Correlated input-port, matter-wave interferometer: Quantum-noise limits to the atom-laser gyroscope

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I derive the quantum phase-noise limit to the sensitivity of a Mach-Zehnder interferometer in which the incident quantum particles enter via both input ports. I show that if the incident particles are entangled and correlated properly, then the phase sensitivity scales asymptotically like the Heisenberg-limited $\Delta\varphi = O(1/N)$, for large N, where N is the number of particles incident per unit time. (In a one-input-port device, the sensitivity can be at best $\Delta\varphi = 1/\sqrt{N}$.) My calculation applies to bosons or fermions of arbitrary integer or half-integer spin. Applications to optical, atom-beam, and atom-laser gyroscopes are discussed—in particular, an atom-laser can be used to obtain the required entanglements for achieving this Heisenberg-limited sensitivity with atomic matter waves. [S1050-2947(98)06506-8]

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I. INTRODUCTION

In an earlier paper, Scully and I gave a general proof that the quantum-limited phase sensitivity $\Delta\varphi$ of a one-input-port, Mach-Zehnder interferometer, is precisely $\Delta\varphi = 1/\sqrt{N}$, where N is the total number of quantum particles (in Fock states) that have passed one at a time through the device in a given unit of time [1]. Our result was very general in that it applied equally well to fermions or bosons of arbitrary half-integer or integer spin, respectively.

In comparison to this Fock-state result, it is well known that in an optical interferometer, in which light in a coherent state $|\alpha\rangle$ enters via only *one* port, that the phase sensitivity scales as $\Delta\varphi = O(1/\sqrt{\bar{n}})$, where $\bar{n} = |\alpha|^2$ is the mean number of photons to have passed through the interferometer [2]. It would seem that any desired sensitivity $\Delta\varphi$ could be attained by simply increasing the laser power and hence $\bar{n}$. However, since $\Delta\varphi$ scales only slowly as $1/\sqrt{\bar{n}}$, the laser power rapidly becomes so large that the power fluctuations at the interferometer's mirrors introduce additional noise terms that eventually limit the device's overall sensitivity [3,4]. Steady improvements in optical laser gyroscope designs indicate that quantum-noise fluctuations such as these will be the dominant effect limiting laser gyroscope accuracy in the near future [5]. Much of the early interest in coherent photon-state squeezing centered around overcoming this signal-to-noise roadblock. In the early 1980s Caves [3], as well as Bondurant and Shapiro [4], demonstrated that when phase-squeezed coherent states are fed into *both* input ports of the interferometer, then phase sensitivity can asymptotically approach $\Delta\varphi = O(1/\bar{n})$, for large $\bar{n}$. This is a great achievement in that the total laser power required for a given amount of phase sensitivity $\Delta\varphi$ is greatly reduced.

In the late 1980s and early 1990s there was a flurry of interest in a proposal by Shapiro and co-workers [6] of a many-photon superposition state whose phase sensitivity was claimed to scale asymptotically as a remarkable $\Delta\varphi = O(1/\bar{n}^2)$. Schleich and I studied some of the interesting properties of this ''Shapiro'' state [7], and I also proposed a related photon state that apparently had the same $\Delta\varphi$ scaling law [8]. However, in the mid-1990s, Lane, Braunstein, and Caves showed that—when all of the steps in the phase measurement process were properly accounted for—the Shapiro and my related state apparently had a sensitivity no greater than $\Delta\varphi = O(1/\bar{n})$; that obtained by ordinary squeezing of a coherent state [9]. The conventional wisdom now seems to be that $\Delta\varphi = O(1/\bar{n})$ is the best one can do with states in which many-photon number states are superposed in a given boson mode—and this sensitivity is often called ''Heisenberg limited,'' by applying the uncertainty principle to the number and phase difference operators between the two input ports of the interferometer.

As early as 1986, Yurke had considered the question of phase-noise reduction using correlated spin-$\frac{1}{2}$ fermions—in Fock states—incident upon both input ports of a Mach-Zehnder interferometer [10]. For Fock states—unlike coherent states—there are no number fluctuations, and for fermions only one number state can be occupied at a time, due to the Pauli-exclusion principle. This rules out squeezing in the conventional sense. Nevertheless, Yurke was able to show that if N spin-$\frac{1}{2}$ fermions entered into each input port of the interferometer in nearly equal numbers—and in a highly correlated and entangled fashion—then it was indeed possible to obtain an asymptotic phase sensitivity of $\Delta\varphi = O(1/N)$, for large N. This should be compared to the $\Delta\varphi = 1/\sqrt{N}$ that is the best one can do using only one input port [1,10,11]. As intriguing as Yurke's result is, his proof relied heavily on the utilization of properties of the su(2) spin-angular-momentum algebra for a collection of N, spin-$\frac{1}{2}$ fermions. In addition, Yurke seems to have envisioned either neutrons or electrons for his correlated input beams, and it is difficult to imagine how the requisite, cross-port entanglements might actually be made experimentally.

Shortly after Yurke's paper was published [10], there appeared a second, related paper by Yurke, McCall, and

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Klauder (YMK) for photons. In this YMK work [11], a similar su(2) formalism was developed for a *bosonic*, Fock-state, correlated-input-port interferometer. In particular, they showed that a phase sensitivity of $\Delta\varphi = O(1/N)$ could be obtained using correlated photons emanating from a nonlinear, optical four-wave mixing device [11].

In this current paper, I will show that it is possible to discuss both fermions and bosons on an equal footing in a simple fashion that does not utilize the su(2) spin-algebra representation. Even though the su(2) formalism is mathematically elegant, the method I will use here makes some aspects of the physical interpretation more transparent. Here, I wish to discuss gyroscopic interferometers using atomic beams or atom lasers as my input beams, and so treating fermions and bosons with the same formalism seems the sensible thing to do.

I will digress here only briefly to mention that the su(2) angular-momentum algebra treatment, of N particles passing through a Mach-Zehnder interferometer, is isomorphic to that of the Dicke manifold [12] for studying the properties of a collection of N spin-$\frac{1}{2}$ particles or two-level atoms [11,13,14]. Hence, using this formalism it is possible to produce coherent and squeezed-coherent atomic ensemble states [13–16] that can be used to lower the quantum-noise limits to atomic Ramsey spectroscopy. In addition, Bollinger *et al.*, as well as Bouyer and Kasevitch (working with degenerate gases) have shown that the correlated Fock-state techniques employed by Yurke and co-workers, to reduce interferometer noise to $\Delta\varphi = O(1/N)$, can also be applied to spectroscopic noise [17]. This is due to the fact that the Ramsey spectroscopic technique for N atoms is isomorphic to Mach-Zehnder interferometry with N particles [11,13,14].

In this paper I will demonstrate that it is possible to achieve an asymptotic interferometer phase sensitivity of $\Delta\varphi = O(1/N)$, using either fermions or bosons of arbitrary half-integer or integer spin, respectively. Only three requirements need be met to attain this level of sensitivity: (1) the particles must be in Fock states; (2) they must be incident on both input ports in roughly equal numbers; and (3) they must enter the two ports in a highly entangled or correlated fashion. The immediate ramification of this result is that it opens the way for a new type of correlated-input-port interferometer using bosonic atom-matter waves emanating from an atom laser. Correlated beams of bosonic atom-matter waves have recently been produced in the atom-laser experiment of Ketterle and co-workers [18]. These experiments give a reasonable hope that the input correlations required for correlated-input-port increased phase sensitivity might be generated from such atomic bosonic sources, and I will give some indication herewith how this might be done.

This paper is divided into six sections. In Sec. II I will review the Fock-state formalism for computing the phase sensitivity in a simple Mach-Zehnder interferometry through which N particles pass in unit time. I will use this formalism to recalculate the *one*-input-port, Fock-state, sensitivity limit of $\Delta\varphi = 1/\sqrt{N}$. In Sec. III I will give the calculation for the correlated *two*-input-port, Fock-state interferometer, and show that—for large N—an asymptotic phase sensitivity of $\Delta\varphi = O(1/N)$ is achievable for highly correlated atomic matter waves of fermions or bosons of arbitrary half-integer or integer spin, respectively. This provides a simple formalism for treating the correlated-input-port, atom-laser gyroscope, and that treatment comprises the primary result of the present paper.

In Sec. IV I will show that an analogous correlated-input-port technique using *coherent* bosonic states—rather than Fock states—gives a sensitivity of, at best, only $\Delta\varphi = O(1/\sqrt{\overline{n}})$, where $\overline{n}$ is the mean number of particles. This calculation demonstrates that the use of Fock states is essential for obtaining an asymptotic phase sensitivity of $\Delta\varphi = O(1/N)$ by this approach. In Sec. V I will compare and contrast typical one-input-port optical and matter-wave gyroscopes to potential two-input-port devices. In particular, a correlated-two-input-port laser gyroscope could be as much as eight orders of magnitude more sensitive to rotations than an equivalent one-port device. Similarly, a two-port, atom-laser gyroscope could be six orders of magnitude more sensitive than a one-port atom-beam gyroscope, and a remarkable *ten* orders of magnitude more sensitive than a comparable one-input-port ring-laser device. Finally, in Sec. VI I will summarize and conclude.

II. ONE-INPUT-PORT, FOCK-STATE INTERFEROMETER

In this section I will recalculate the phase sensitivity $\Delta\varphi$ of a one-input-port, Mach-Zehnder interferometer [1,10,11], depicted in Fig. 1. A stream of N, Fock-state particles are incident upon the upper input port that I shall call A. At the first beam splitter, $\mathcal{S}_1$, the particles are split evenly into either upper path U of length ℓ_U, or lower path L, of length ℓ_L. They reflect off either upper mirror $\mathcal{M}_U$ or lower mirror $\mathcal{M}_L$, and then recombine at beam splitter $\mathcal{S}_2$. Finally, the particles emerge from either upper output port C or lower output port D, whereupon they strike upon upper detector $\mathcal{D}_U$ or lower detector $\mathcal{D}_L$. Without loss of generality, I will assume that the particles are sufficiently well collimated to be in single spatial modes, and I assume their velocity spread is negligible, so that all particles have the same constant wave number k [10]. Let $\hat{a}$ and $\hat{b}$ be the annihilation operators for each particle in input modes A and B, with corresponding creation operators $\hat{a}^\dagger$ and $\hat{b}^\dagger$, respectively. Similarly, let $\hat{c}$ and $\hat{d}$ (and their Hermitian conjugates) be the operators corresponding to output ports C and D, respectively. The operators of each mode obey the usual commutation relations,

$$\hat{a}\hat{a}^\dagger \pm \hat{a}^\dagger\hat{a} = 1, \tag{1a}$$

$$\hat{b}\hat{b}^\dagger \pm \hat{b}^\dagger\hat{b} = 1, \tag{1b}$$

$$\hat{c}\hat{c}^\dagger \pm \hat{c}^\dagger\hat{c} = 1, \tag{1c}$$

$$\hat{d}\hat{d}^\dagger \pm \hat{d}^\dagger\hat{d} = 1, \tag{1d}$$

where the plus sign is for bosons and the minus for fermions.

Now there is a simple scattering matrix relationship between the input operators $\hat{a}$ and $\hat{b}$ and the output operators $\hat{c}$ and $\hat{d}$, namely [1,10,11,19],

$$\begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -i(e^{i\mu} - e^{i\nu}) & (e^{i\mu} + e^{i\nu}) \\ (e^{i\mu} + e^{i\nu}) & i(e^{i\mu} - e^{i\nu}) \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}, \tag{2}$$

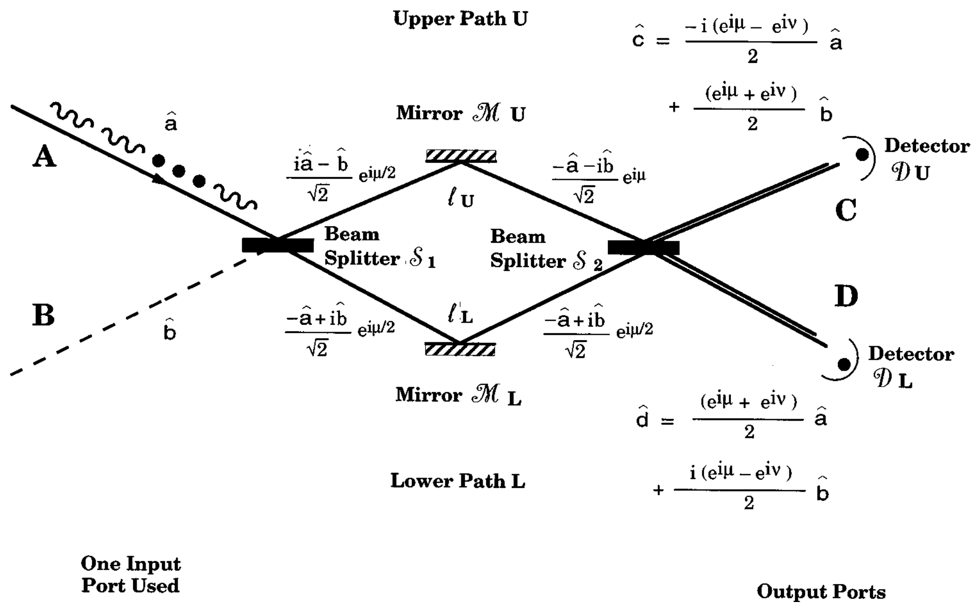

FIG. 1. A one-port particle Mach-Zehnder interferometer. Particles are incident on input port A while only vacuum comes into input port B. The paths taken in the upper and lower branches U and L are of length ℓ_V and ℓ_L, respectively. The phases accumulated along each branch of the interferometer, and at the equidistant detectors, is also shown.

where $\mu \equiv k\ell_U$ and $\nu \equiv k\ell_L$ are the phases accumulated by the particles when traversing the upper or lower paths U or L, respectively. As in Ref. [1], I have also assumed that each beam splitter is 50-50, which is accounted for by the factor of $\frac{1}{2}$ in Eq. (2). In addition, also as in Ref. [1], I have assumed a $\pi/2$ phase shift upon each reflection and a π phase shift upon each beam-splitter transmission, which is responsible for the minus signs and the factors of imaginary $\pm i$.

In general, time-reversal symmetry and parity conservation—as embodied in the Stokes reciprocity relations—implies that, after passage through a *symmetric* beam splitter, the difference in phase between the reflected and transmitted components of the beam *must* be $\pm\pi/2$. That requirement is met in the choice of beam-splitter phases given above. If I assume the more general case of a phase shift α and β on transmission and reflection, respectively, at each of the beam splitters, and a shift γ upon reflection at each of the mirrors, I recover the form of the matrix, Eq. (2)—up to an unimportant overall phase factor—provided I demand $\alpha - \beta = \pm\pi/2$, but independent of the choice of γ. For $\alpha = \pi$, $\beta = \pi/2$, and $\gamma = \pi$, the phases accumulated at the end of each branch of the interferometer and at the equidistant detectors are depicted in Fig. 1.

Carrying out the matrix multiplication, I arrive at an expression relating the input operators to the output number operators, namely,

$$\hat{c}^\dagger\hat{c} = \hat{a}^\dagger\hat{a}\sin^2\varphi/2 + \hat{b}^\dagger\hat{b}\cos^2\varphi/2 + \tfrac{1}{2}(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})\sin\varphi, \tag{3a}$$

$$\hat{d}^\dagger\hat{d} = \hat{a}^\dagger\hat{a}\cos^2\varphi/2 + \hat{b}^\dagger\hat{b}\sin^2\varphi/2 - \tfrac{1}{2}(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})\sin\varphi, \tag{3b}$$

where $\varphi \equiv \mu - \nu = k(\ell_U - \ell_L)$ corresponds to the phase difference between the two interferometer paths—the quantity to be measured. Let us define output-port sum and difference number operators $\hat{N}$ and $\hat{M}$, respectively, by

$$\hat{N} \equiv \hat{d}^\dagger\hat{d} + \hat{c}^\dagger\hat{c}, \tag{4a}$$

$$\hat{M} \equiv \hat{d}^\dagger\hat{d} - \hat{c}^\dagger\hat{c}. \tag{4b}$$

Then, from Eq. (3), it is trivial to reexpress these output sum and difference operators in terms of the input operators $\hat{a}$ and $\hat{b}$, via

$$\hat{N} \equiv \hat{b}^\dagger\hat{b} + \hat{a}^\dagger\hat{a}, \tag{5a}$$

$$\hat{M} \equiv (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})\cos\varphi - (\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})\sin\varphi. \tag{5b}$$

The expression (5a) for the output sum operator $\hat{N}$ expresses conservation of particles. The phase information φ we are measuring is contained in the output difference operator $\hat{M}$, as is clearly seen in Eq. (5b). This quantity $\langle\hat{M}\rangle$ is measured by counting the difference of the detection rate between upper and lower detectors $\mathcal{D}_U$ and $\mathcal{D}_L$, respectively. The root mean square of the quantum phase fluctuations, $\Delta\varphi^2$, associated with this measurement of φ can be obtained from [1,10,11]

$$\Delta\varphi^2 = \frac{\Delta M^2}{|\partial\langle\hat{M}\rangle/\partial\varphi|^2}, \tag{6}$$

where the variance ΔM^2 is defined as usual as

$$\Delta M^2 \equiv \langle\hat{M}^2\rangle - \langle\hat{M}\rangle^2, \tag{7}$$

with the expectation values to be carried out with respect to the appropriate input state $|\psi\rangle$.

A simple heuristic argument for the validity of this equation (6) can be given as follows. Consider a differentiable function $y = y(x)$. I have $y'(x) = dy/dx \cong \Delta y/\Delta x$. The approximation becomes an equality in the limit $\Delta x \to 0$. However, if $y(x)$ is a fluctuating dependent random variable, then I cannot take $\Delta y \to 0$. Hence the best that I can resolve the independent variable x is to within the associated Δx. Hence, given Δy, the minimal resolvable Δx is given implicitly by $|y'(x)| = \Delta y/\Delta x$ or explicitly by $\Delta x = \Delta y/|y'(x)|$. Setting $\langle \hat{M} \rangle \equiv y$ and $\varphi \equiv x$ and squaring gives Eq. (6) for the variance of the minimal detectable phase, $\Delta \varphi^2$, if I identify $\Delta M \cong \Delta \langle \hat{M} \rangle$.

Without yet specifying the form of $|\psi\rangle$, I can expand Eq. (6) in terms of expectation values of various products of the input operators $\hat{a}$ and $\hat{b}$ and their conjugates. This is easily accomplished by taking Eq. (5b) for the output difference operator $\hat{M}$ and inserting it into the expression (6) for $\Delta \varphi^2$, yielding

$$\Delta \varphi^2 = \frac{\Delta X^2 \cos^2 \varphi - (\langle \hat{X} \hat{Y} \rangle - 2\langle \hat{X} \rangle \langle \hat{Y} \rangle + \langle \hat{Y} \hat{X} \rangle) \sin \varphi \cos \varphi + \Delta Y^2 \sin^2 \varphi}{|\langle \hat{X} \rangle \sin \varphi + \langle \hat{Y} \rangle \cos \varphi|^2}, \tag{8}$$

where I have defined the difference operator $\hat{X}$ and the exchange operator $\hat{Y}$ as combinations of the input operators $\hat{a}$ and $\hat{b}$, and their conjugates, namely,

$$\hat{X} \equiv \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}, \tag{9a}$$

$$\hat{Y} \equiv \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}, \tag{9b}$$

with corresponding variances,

$$\Delta X^2 \equiv \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2, \tag{10a}$$

$$\Delta Y^2 \equiv \langle \hat{Y}^2 \rangle - \langle \hat{Y} \rangle^2. \tag{10b}$$

Equation (8) for $\Delta \varphi^2$ will provide the workhorse of this paper. Once an input state $|\psi\rangle$ is given, I merely need to compute the expectation values in Eq. (8) to arrive at a phase sensitivity associated with that input state. As an example, I will calculate the sensitivity associated with N particles entering one at a time in only top input port A, Fig. 1. Such a calculation will reproduce my earlier result with Scully [1], or that of Yurke [10]. The appropriate input state $|\psi\rangle$ for this initial condition is the Hilbert space product

$$|\psi\rangle_{\mathrm{I}} \equiv |N\rangle_A |0\rangle_B, \tag{11}$$

indicating N particles entering port A and vacuum in port B. Writing the input state in this form assumes that only one fermion is in a Fock state at a time—in order to satisfy the exclusion principle. (This restriction obviously can be relaxed for bosons.) I now compute the pieces of Eq. (8) for $\Delta \varphi^2$, in terms of the expectation values and variances of the operators $\hat{X}$ and $\hat{Y}$, defined in Eqs. (9) and (10). First,

$$\begin{aligned}\langle \hat{X} \rangle_{\mathrm{I}} &= {}_B\langle 0|{}_A\langle N|\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}|N\rangle_A|0\rangle_B \\ &= {}_A\langle N|\hat{a}^\dagger \hat{a}|N\rangle_A \, {}_B\langle 0|0\rangle_B - {}_A\langle N|N\rangle_A \, {}_B\langle 0|\hat{b}^\dagger \hat{b}|0\rangle_B \\ &= (N)(1) - (1)(0) = N. \end{aligned} \tag{12}$$

Similarly, I have

$$\langle \hat{Y} \rangle_{\mathrm{I}} = 0, \tag{13}$$

$$\langle \hat{Y}^2 \rangle_{\mathrm{I}} = N, \quad \langle \hat{X}^2 \rangle_{\mathrm{I}} = N^2, \tag{14}$$

$$\langle \hat{X} \hat{Y} \rangle_{\mathrm{I}} = \langle \hat{Y} \hat{X} \rangle_{\mathrm{I}} = 0. \tag{15}$$

Inserting these expressions (11)–(14) into Eq. (8) for the phase variance $\Delta \varphi^2$, I obtain

$$\Delta \varphi_{\mathrm{I}}^2 = \frac{N \sin^2 \varphi}{|N \sin \varphi|^2} = \frac{1}{N}, \tag{16}$$

an expression that is completely independent of the measured phase φ. Hence, the phase sensitivity $\Delta \varphi_{\mathrm{I}}$ for the one-input-port interferometer is precisely

$$\Delta \varphi_{\mathrm{I}} = \frac{1}{\sqrt{N}}, \tag{17}$$

reproducing the earlier calculations [1,10,11]. This is just the classical Poisson noise associated with the random switching of the particles between interferometer branches.

I now proceed to the next section where I show how to use two ports with correlated inputs in order to reduce the noise to $\Delta \varphi = O(1/N)$.

III. CORRELATED-TWO-INPUT-PORT, FOCK-STATE INTERFEROMETER

Consider Fig. 2 for a Mach-Zehnder interferometer in which the particles are incident upon both input ports. From the work of Yurke with spin-$\frac{1}{2}$ particles, I have a phase sensitivity of $\Delta \varphi = O(1/N)$ when approximately equal numbers of particles enter each port in a highly correlated fashion, and the phase difference φ to be measured is set to zero. Since a zero phase can always be obtained by monitoring a null in the interference pattern with a feedback mechanism [11], without loss of generality, I may set $\varphi = 0$ in Eq. (8) for the phase variance $\Delta \varphi^2$ to obtain

$$\Delta \varphi^2|_{\varphi=0} = \frac{\Delta X^2}{|\langle \hat{Y} \rangle|^2}, \tag{18}$$

which greatly simplifies my calculation at the outset.

To see why this choice of $\varphi = 0$ for the measured phase reduces the noise, consider the difference and exchange operators $\hat{X}$ and $\hat{Y}$, Eqs. (9a) and (9b), respectively. Because $\hat{X}$ is the difference of two number operators, the quantities $\langle \hat{X} \rangle$ and $\langle \hat{X}^2 \rangle$ can be made to be small integer constants, independent of N, if input states of approximately equal number of particles $N/2$ are used. This is not true for $\langle \hat{Y} \rangle$ and $\langle \hat{Y}^2 \rangle$

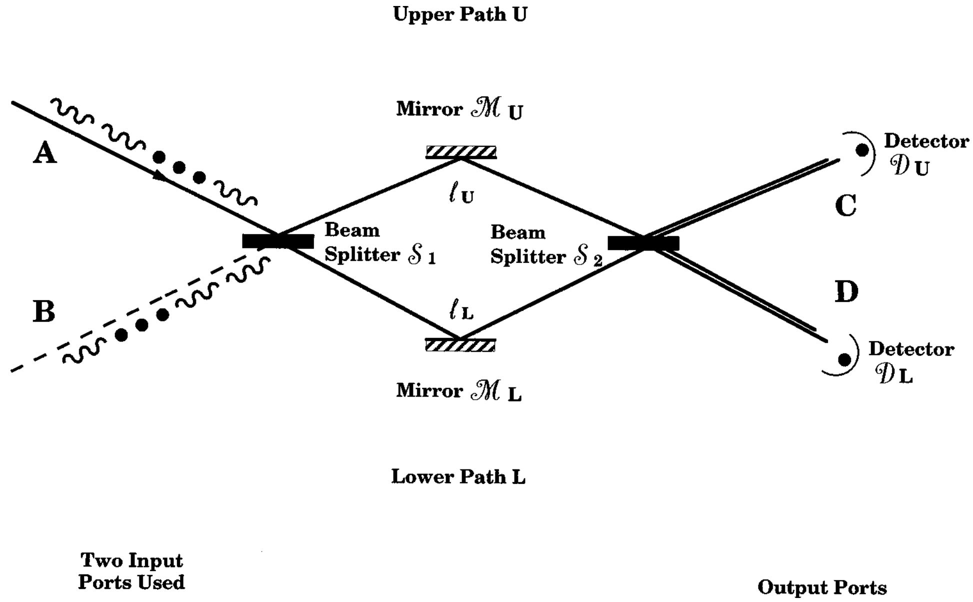

FIG. 2. Same as in Fig. 1, but now entangled correlated particles are incident on both input ports A and B.

since $\hat{Y}$ is the sum of two exchange operators. Hence, in the general expression for $\Delta\varphi^2$, Eq. (8), I wish to choose a value of φ that removes all factors of $\langle\hat{Y}\rangle$ and $\langle\hat{Y}^2\rangle$ from the numerator—in order to make $\Delta\varphi^2$ as small as possible. The choice $\varphi=0$ does this, since all the undesirable terms are multiplied by factors of $\sin\varphi$. Now I choose a highly correlated-input state with roughly equal numbers of particles entering each port. Again, using Yurke's result for spin-$\frac{1}{2}$ fermions as our guide, we make the ansatz

$$|\psi\rangle_{\rm II}\equiv\frac{1}{\sqrt{2}}\left\{\left|\frac{N+1}{2}\right\rangle_A\left|\frac{N-1}{2}\right\rangle_B+\left|\frac{N-1}{2}\right\rangle_A\left|\frac{N+1}{2}\right\rangle_B\right\}$$
$$\equiv\frac{1}{\sqrt{2}}\{|N_+\rangle_A|N_-\rangle_B+|N_-\rangle_A|N_+\rangle_B\},\qquad(19)$$

where the total number of incident particles, N, is taken to be odd and we define

$$N_\pm\equiv(N\pm1)/2.\qquad(20)$$

A physical picture of why I might expect such a state to exhibit such a small amount of phase noise can be found in the YMK paper and also the recent work by Kim, Pfister, Noh, Holland, and Hall [11]. The idea is to minimize the quantum uncertainty in su(2), angular momentum, Heisenberg uncertainty relations.

A simple argument for this choice of input state can be given as follows. Already I have chosen $\varphi=0$ in order to make the numerator of Eq. (8) for $\Delta\varphi^2$ as small as possible. I see that I can make this expectation value as small as possible again by choosing an input state with approximately equal numbers of particles near $N/2$ in each port—and with a quantum uncertainty over *which* particle number is associated with *which* port, as in Eq. (19). In this case, ΔX^2 will be a small constant integer—independent of N.

In order to evaluate the expression (18) for $\Delta\varphi^2|_{\varphi=0}$, I need the following expectation values:

$$\langle\hat{X}\rangle_{\rm II}=0,\qquad(21)$$

$$\langle\hat{Y}\rangle_{\rm II}=N_+,\qquad(22)$$

$$\langle\hat{X}^2\rangle_{\rm II}=(N_+-N_-)^2=1,\qquad(23)$$

$$\Delta X^2_{\rm II}=1,\qquad(24)$$

which are computed using the input state $|\psi\rangle_{\rm II}$, Eq. (19), and the operator definitions of Eqs. (9) and (10). An important point to note is that $\Delta X^2_{\rm II}$ is a constant on the order of unity independent of N, as desired. It is this fact that drives the change in the phase-noise power law. Inserting Eqs. (21)–(24) into Eq. (18) for the variance at $\varphi=0$ yields

$$\Delta\varphi^2_{\rm II}|_{\varphi=0}=\frac{1}{N_+^2},\qquad(25)$$

and hence

$$\Delta\varphi_{\rm II}|_{\varphi=0}=\frac{1}{N_+}=\frac{2}{N+1}=O(1/N),\qquad(26)$$

which shows the required Heisenberg-limited scaling of sensitivity with N. This is then the primary result of this paper. As it turns out, all the dependence on the fermionic or bosonic nature of the particles appears in the general Eq. (8) for $\Delta\varphi^2$ only in terms proportional to $\sin\varphi$. Hence, these statistics-dependent terms always vanish for the choice of $\varphi=0$, clinching my argument that the power law in Eq. (26) holds for either bosons or fermions—provided the correlated-input state has the form of $|\psi\rangle_{\rm II}$ in Eq. (19).

Also, by inspecting Eq. (23) for $\langle\hat{X}^2\rangle_{\rm II}$, I see that I could have chosen $N_+=(N+i)/2$ and $N_-=(N-j)/2$ so long as both i and j are small integers independent of N. Bouyer and Kasevitch have shown that the $1/N$ power-law sensitivity is robust, even in spite of small variations in i or j. However,

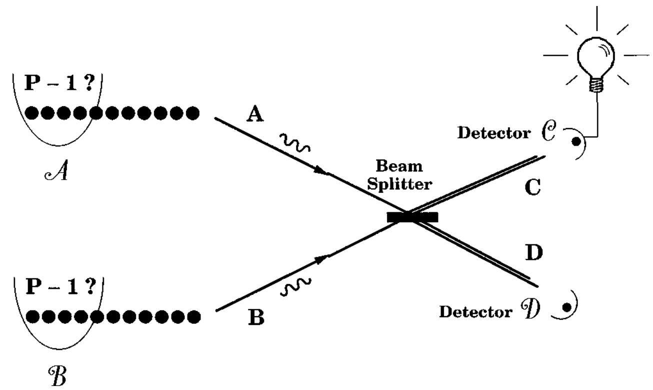

FIG. 3. In a dual Bose condensate, I start with P particles in each condensate, with a Hilbert product state $|\psi\rangle_{\text{before}}=|P\rangle|P\rangle$. Particles are allowed to be incident on input ports A and B to the beam splitter. The first click of either detector projects the dual condensate into an entangled state $|\psi\rangle_{\text{after}}=\{|P-1\rangle|P\rangle+|P\rangle|P-1\rangle\}/\sqrt{2}$, which is needed for Heisenberg-limited interferometry, as per Eq. (19), if I take $N=2P-1$.

there seems to be no obvious mechanism for producing such a correlated state, Eq. (19), using fermions [10].

For photons, one can make such states using four-wave mixers, degenerate-parametric amplifiers, or possibly parametric downconverters—at least in theory [11]. To my knowledge, this effect—the reduction of the phase-noise power law using correlated Fock states—has never been seen experimentally. The theoretical proposals to implement this noise reduction using nonlinear optical devices, such as four-wave mixers, apparently are difficult to implement in the laboratory. Perhaps other techniques of producing entangled photon states of type Eq. (19) might be employed—such as parametric down-conversion. The payoff in producing a bright source of such entangled photons would be enormous—particularly in the application to optical interferometric devices such as laser gyroscopes and laser-interferometer gravity wave detectors. What I would like to emphasize here is that recent experimental [18] and theoretical work [20,21] on the interference of Bose condensates seems to indicate that such an entangled state as Eq. (19) is feasibly manufacturable with atoms. On the experimental side is the recent demonstration of long-range coherence in the atom-laser experiment of Ketterle and co-workers. On the theoretical side are a series of papers demonstrating that the first-order phase coherence of a condensate can manifest itself in interference experiments in which the interfering condensates are treated by a Fock-state formalism—rather than a coherent-state analysis [20–23]. In particular, Javanainan and Yoo gave a convincing argument that a Fock-state approach suffices, which seems intuitive since the total number of atoms in the condensate is measurable—at least in principle [20]. Of particular interest for my argument here is the theoretical work of Wong, Collett, and Walls (WCW) [20], and Castin and Dalibard (CD) [23], that suggests that entangled states such as $|\psi\rangle_{\text{II}}$, Eq. (19), can be readily made by performing a selective measurement on two interfering condensates. I will show why this is so, in a simple gedanken experiment.

Consider Fig. 3, adapted from the CD paper [23]. Two Bose-Einstein condensates $\mathcal{A}$ and $\mathcal{B}$ with P particles each are output coupled via the upper and lower channels A and B, respectively, and made incident on a 50-50 beam splitter $\mathcal{S}$, as shown. The upper and lower output channels, U and L, respectively, lead directly to the upper and lower detectors $\mathcal{D}_U$ and $\mathcal{D}_L$, respectively. I will assume that the dual condensate system is in a Fock state, and hence the wave function is represented by the Hilbert space direct product

$$|\psi\rangle_{\text{III}}=|P\rangle_{\mathcal{A}}|P\rangle_{\mathcal{B}}. \tag{27}$$

The first click at either detector $\mathcal{D}_L$ or $\mathcal{D}_U$ projects the condensate wave function into

$$\begin{aligned}|\psi\rangle_{\text{IV}}&=\frac{1}{\sqrt{2}}\,(\hat{a}+\hat{b})|P\rangle_A|P\rangle_B\\&=\frac{1}{\sqrt{2}}\,\{|P-1\rangle_A|P\rangle_B+|P\rangle_A|P-1\rangle_B\},\end{aligned} \tag{28}$$

which is precisely the requisite correlated-input state $|\psi\rangle_{\text{II}}$ of Eq. (19), once I make the identification that $N=2P-1$. The entanglement arises because I cannot now—even in principle—say which condensate the detected atom came from, nor which atom in that condensate it was.

As Yurke has pointed out [10], it is difficult to see how one might construct such a correlated-input state as $|\psi\rangle_{\text{II}}$, Eq. (19), with massive fermionic particles such as neutrons. Here I show what an atom laser can do that an ordinary atomic beam cannot.

Although this gedanken method of preparing the requisite correlated state may seem a bit contrived, recently Spekkens and Sipe have shown that precisely such states can be prepared by forming a dual condensate in a double potential well with an adiabatically adjustable barrier height [24].

In the next section, I will demonstrate why using a correlated Fock—rather than a coherent—state is important for achieving this type of sensitivity by this mechanism.

IV. CORRELATED-TWO-INPUT-PORT, COHERENT-STATE INTERFEROMETER

The question arises, what is so important about using a Fock-state description to attain the phase-noise sensitivity of $\Delta\varphi=O(1/N)$? To answer this, I thought it would be instructive to try and replicate the argument of the preceding section using correlated *coherent* states of bosonic degenerate particles rather than Fock states. Of course, coherent-state phase-noise squeezing techniques have long been known to produce interferometers with a phase sensitivity of $\Delta\varphi=O(1/\overline{n})$, where $\overline{n}=|\alpha|^2$ is the mean photon number of a coherent state $|\alpha\rangle$ [3,4]. In addition, Knight [25], as well as Schleich and co-workers [26], have shown that a type of phase squeezing can occur when two coherent states of different particle number, but of the same phase, are superposed properly. Furthermore, recent experimental evidence supports the conclusion that a Bose-Einstein condensate ground state is a number-squeezed coherent state [27]. Hence, it seems plausible that the scheme used in the preceding section, to reduce the phase noise of a superposition of Fock states of different total particle number, might well be employed to a superposition of coherent states of different mean particle number. I will see if this type of correlated-input-port state manipulation would work instead to achieve the desired phase sensitivity of $\Delta\varphi = O(1/\overline{n})$.

First, I will calculate the sensitivity of a Mach-Zehnder interferometer in which a coherent state $|\alpha\rangle$ is incident on only one input port, as in the setup in Fig. 1. The phase sensitivity is still obtained from Eq. (8) for $\Delta\varphi^2$, with the expectation values taken with respect to the two coherent states

$$|\psi\rangle_{\rm V} = |\alpha\rangle_A |0\rangle_B, \tag{29}$$

where we have the vacuum coherent state $|0\rangle_B$ incident on the input port B. We need to compute the expectation values used in Eq. (8) for $\Delta\varphi^2$. I first consider

$$\begin{aligned}\langle \hat{X}\rangle_{\rm V} &= {}_B\langle 0|_A\langle\alpha|\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}|\alpha\rangle_A|0\rangle_B\\ &= {}_A\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle_A - {}_B\langle 0|\hat{b}^\dagger\hat{b}|0\rangle_B\\ &= |\alpha|^2 - 0 = \overline{n},\end{aligned} \tag{30}$$

which is the mean number of particles in $|\alpha\rangle$. In a similar fashion I have

$$\langle\hat{Y}\rangle_{\rm V} = 0, \tag{31}$$

$$\langle\hat{X}^2\rangle_{\rm V} = \overline{n}^2 + \overline{n}, \quad \langle\hat{Y}^2\rangle_{\rm V} = \overline{n}, \tag{32}$$

$$\langle\hat{X}\hat{Y}\rangle_{\rm V} = \langle\hat{Y}\hat{X}\rangle_{\rm V} = 0, \tag{33}$$

where I have made use of the coherent-state properties [28]

$$\langle\alpha|\alpha\rangle = 1, \tag{34a}$$

$$\langle\alpha|\hat{N}_A|\alpha\rangle = |\alpha|^2 \equiv \overline{n}, \tag{34b}$$

$$\langle\alpha|\hat{N}_A^2|\alpha\rangle = \overline{n}^2 + \overline{n}, \tag{34c}$$

and where I have defined the number operator $\hat{N}_A \equiv \hat{a}^\dagger\hat{a}$. Inserting Eqs. (31)–(33) into the general expression (8) for $\Delta\varphi^2$, I have

$$\Delta\varphi_{\rm V}^2 = \frac{[(\overline{n}^2+\overline{n}) - \overline{n}^2]\cos^2\varphi - 0 + \overline{n}\sin^2\varphi}{|\overline{n}\sin\,\varphi + 0|^2} = \frac{1}{\overline{n}\sin^2\varphi}, \tag{35}$$

which is minimal at $\varphi = \pi/2$ (and odd integer multiples thereof). Hence, I have

$$\Delta\varphi_{\rm V}|_{\varphi=\pi/2} = \frac{1}{\sqrt{\overline{n}}}, \tag{36}$$

the expected minimal uncertainty for the one-input-port, coherent-state, Mach-Zehnder interferometer.

Notice that this result is not independent of the choice of phase, as was the Fock-state phase noise $\Delta\varphi_{\rm I}$, Eq. (17). This is because the variance ΔX^2 is much different for the two types of states. In particular, $\Delta X_{\rm I}^2 = 0$ for the Fock input state, $|\psi\rangle_{\rm I} = |N\rangle_A|0\rangle_B$, because there are no number fluctuations in the Fock state $|N\rangle$. Hence, the $\cos^2\varphi$ factor in the numerator of $\Delta\varphi^2$, Eq. (8), vanishes—allowing the remaining factors of $\sin^2\varphi$ to cancel identically. In contrast, the variance $\Delta X_{\rm V}^2 = \overline{n} \neq 0$, due to the number uncertainty of the coherent state $|\alpha\rangle$. Hence, the $\cos^2\varphi$ does not vanish and the φ dependence does not cancel. I will now show how this additional number uncertainty affects the performance of a correlated-two-input-port, coherent-state device.

To mimic the entangled Fock state $|\varphi\rangle_{\rm II}$, Eq. (19), as much as possible, I construct a coherent correlated-input state $|\psi\rangle_{\rm VI}$ as

$$\begin{aligned}|\psi\rangle_{\rm VI} &\equiv \frac{1}{\sqrt{2}}\{|\alpha+\delta\rangle_A|\alpha-\delta\rangle_B + |\alpha-\delta\rangle_A|\alpha+\delta\rangle\}\\ &\equiv \frac{1}{\sqrt{2}}\{|\alpha_+\rangle_A|\alpha_-\rangle_B + |\alpha_-\rangle_A|\alpha_+\rangle_B\},\end{aligned} \tag{37}$$

where I assume $|\delta| \ll |\alpha|$ with $|\delta|^2 \cong 1$. Without loss of generality, I take α and δ to be real, which is equivalent to setting the absolute (and arbitrary) phase of the state $|\alpha_\pm\rangle \equiv |\alpha \pm \delta\rangle$ to zero. With this input state $|\psi\rangle_{\rm VI}$, I compute the following requisite expectation values:

$$\langle\hat{X}\rangle_{\rm VI} = \langle\hat{N}_A\rangle_{\rm VI} - \langle\hat{N}_B\rangle_{\rm VI} = 0, \tag{38}$$

which can be obtained easily by noting that the state $|\psi\rangle_{\rm VI}$, Eq. (37), is symmetric under the interchange of $A \leftrightarrow B$, and hence $\langle\hat{N}_A\rangle_{\rm VI} = \langle\hat{N}_B\rangle_{\rm VI}$, where I define $\hat{N}_A = \hat{a}^\dagger\hat{a}$ and $\hat{N}_B = \hat{b}^\dagger\hat{b}$, as before. Hence, this is analogous to $\langle\hat{X}\rangle_{\rm II} = 0$, as per Eq. (21) for the correlated Fock state $|\psi\rangle_{\rm II}$, Eq. (19). Next I have

$$\begin{aligned}\langle\hat{Y}\rangle_{\rm VI} &= \tfrac{1}{2}\{{}_B\langle\alpha_-|_A\langle\alpha_+| + {}_B\langle\alpha_+|_A\langle\alpha_-|\}(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})\\ &\quad\times\{|\alpha_+\rangle_A|\alpha_-\rangle_B + |\alpha_-\rangle_A|\alpha_+\rangle_B\}\\ &= 2(\alpha^2 - \delta^2) + 2(\alpha^2+\delta^2)e^{-\delta^2},\end{aligned} \tag{39}$$

where I have made use of $\alpha_\pm \equiv \alpha \pm \delta$, with α and δ real, and I have also used the coherent-state properties [28]

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle = \alpha^*\langle\alpha| = \langle\alpha|\hat{a}^\dagger, \tag{40a}$$

$$|\langle\alpha|\beta\rangle|^2 = |e^{-|\alpha|^2/2 - |\beta|^2/2 + \alpha^*\beta}|^2 = e^{-|\alpha-\beta|^2}, \tag{40b}$$

for arbitrary coherent states $|\alpha\rangle$ and $|\beta\rangle$. Hence, I see that $\langle\alpha_+|\alpha_-\rangle = \langle\alpha_-|\alpha_+\rangle = e^{-2\delta^2}$, with $\delta \cong 1$. Therefore I see that $\langle\hat{Y}\rangle_{\rm VI} = O(\alpha^2) = O(\overline{n})$, the leading term. This result should be compared to the correlated, two-port, Fock-state expectation of Eq. (22), namely, $\langle\hat{Y}\rangle_{\rm II} = N+1 = O(N)$. So far, so good—the analogy between the correlated two-port Fock and coherent states seems to be holding with the identification $N \leftrightarrow \overline{n}$.

Notice at this point that a choice of phase difference $\varphi = \pi/2$, as was made in the one-port coherent input calculation, does not minimize the phase noise for the correlated two-import coherent state. This is clear in the general expression, Eq. (8), for the phase variance $\Delta\varphi^2$, where now I have $\langle\hat{X}\rangle_{\rm IV} = 0$. A choice of $\varphi = \pi/2$ now would make $\Delta\varphi^2$ formally singular and maximize the phase noise. So I take $\varphi = 0$, using a similar argument as in the two-port Fock calculation. Hence, once again the expression for $\Delta\varphi^2$ simplifies to that of Eq. (18), and I see that I need to calculate only $\Delta X^2_{\rm VI}$. Using the fact that $|\psi\rangle_{\rm VI}$ is symmetric under $A \leftrightarrow B$ interchange, I may write

$$\begin{aligned}\langle \hat{X}^2\rangle_{\rm VI} &= \langle \hat{N}_A^2 - \hat{N}_A\hat{N}_B - \hat{N}_B\hat{N}_A + \hat{N}_B^2\rangle_{\rm VI}\\ &= 2\langle \hat{N}_A^2\rangle_{\rm VI} - 2(\hat{N}_A\hat{N}_B)_{\rm VI}\\ &= 2(\alpha^2+\delta^2+8\alpha^2\delta^2) + 2(\alpha^2-\delta^2)e^{-\delta^2}, \end{aligned} \tag{41}$$

where I have made use of the coherent-state properties, Eq. (40), and have used the fact that the particles are bosons—as they must be for a coherent state—when commuting the raising and lowering operators. Hence,

$$\begin{aligned}\Delta X^2_{\rm VI} &= \langle \hat{X}^2\rangle_{\rm VI} - \langle \hat{X}\rangle^2_{\rm VI} = \langle \hat{X}^2\rangle_{\rm VI} - 0\\ &= O(\alpha^2) = O(\overline{n}),\end{aligned} \tag{42}$$

and now I see a difference between this two-port coherent result and the two-port Fock-state result, Eq. (24), namely, $\Delta X^2_{\rm II} = 1$ for the two-port Fock state. The fact that the correlated-two-port, Fock-state variance for $\Delta X^2_{\rm II}$ is independent of N is directly a result of the fact that a Fock state has zero number fluctuations. The fact that the two-port, coherent-state variance $\Delta X^2_{\rm VI}$ is of order $\overline{n}$ is due to the fact that coherent-state number fluctuations are of this same order $\overline{n}$. This extra source of fluctuations will spoil my attempt to improve the power law for the two-port, coherent-state phase noise by this mechanism, as will be seen. Inserting Eqs. (39) and (42) for $\langle \hat{Y}\rangle_{\rm VI}$ and $\Delta X^2_{\rm VI}$, respectively, into the expression (18) for the phase variance $\Delta\varphi^2$, I get

$$\begin{aligned}\Delta\varphi^2_{\rm VI}\big|_{\varphi=0} &= \frac{(\alpha^2+\delta^2+8\alpha^2\delta^2)+(\alpha^2-\delta^2)e^{-\delta^2}}{[(\alpha^2+\delta^2)e^{-\delta^2}+2(\alpha^2-\delta^2)]^2} = O(1/\alpha^2)\\ &= O(1/\overline{n})\end{aligned} \tag{43}$$

and hence

$$\Delta\varphi_{\rm VI}\big|_{\varphi=0} = O(1/\sqrt{\overline{n}}), \tag{44}$$

for large $\overline{n}$, and there is no improvement in power-law scaling, over the coherent one-port result, Eq. (36).

I think this exercise with correlated-two-port, coherent states demonstrates the need for correlated Fock states, for reducing the power law of the phase noise via this correlated-two-input-port mechanism. I apply this Fock-state result in the next section to compare and contrast the sensitivity of one- and two-input-port optical and matter-wave interferometers operating as inertial gyroscopes.

V. ATOM-LASER GYROSCOPES

In this section I will review the theory of gyroscopy based on Mach-Zehnder interferometry. The treatment is similar to that of the Sagnac effect for massive particles found in the paper by Scully and me [1]. Consider in Fig. 4 an idealized circular interferometer used as a rotation sensor. From the figure, I can see easily that the path difference $\delta\ell$ for particles on the upper and lower paths, U and L, respectively, is given by $\delta\ell = 2r\Omega t$, where r is the radius of the circular interferometer, Ω the angular velocity about the center in the plane of the interferometer, and $t = \pi r/v$ is the transit time of the particle through the arms of the interferometer for particles of velocity v [29]. (A fully relativistic treatment, required for massless particles such as photons, can be found also in the paper by Gea-Banacloche and co-workers, Ref. [29].) From these expressions, I can compute the Sagnac phase difference φ as

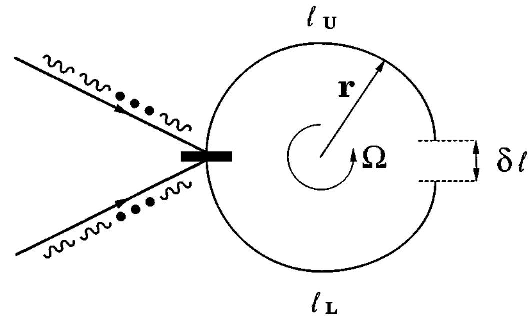

FIG. 4. Interferometer as gyroscope. The path difference between the upper and lower branches of the gyroscope is given by $\delta\ell = \pi r^2\Omega/v$, as shown. This difference is measured as a Sagnac phase shift φ in the operation of the gyroscope. In general, the area $A = \pi r^2$ may be replaced with that of an arbitrarily shaped gyroscope which has the same area.

$$\varphi = k(\ell_U - \ell_L) = \frac{2\pi r^2\Omega}{\lambda\!\!\!^{-} v} = \frac{2A\Omega}{\lambda\!\!\!^{-} v}, \tag{45}$$

where k is the particle wave number, ℓ_U and ℓ_L the path lengths of the upper and lower areas of the interferometer, respectively, $\lambda\!\!\!^{-} \equiv \hbar/mv$ is the circular de Broglie wavelength for a particle of mass m; and $A = \pi r^2$ is the area enclosed by the interferometer. Hence, the phase signal φ that I am trying to detect is given by

$$\varphi = \frac{2Am\Omega}{\hbar}, \tag{46}$$

and it turns out that this expression is independent of the actual interferometer shape—so long as A is the total area enclosed by the arms [7]. It also turns out that—after a fully relativistic treatment [29]—the exact same expression (46) holds for massless particles such as photons, as long as we define an effective photon ''mass'' m_γ by

$$m_\gamma = \frac{\hbar\omega}{c^2}. \tag{47}$$

Using this effective mass, I see that the ''mass'' of an optical photon corresponds to only a few electron volts, while the mass of an atom is on the order of 10^3 MeV. It is this mass-enhancement factor that is largely responsible for the greatly increased sensitivity of matter-wave over light-wave gyroscopes.

The use of one-port, atom-beam, atomic matter-wave interferometers has been demonstrated by Pritchard and co-workers [30] and also by Kasevich and co-workers [31]. Pritchard and co-workers use material gratings for the atom-beam splitters and mirrors, and they measure the $1/\sqrt{N}$

TABLE I. Compared and contrasted are different properties of one- and two-port matter-wave and optical gyroscopes in the terms of their sensitivity to phase differences—or equivalently—rotation rates. We see that the high mass of atoms initially contributes an increase of sensitivity of 10^{10}, but that the low atomic beam intensity, compared to photon beams, removes some of this advantage, as does the reduced number of round-trips possible in an atom interferometer.

	Matter	Laser	One-port atom-to-light factor	Two-port matter-to-light factor	Two-port to one-port atom	Two-port to one-port light	Two-port atom to one-port light
Mass factor	10^4 MeV	1 eV	10^{10}	10^{10}	1	1	10^{10}
Flux	$N=10^{12}\ \dfrac{\text{particles}}{\text{sec}}$	$N=10^{16}\ \dfrac{\text{photons}}{\text{sec}}$	$\sqrt{10^{12}/10^{16}}=10^{-2}$	$\dfrac{10^{12}}{10^{16}}=10^{-4}$	$\dfrac{10^{12}}{\sqrt{10^{12}}}=10^{6}$	$\dfrac{10^{16}}{\sqrt{10^{16}}}=10^{8}$	$\dfrac{10^{12}}{\sqrt{10^{16}}}=10^{4}$
Round-trips	1	10^4	10^{-4}	10^{-4}	1	1	10^{-4}

power law for phase sensitivity, Eq. (17), as I have predicted for a one-input-port device. Kasevich and co-workers demonstrate the best absolute atomic matter-wave sensitivity to rotation seen to date, namely 2×10^{-8} (rad/s)/$\sqrt{\text{Hz}}$. (The Earth's rotation rate is $\Omega_E=7.3\times10^{-5}$ rad/s.) In this section, I would like to estimate how much better I could expect a comparable, correlated-two-input-port device to perform.

If N is the total number of particles passing through the gyroscope in unit time, then I can write

$$N=JT, \tag{48}$$

where J is the particle flux and T the measurement or integration time. Combining Eq. (46) for the phase with this expression (48), I can write two expressions for the minimum detectable rotation rate Ω; one for a one-port device, and a second for a correlated two port, using Eqs. (17) and (26), respectively,

$$\Omega_{\text{one-port}}=\frac{\hbar}{2Am}\frac{1}{\sqrt{JT}}=\frac{\hbar}{2Am\sqrt{N}}, \tag{49}$$

$$\Omega_{\text{two-port}}=\frac{\hbar}{Am}\frac{1}{JT}=\frac{\hbar}{AmN}, \tag{50}$$

where A is the gyroscope area and m the particle mass, as before. These expressions hold for photons if I take $m=m_\gamma$, Eq. (47), and also identify the flux J with the optical power P via

$$J=\frac{P}{\hbar\omega}. \tag{51}$$

Although these formulas are useful to calculate the minimum detectable rotation rate for a given device, to compare one-input-port to two-input-port devices, it is handy to leave Eqs. (49) and (50) in terms of N instead of JT. For an integration time of 1 sec, Table I compares and contrasts typical mass-enhancement factors for an atom matter wave over an optical gyroscope [1]. I show that, from the mass-enhancement factor alone, an increase in sensitivity of 10^4 can be expected by using atoms rather than photons in an equivalent one-port device. In the fourth column I show only an enhancement of 10^2 can be expected for a two-port atom matter wave over an optical correlated-two-input device, due to the change in the scaling laws with N. However, in the fifth column, I note that a two-port atom-laser gyroscope can be 10^6 times more sensitive than an equivalent one-port atom-beam gyroscope.

Because of the higher photon flux in an optical interferometer, I show in the sixth column of Table I that a correlated two-port optical gyroscope is 10^8 times more sensitive than an equivalent optical one-port gyroscope. This fact alone should serve as impetus to the quantum optics community to try and find an all-optical implementation of a correlated-two-port, photon interferometer using four-wave mixing, parametric down-conversion, or some other optically nonlinear process [11].

Finally, in the last column, I contrast a correlated-two-input-port atom gyroscope with a comparable, one-input-port photon gyroscope. I show that an amazing ten orders of magnitude increase in sensitivity can be expected. As one-port optical gyroscopes become more, and more limited by the shot-noise scaling law of $\Delta\varphi=1/\sqrt{N}$—as opposed to other technical sources of noise—it becomes clear that correlated input-port gyroscopy is a field that warrants further serious theoretical and experimental investigation.

VI. SUMMARY

In this paper I have given a very general proof that a correlated-two-input-port, Fock-state, Mach-Zehnder interferometer has a phase detection sensitivity $\Delta\varphi$ that scales asymptotically as $\Delta\varphi=O(1/N)$, Eq. (26), where N is the number of particles passing through the device in unit time. This is to be compared with the usual shot-noise limit of $\Delta\varphi=1/\sqrt{N}$, Eq. (17), that is the best one can do with a one-port device. The result is independent of the particle statistics, and applies equally well to bosons or fermions, so long as they are amenable to a Fock-state treatment. This treatment can always be applied to bosons, and the application to fermions entails the caveat that only one fermionic particle be in the interferometer at a time, so that the Pauli-exclusion principle is obeyed. Important in obtaining this increased sensitivity is the use of a correlated, entangled, Fock input state of the form given in Eq. (19). It is difficult to imagine how such entangled input states can be made using fermions. However, for photons, correlated states such as this can be made using such processes as four-wave mixing, parametric down-conversion, or perhaps some other nonlinear optical process [11].

For massive atomic bosonic particles, I have shown that such entangled input states, Eq. (19), can in principle be generated from suitably prepared dual Bose condensates. This idea has additional support from other recent theoretical papers on interference between condensates [17,24]. I have also shown that an analogous enhancement of sensitivity does *not* occur for a correlated-two-input-port, interferometer if the inputs are in comparable coherent states. The important point that Bose condensates are in fact amenable to a Fock-state—rather than a coherent-state—treatment has been recently advanced rather forcefully in a number of theoretical papers [20–23,32]. The use of a dual condensate in the recent atom-laser experiments of Ketterle and co-workers allows me to conjecture that a suitably prepared atom-laser source could provide the necessary entangled input state, Eq. (19), needed to change the power-law scaling. For this reason, I have adopted the not-so-whimsical moniker of the ''atom-laser gyroscope,'' as I used in the title of this work.

Finally, in the penultimate section above, I compared and contrasted one- and two-input-port matter- and light-wave interferometers, used as gyroscopes. As summarized in Table I: (1) a two-port atom-laser device can be 10^6 times more sensitive than a comparable one-port atom-beam gyroscope; (2) a two-port optical interferometer can be 10^8 times more sensitive than a one-port photon gyroscope; and (3) a two-port atom-laser gyroscope can be an amazing ten orders of magnitude more sensitive to rotations than a comparable one-port optical gyroscope.

Current state-of-the-art one-port optical gyroscopes are operating close to their shot-noise limit of sensitivity of $\Delta\varphi = 1/\sqrt{N}$. A breakthrough in device sensitivity to inertial effects can be had by utilizing atom waves, correlated-input-ports, or both.

As I mentioned in the Introduction, the mathematics of N particles passing through a Mach-Zehnder interferometer is formally isomorphic to that of N two-level atoms passing through a Ramsey interferometer. Hence, entangled states that improve the signal-to-noise ratio will do so for both types of interferometers. However, implementing this type of procedure experimentally is, of course, much different for each of these scenarios. Recently, Ekert and co-workers have considered applying quantum computing techniques of entanglement generation and decoherence error correction to the Ramsey interferometer [33]. For example, entangled states as in Eq. (19) can be obtained for a collection of N two-level atoms by applying a sequence of N controlled-NOT gates to the initial Fock ground state $|N/2\rangle|N/2\rangle$. Recent work by Cerf, Adami, and Kwiat seems to indicate that all-optical quantum computing techniques such as error correction may be applied to the optical Mach-Zehnder interferometer, by treating the device as only one of many elements in an extended, optical, quantum-logic circuit [34].

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