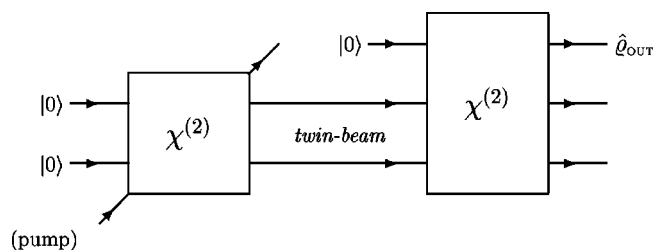

FIG. 1. Scheme of generation of phase-coherent states.

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[19,20]. In this paper we show that the interaction scheme sketched in Fig. 1 is indeed effective for the generation of PCSs.

The paper is structured as follows. In Sec. II we briefly describe our approach to the evaluation of the dynamics of the three-wave interactions and discuss the validity of the parametric approximation in the generation of twin-beam state. In Sec. III we analyze the performances of the twin-beam up-conversion in producing phase-coherent states in the second stage of the scheme of Fig. 1. Finally, Sec. IV closes the paper with some concluding remarks.

II. DYNAMICS OF THE THREE-WAVE INTERACTIONS

The overall input state describing the three involved modes can be written in the Fock basis as

$$|\psi_0\rangle = \sum_{n_1,n_2,n_3}^{N} c_{n_1,n_2,n_3} |n_1,n_2,n_3\rangle, \tag{3}$$

where N is an arbitrarily large integer, which denotes the largest non-negligible Fock component. In order to compute the dynamical evolution of $|\psi\rangle$,

$$|\psi_t\rangle = \exp(-it\hat{H})|\psi_0\rangle, \tag{4}$$

one should, in principle, diagonalize the full Hamiltonian matrix in the Fock basis. This becomes a very difficult task when the truncation N of the Fock space increases, becoming unrealistic for N exceeding few teens. However, one can notice that the Hamiltonian (1) admits two independent constants of motion. For the sake of convenience we choose them as

$$\hat{S} = \frac{1}{2}[a^\dagger a + b^\dagger b + 2c^\dagger c], \quad \hat{K} = a^\dagger a + c^\dagger c. \tag{5}$$

Conservation of $\hat{S}$ and $\hat{K}$ means that subspaces corresponding to given eigenvalues of these quantities are invariant under the action of the Hamiltonian (1) as $[\hat{H},\hat{S}]=0$ and $[\hat{H},\hat{K}]=0$. In other words, the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$ can be decomposed into the direct sum of subspaces that are invariant under the action of the unitary evolution operator in Eq. (4). Such a decomposition can be written as

$$\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c = \bigoplus_{s=0}^{N} \bigoplus_{k=0}^{s} \mathcal{H}_{sk}, \tag{6}$$

where

$$\mathcal{H}_{sk} = \text{Span}\{|k-n\rangle \otimes |s-k-n\rangle \otimes |n\rangle\}, \tag{7}$$

$$n \in [0,\min(k,s-k)],$$

Span{ } denoting the Hilbert space linearly spanned by the orthogonal vectors within the curly brackets and $|n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \equiv |n_1,n_2,n_3\rangle$ representing the state that is simultaneously the eigenvector of the number operator of the three modes. The Hamiltonian (1) can be consistently rewritten as

$$\hat{H} = \sum_{s,k} \hat{h}_{sk} , \tag{8}$$

with $\hat{h}_{sk}$ acting on $\mathcal{H}_{sk}$ only. Correspondingly, the representation for $|\psi_0\rangle$ can be written as

$$|\psi_0\rangle = \sum_{s=0}^{N} \sum_{k=0}^{s} \sum_{n=0}^{\min(k,s-k)} c_{k-n,s-k-n,n} |k-n,s-k-n,n\rangle , \tag{9}$$

which emphasizes the invariant subspaces structure. In this way one needs to diagonalize the Hamiltonian only *inside* each invariant subspace, thus leading to a considerable saving of resources.

As a first application of the above method we look for the conditions under which the parametric approximation is justified in describing the process of frequency down-conversion between a strong semiclassical pump and the vacuum. The input state is given by $|\psi_0\rangle = |0,0,\alpha\rangle$, $|\alpha| \gg 1$ being the amplitude of a considerably excited coherent state. In the parametric approximation the pump mode c in the Hamiltonian (1) is replaced by the c number α, thus neglect-
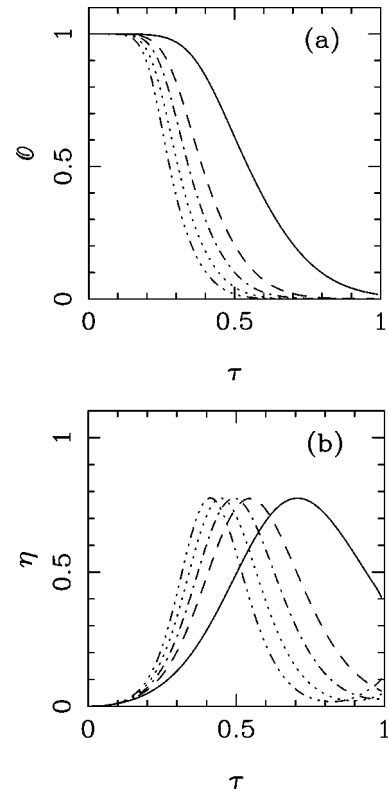

FIG. 2. (a) Overlap $\mathcal{O} = \sqrt{\langle \chi | \hat{\varrho}' | \chi \rangle}$ between the state $\hat{\varrho}'$ coming from the exact evolution and the twin beam $|\chi\rangle$ expected within the parametric approximation, as a function of the scaled time $\tau = \kappa t$ for different values of the pump input photon number. (b) Energy conversion rate η as a function of the scaled time for different values of the pump input photon number. In both plots different line styles denotes different values of pump intensity: $\langle \hat{n}_c \rangle = 81$ (dot-dot-dashed), $\langle \hat{n}_c \rangle = 64$ (dotted), $\langle \hat{n}_c \rangle = 49$ (dot-dashed), $\langle \hat{n}_c \rangle = 36$ (dashed), and $\langle \hat{n}_c \rangle = 16$ (solid). The interaction time leading to maximum conversion rate follows the relation $\tau_{opt} \propto \langle \hat{n}_c \rangle^{-1/3}$.
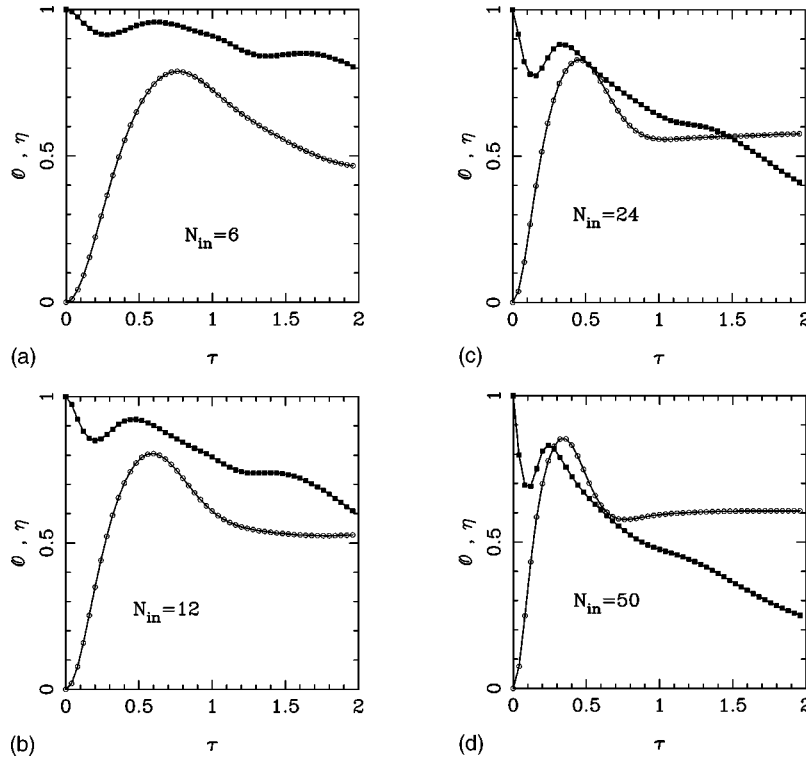
FIG. 3. Behavior of overlap $\mathcal{O}$ (solid squares) and conversion rate η (open circles) as a function of the scaled interaction time for different intensities of the incoming twin beam.

ing its quantum fluctuations as well as its depletion. Within such approximation the dynamics of the input state $|0,0\rangle$ is governed by the evolution operator

$$\hat{U} = \exp\{\zeta a^{\dagger}b^{\dagger} - \bar{\zeta}ab\}, \tag{10}$$

where $\zeta = -i\kappa t\alpha$, t is the interaction time, and κ is the coupling constant containing the nonlinear susceptibility. The evolution governed by Eq. (10) can be easily computed by means of the Baker-Haussdorff-Campbell formula for the su(1,1) Lie algebra [18,21,22] and the output is represented by the twin-beam state

$$|\chi\rangle = \sqrt{1-|\chi|^2} \sum_{n=0}^{\infty} \chi^n |n,n\rangle, \tag{11}$$

where

$$\chi = -i \tanh(\kappa t|\alpha|) e^{i\phi_{\alpha}}. \tag{12}$$

In order to check the theoretical results predicted by the parametric approximation we consider the overlap

$$\mathcal{O} = \sqrt{\langle\chi|\hat{\varrho}'|\chi\rangle}$$

between the state $\hat{\varrho}'$ coming from the exact evolution and the expected twin beam $|\chi\rangle$. In Fig. 2(a) we show the behavior of the overlap as a function of the scaled interaction time $\tau = \kappa t$ for different values of the pump input power. In order to evaluate the efficiency of the process we also consider the energy conversion rate η, which is defined as

$$\eta = \frac{1}{2} \frac{\text{Tr}[\hat{\varrho}' (\hat{n}_a + \hat{n}_b)]}{\text{Tr}[\hat{\varrho}^{in}\hat{n}_c]}, \tag{13}$$

where $\hat{\varrho}^{in} = |\psi_0\rangle\langle\psi_0|$. In Eq. (13) η runs between zero and one, the factor 1/2 coming from frequency conversion. In Fig. 2(b) we show the behavior of η as a function of the scaled interaction time τ for different values of the input power, as in Fig. 2(a). It is apparent that the parametric approximation is valid also for moderate input power and that one has a considerably wide range of values of the interaction time leading to an overlap very close to unity; the weaker the pump, the larger this range. On the other hand, these values of the interaction time correspond to a low conversion rate. In addition, we note that the interaction time leading to the maximum conversion rate follows the relation $\tau_{opt} \propto \langle\hat{n}_c\rangle^{-1/3}$.

III. TWIN-BEAM UP-CONVERSION

In this section we analyze the second step of the PCS generation setup reported in Fig. 1, namely, the three-wave interaction starting from the twin-beam input state

$$|\chi\rangle = \sqrt{1-|\chi|^2} \sum_{n=0}^{\infty} \chi^n |n,n,0\rangle. \tag{14}$$

The complex amplitude χ is confined in the unit circle and the mean photon number pertaining to the state (14) is given by
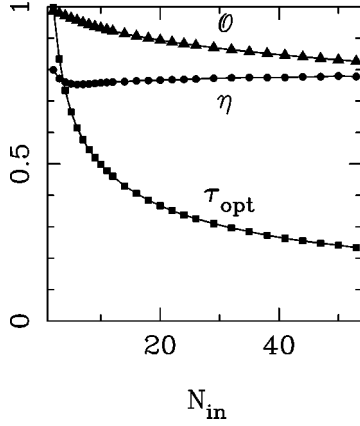
FIG. 4. Optimized overlap along with the corresponding interaction time and conversion rate as a function of the twin-beam input photons.

$$\langle\chi|\hat{n}_a+\hat{n}_b+\hat{n}_c|\chi\rangle=2|\chi|^2/(1-|\chi|^2).$$

The synthesis of the PCS (2) starting from $|\chi\rangle$ would be easily achieved by having at our disposal a device that performs the photon number recombination

$$|n,n,0\rangle\rightarrow|0,0,n\rangle.$$

Such a transformation has been analyzed in Ref. [18] and has been shown to correspond to the interaction Hamiltonian

$$\hat{H}_r=a^\dagger b^\dagger(b^\dagger b+1)^{-1/2}c+c^\dagger(b^\dagger b+1)^{-1/2}ab. \quad (15)$$

Unfortunately, the Hamiltonian (15) cannot be realized by known optical devices. However, one may notice that the perfect number recombination $|1,1,0\rangle\rightarrow|0,0,1\rangle$ is performed by the Hamiltonian (1), which suggests that one should substitute the intensity-dependent factor in Eq. (15) by its expectation value. In spite of this rather crude approximation, the trilinear interaction (1) has been shown [19,20] to provide a good approximation of the photon recombination in the case of a single photon number state at the input. Here we analyze the case of the input twin-beam state (14). Our aim is to demonstrate that the scheme of Fig. 1 is indeed effective in synthesizing a PCS. As a parameter to evaluate the effectiveness of PCS synthesis we use the overlap $\mathcal{O}=\sqrt{\langle\lambda|\hat{\varrho}^{out}|\lambda\rangle}$ between the state

$$\hat{\varrho}^{out}=\mathrm{Tr}_{ab}[\exp(-it\hat{H})|\chi\rangle\langle\chi|\exp(it\hat{H})], \quad (16)$$

exiting the $\chi^{(2)}$ crystal in the mode c and a theoretical PCS $|\lambda\rangle$ corresponding to the same mean photon number. In order to evaluate the efficiency of the process we also consider the conversion rate η, defined as

$$\eta=2\frac{\mathrm{Tr}(\hat{\varrho}^{out}\hat{n}_c)}{\langle\chi|(\hat{n}_a+\hat{n}_b)|\chi\rangle}. \quad (17)$$

In Fig. 3 we show the behavior of the overlap $\mathcal{O}$ and the conversion rate η as a function of the scaled interaction time

for different intensity of the incoming twin beam. A remarkable fact is apparent: Interaction times corresponding to a high conversion rate also optimize the overlap between the outgoing state and the theoretical PCS. This means that the up-conversion, although only approximated, produces a recombination process that is at the same time efficient and quite precise. One should also mention that for the same interaction times one has a small degree of mixing, indicating that the outgoing states are quite pure, and minimum reciprocal peak likelihood, thus confirming good phase-coherence properties.

In Fig. 4 we show the maximum overlap, along with the corresponding interaction time and conversion rate, as a function of the twin-beam input energy $N_{in}=\langle\hat{n}_a+\hat{n}_b\rangle$. The overlap $\mathcal{O}$ slowly decreases with respect to the input energy N_{in}, whereas the conversion rate η is almost independent of this quantity, saturating to a value close to 80%. This results in a reliable generation of PCSs with overlap between 80% and 100%, for outgoing states with energy $N_{out}=\langle\hat{n}_c\rangle$ up to $N_{out}=20$ mean photon number. The corresponding reciprocal peak likelihood $\delta\phi$ shows the scaling $\delta\phi\propto N_{out}^{-3/4}$, which, though worse than the ideal PCS performances $\delta\phi\propto N_{out}^{-1}$, is far superior to the coherent-state level $\delta\phi\propto N_{out}^{-1/2}$.

The interaction time τ_{opt}, which corresponds to the maximum overlap, decreases with the input energy N_{in}. By a best fit to the data in Fig. 4 we obtained the scaling power law $\tau_{opt}\simeq1.4N_{in}^{-0.45}$. Remarkably, the same scaling is observed as a function of the output energy N_{out}, with only a slight change in the proportionality constant $\tau_{opt}\simeq0.9N_{out}^{-0.45}$.

IV. CONCLUSION

In this paper we have suggested a scheme to generate the phase-coherent states introduced in Ref. [10]. The setup involves two $\chi^{(2)}$ nonlinear crystals and is based on parametric amplification of the vacuum followed by up-conversion of the resulting twin beam, the up-conversion playing the role of an approximate photon number recombination.

We found that parametric approximation in down-conversion of the vacuum state is valid also for moderate input power and that one has a considerably wide range of values of the interaction time leading to an overlap very close to unity, the weaker the pump, the larger this range. However, these values of the interaction time correspond to a low conversion rate. On the other hand, we found that the up-conversion process involved in the second step of the scheme is both power efficient and quite precise in the generation of PCSs. It is a remarkable fact that the range of interaction times leading to a high conversion rate also optimizes the overlap between the outgoing state and the theoretical PCSs. We have explored the case of twin-beam input photon number ranging from 0 to 54 and we have observed a conversion rate about 80%, with an overlap with ideal PCSs between 80% and 100%. This corresponds to a reliable generation of PCSs up to $N_{out}=20$ photons at the output.

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