Thermal dephasing and the echo effect in a confined Bose-Einstein condensate

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Thermal fluctuations of the normal component induce dephasing—reversible damping of the low-energy collective modes of a confined Bose-Einstein condensate. The dephasing rate is calculated for the isotropic oscillator trap, where the Landau damping is expected to be suppressed. This rate is characterized by a steep temperature dependence. It is weakly amplitude dependent, and is sensitive to the total number of atoms in the trap. The value of the rate belongs to the range of the damping rates observed by Jin *et al.* [Phys. Rev. Lett. **77**, 420 (1996)]. We suggest that a reversible nature of the damping caused by the thermal dephasing in the isotropic trap can be tested by the echo effect. A reversible nature of the Landau damping is also discussed, and a possibility of observing the echo effect in an anisotropic trap is considered as well. The parameters of the echo for the isotropic trap are calculated in the weak echo limit. Results of the numerical simulations of the echo are also presented. [S1050-2947(98)02906-0]

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I. INTRODUCTION

Properties of trapped ultracold atomic gases demonstrating the phenomenon of Bose-Einstein condensation [1–3] have been attracting much attention. A confined geometry of the cloud leads to circumstances under which new phenomena, such as, e.g., a specific scaling in the thermodynamical properties [4], and two-step condensation [5] can be anticipated. It was also suggested that the phase of the confined condensate should demonstrate the phenomenon of collapses and revivals [6] and the phase diffusion effect [7].

Recently it was discussed in Refs. [8,9] that the dynamical quantum behavior of an atomic cloud consisting of a finite number of bosons is very different from the properties of infinite systems. Specifically, the amplitude of a normal mode of the confined condensate should demonstrate quantum dephasing, which results in an apparent damping of the mode even at zero temperature T. The rate of such damping is determined by the interparticle interaction, and turns out to be linearly dependent on the mode amplitude, if it is small [8,9]. A possibility of the mode revival has also been predicted [8,9]. A current experimental situation does not allow us to resolve this effect because of the presence of the normal component, which introduces a substantial damping on its own.

Damping of the normal modes at finite T of the confined condensate was experimentally studied in Refs. [10] and [11]. Recently, it was stressed by Pitaevskii [12] (see also Refs. [13–16]) that the damping in the trap containing condensate is essentially Landau damping (LD), which is collisionless in nature. In other words, it is the dephasing of the collective modes rather than their irreversible dissipation [17]. It is worth noting that the reversible nature of the LD in the classical uniform plasma can be revealed in the plasma echo effect [17].

Theoretical approaches [13–16] are based on applying the standard results of the perturbation theory developed for an infinite and uniform medium. Accordingly, a discrete structure of the quasiparticle spectrum is tacitly replaced by an effective continuum [13–16]. In recent work [18], it was emphasized that the existence of the LD is directly related to the presence of randomness in the spectrum of the anisotropic traps. Conversely, in the isotropic trap the LD should be suppressed because the quasiparticle spectrum is regular [18].

In this paper, we suggest that in a confined condensate where the LD is suppressed, the damping of the collective modes can still be observed. Specifically, thermal fluctuations of the population factors of the normal component induce a reversible dephasing of the collective oscillations. The nature of such a damping is similar to that of the dephasing observed in quantum dots (see Ref. [19]). We show that, while being essentially zero for a traditional uniform condensate, the rate of such a dephasing $1/\tau_d$ in the trap containing 10^3–10^4 atoms can be comparable to the rate of the damping observed experimentally in Ref. [10].

We also suggest that the reversibility of the damping in a confined condensate can be tested in a kind of phonon echo experiment (see Ref. [20]), when two consecutive external pulses imposed on the trap induce a third pulse—the echo—at a time approximately equal to twice the time interval between the first two pulses. In this paper we analyze the case of the isotropic trap, where the LD is not expected to be an important mechanism of dissipation. In a future work we will investigate the echo in strongly anisotropic traps, where the main mechanism of damping is the LD.

II. THERMAL FLUCTUATIONS AND DEPHASING IN THE ATOMIC TRAP

In Refs. [8,9] it was shown that a collective mode of the confined condensate should exhibit a dephasing caused by the interparticle interaction. In the case $T=0$, the dephasing is produced by the particles forming the mode. Below, we will consider a similar effect caused by the interaction between a low-energy collective mode and the quasiparticles forming a normal cloud.

Let us discuss general reasons for the thermally induced dephasing in the presence of the interparticle interaction. We will especially clarify why this effect is not significant for an

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infinite condensate, where the only cause of the dephasing should be the LD. The following analysis is based on the approach suggested by Pitaevskii in Ref. [9] for the case $T=0$. We extend this analysis for $T\neq 0$.

If an external modulation at some frequency ω_0 has excited a system, the many-body time-dependent wave function $|t\rangle$ constructed in terms of the quasiparticle states $|N_0,N_1,\ldots\rangle$ acquires the form

$$|t\rangle=\sum_{N_0,N_1,\ldots} C_{N_0,N_1,\ldots}e^{-iE_{N_0,N_1,\ldots}t}|N_0,N_1,\ldots\rangle, \tag{1}$$

where N_n is a population number for the nth ($n=0,1,2,\ldots$) quasiparticle state; $E_{N_0,N_1,\ldots}$ stands for the energy of the state $|N_0,N_1,\ldots\rangle$, and $C_{N_0,N_1,\ldots}$ denotes the normalized coefficients ($\Sigma_{N_0,N_1,\ldots}|C_{N_0,N_1,\ldots}|^2=1$). For the case of weak interaction between quasiparticles, the external drive, which is in resonance with the energy of, e.g., the zeroth quasiparticle state, creates a coherent mixture of the quasiparticles in the zeroth state and does not significantly affect the other states. Accordingly, one can factorize $C_{N_0,N_1,\ldots}$ as

$$C_{N_0,N_1,\ldots}=C^{(0)}_{N_0}C^{(1)}_{N_1}C^{(2)}_{N_2},\ldots, \tag{2}$$

and assume that [9]

$$|C^{(0)}_{N_0}|^2=\frac{\overline{N}_0^{N_0}}{N_0!}\exp(-\overline{N}_0), \tag{3}$$

with the rest of the C coefficients corresponding to the number states ($C^{(n)}_{N_n}=1$ for some $N_n=N'_n$, and $C^{(n)}_{N_n}=0$ otherwise). In Eq. (3) $\overline{N}_0$ is given by the value of the classical amplitude of the resonant mode [8,9].

The expectation value of the single-particle operator $A(t)$, which changes the number of quasiparticles in the given state by 1, acquires a resonant contribution from the zeroth state. This is

$$\langle t|A|t\rangle=\sum_{N_0} C^{(0)*}_{N_0+1}C^{(0)}_{N_0}\overline{A}\exp[i(E_{N_0+1,N_1,\ldots}-E_{N_0,N_1,\ldots})t]+\text{c.c.}, \tag{4}$$

where $\overline{A}$ is a corresponding single-particle matrix element. In the case of a weak interaction between quasiparticles, one can expand the energy in accordance with the Landau theory of quantum liquids as

$$E_{N_0,N_1,\ldots}=\sum_n \omega_nN_n+\tfrac{1}{2}\sum_{mn} g_{mn}N_mN_n+o(N_iN_jN_k), \tag{5}$$

where the coefficients g_{mn} arise due to interaction between quasiparticles.

At finite T, solution (4) should be averaged over the initial values of N_n with $n>0$, distributed in accordance with a thermal ensemble. Substituting Eq. (5) into Eq. (4), and neglecting the term $\sim N_0^2$ significant for very small T only (see Refs. [8,9]), one obtains

$$\langle t|A|t\rangle_T=\overline{A}e^{i\omega_0t}\sum_{N_0} C^{(0)*}_{N_0+1}C^{(0)}_{N_0}\times\left\langle\exp\left(it\sum_{n>0} g_{0n}N_n\right)\right\rangle_T+\text{c.c.}, \tag{6}$$

with $\langle\cdots\rangle_T$ denoting the thermal average with respect to the population factors N_n.

In what follows we will assume that N_n are distributed in accordance with the grand canonical distribution. As will be discussed below, this assumption is valid as long as the mean number of atoms in the condensate N_c is sufficiently large. Thus, after straightforward calculations, one finds

$$\left\langle\exp\left(it\sum_{n>0} g_{0n}N_n\right)\right\rangle_T=\exp\left(-\sum_{n>0}\ln\frac{1-\exp[ig_{0n}t-\omega_n/T]}{1-\exp(-\omega_n/T)}\right). \tag{7}$$

We note that in a large system the matrix elements g_{0n} are scaled as $g_{0n}\sim 1/V$ by the effective volume V of the system. Accordingly, in the formal limit $V\to\infty$, one should expand $\exp(ig_{0n}t)$ in Eq. (7) up to the first term $\sim 1/V$ only. Then the smallness of $1/V$ will be compensated for by the summation $\Sigma_n\sim V$. This results in

$$\left\langle\exp\left(it\sum_{n>0} g_{0n}N_n\right)\right\rangle_T=\exp\left(it\sum_{n>0} g_{0n}\overline{N}_n\right), \tag{8}$$

where $\overline{N}_n$ denotes a Bose distribution of the noninteracting quasiparticles. As one can see, in the infinite system fluctuations of the numbers of the quasiparticles do not cause any dephasing. Instead, the frequency ω_0 experiences a thermal shift $\omega_0\to\omega_0+\Sigma_ng_{0n}\overline{N}_n$, and the only cause of the dephasing is the LD.

In the case of a finite system, one should keep higher-order terms in Eq. (7). As will be shown below, the dephasing rate in the oscillator trap contains a smallness $a/r_{\text{trap}}\ll 1$, where a and r_{trap} stand for the scattering length $\sim 10^{-7}$–10^{-6} cm, and some typical scale $\sim 10^{-4}$ cm associated with the trapping potential, respectively. Therefore, as long as $a/r_{\text{trap}}\ll 1$, one can neglect the higher terms $o(a^2/r^2_{\text{trap}})$. Such an approximation corresponds to expanding $\exp(ig_{0n}t)\approx 1+ig_{0n}t-g^2_{0n}t^2/2+o(g^3_{0n})$. This results in Eq. (7) being rewritten as

$$\left\langle\exp\left(it\sum_{n>0} g_{0n}N_n\right)\right\rangle_T=\exp\left(i\sum_{n>0} g_{0n}\overline{N}_nt-t^2/\tau_d^2\right), \tag{9}$$

where the notation

$$\frac{1}{\tau_d}=\sqrt{\frac{1}{2}\sum_n g^2_{0n}\overline{N}_n(1+\overline{N}_n)} \tag{10}$$

for the dephasing rate is introduced.

Equations (9) and (10) indicate that a collective mode of the confined atomic condensate should exhibit a dephasing induced by thermal fluctuations of the normal component. Below, we suggest a model of dephasing of a collective mode of the confined condensate in the Thomas-Fermi limit, and calculate the value of τ_d for the isotropic trap.

III. ADIABATIC EFFECTIVE ACTION FOR LOW-ENERGY COLLECTIVE MODES

In order to obtain $1/\tau_d$, one should find the coefficients g_{0n}, and perform the summation in Eq. (10). An explicit expression for g_{0n} can be derived from a many-body Hamiltonian H taken in the standard form

$$H=\int d\mathbf{r}\Psi^{\dagger}(H_1-\mu)\Psi+H_{\rm int},$$

$$H_1=-\frac{\hbar^2}{2m}\Delta+\sum_{i=1,2,3}\frac{1}{2}m\omega_i^2r^2,$$

$$H_{\rm int}=\frac{1}{2}u_o\int d\mathbf{r}\ \Psi^{\dagger}\Psi^{\dagger}\Psi\Psi, \tag{11}$$

where the Bose operators Ψ and $\Psi^{\dagger}$ obey the usual Bose commutation rule; μ is the chemical potential; ω_i denotes three frequencies characterizing the trapping potential; and the interaction constant $u_o=4\pi\hbar^2a/m$ is expressed in terms of the scattering length a and the atomic mass m. As usual, in the presence of the condensate, one uses the conventional representation

$$\Psi=\Phi_c+\Psi', \tag{12}$$

where Φ_c is a classical condensate wave function giving the condensate density $n_c=|\Phi_c|^2$, and normalized to the number N_c of atoms in the condensate; Ψ' stands for the excitation part. For the latter, the Bogolubov representation should be employed as

$$\Psi'=\sum_n\ (U_na_n+V_n^*a_n^{\dagger}), \tag{13}$$

where a_n destroys a quasiparticle on the level having the energy E_n, and (U_n,V_n) is an eigenvector of the linearized Bogolubov equations

$$E_nU_n=H_1'U_n+u_0\Phi_c^2V_n,\quad -E_nV_n=H_1'V_n+u_0\Phi_c^{2*}U_n,$$
$$H_1'=H_1+2u_0|\Phi_c|^2-\mu. \tag{14}$$

Expressed in terms of the quasiparticle operators a_n and $a_n^{\dagger}$, Hamiltonian (11) acquires the form

$$H=\sum_n\ E_na_n^{\dagger}a_n+H_{\rm int}, \tag{15}$$

where the interaction part $H_{\rm int}$ contains the terms

$$H_{\rm int}'=\sum_{mnk}\ (g_{mnk}a_m^{\dagger}a_na_k+\text{H.c.})+\sum_{mnkl}\ g_{mnkl}a_m^{\dagger}a_n^{\dagger}a_ka_l, \tag{16}$$

which could be identified with the terms $\sim N_mN_n$ in Eq. (5). Specifically, performing calculations in the first and second orders of the perturbation theory with respect to $H_{\rm int}$ given by Eqs. (11)–(13), one finds the coefficients g_{kn} in Eq. (5) as

$$g_{kn}=4g_{knkn}+2\sum_m\left[\frac{|g_{mnk}|^2}{E_n+E_k-E_m}-\frac{|g_{knm}|^2}{E_n-E_k+E_m}-\frac{|g_{nkm}|^2}{E_k-E_n+E_m}\right], \tag{17}$$

where the coefficients g_{knkn} and g_{mnl} are expressed explicitly in terms of the Bogolubov amplitudes in Eq. (13) as

$$g_{knkn}=\frac{u_0}{2}\int d\mathbf{r}[(|U_k|^2+|V_k|^2)(|U_n|^2+|V_n|^2)+(U_k^*V_kV_n^*U_n+\text{c.c.})] \tag{18}$$

and

$$g_{mnk}=u_0\int d\mathbf{r}\ \Phi_c[U_k(U_m^*U_n+V_m^*V_n+U_m^*V_n)+V_k(U_m^*U_n+V_m^*V_n+V_m^*U_n)], \tag{19}$$

and Φ_c is taken real.

We note that in the case of a continuum or quasicontinuum spectrum [18], the sum in Eq. (17) makes the main contribution to the imaginary part corresponding to the LD in the lowest order of the perturbation theory [13–15,18]. In the isotropic trap, this imaginary contribution is not significant [18], which implies that the LD is suppressed. Therefore, the real value of g_{kn} [Eq. (17)] could be employed in Eq. (10) for calculating $1/\tau_d$. Unfortunately, such an approach leads to an expression $1/\tau_d$ which formally diverges at low energies, despite the natural expectation that the main contribution to the dephasing is produced by high-energy excitations. Accordingly, rate (10) acquires the incorrect T dependence $1/\tau_d\sim\sqrt{T}$ for $T>\mu$. In fact, this divergence can be eliminated by a proper renormalization of the vertex in the low-energy region, where an adequate description relies on the hydrodynamical approach [21]. In order to solve this problem for the low-energy collective modes, we employ a simple scaling procedure which yields a description of the effect of thermal dephasing in closed form at finite T in the limit of large N.

First, we note that the dephasing effect discussed here is an adiabatic process when the high-energy component follows the evolution of the low-energy collective mode without dissipating energy. As discussed in Ref. [22], the low-energy modes at $T=0$ can be viewed as a time-dependent scaling $r_i\rightarrow r_i/b_i$ of the coordinates r_i by some time-dependent scaling variables $b_i=b_i(t)$. Furthermore, it has been shown [22] that, if one ignores the kinetic-energy term, such a scaling approach is exact for any given initial state of the many-body wave function. As will be seen below, this implies that no dephasing of the low-energy scaling modes should occur in such an approximation. The dephasing is induced by the kinetic-energy terms, which are, however, small in the Thomas-Fermi limit [22]. Such smallness implies that one can still consider the scaling variables b_i as proper collective degrees of freedom, whose dynamics is modified in the presence of the kinetic terms. Below, we will derive an adiabatic classical action for the b variables. In order to do this, we treat the Ψ operator as a classical field by means of considering the a and $a^\dagger$ operators in Eq. (13) as c numbers. Then we employ the scaling ansatz [22]

$$\Psi(r_i,t)\to\frac{e^{i\varphi}}{\sqrt{b_1b_2b_3}}\Psi\left(\frac{r_i}{b_i},t'(t)\right),\quad \varphi=\frac{m}{2\hbar}\sum_i\frac{\dot{b}_i}{b_i}r_i^2, \tag{20}$$

where t' (t) is some function of time determined in terms of the variables b_i and their time derivatives $\dot{b}_i$. In what follows, we assume that the scale invariant shape of Ψ is given by Eqs. (12)–(14) obtained for $b_i=1$. In this manner we eliminate the nonadiabatic processes induced due to $\dot{b}_i\neq 0$. Consequently, one can derive an effective classical action $S_b=S[b_i,\dot{b}_i]$ for the variables b_i by means of performing the scaling transformation (20) in the full classical action

$$S=\int dt\left\{\int d\mathbf{r}\,\frac{i}{2}\,(\Psi^*\dot{\Psi}-\text{H.c.})-H\right\}. \tag{21}$$

Note that a substitution of forms (12) and (13) is to be done in Eq. (11) and, then, in Eq. (21). Then the off-diagonal products $a_n^*a_m$ and a_na_m $(m\neq n)$ should be eliminated in the adiabatic approximation because they oscillate in time. The diagonal terms $a_n^*a_n$, which do not oscillate, should be retained and then identified with the population factors. Let us denote the action obtained as a result of such a procedure as $S_b=\langle S\rangle$. Then we find

$$S_b=\int dt\left\{\sum_i\left[\frac{m}{2}R_i(\dot{b}_i^2-\omega_i^2b_i^2)-\frac{P_i}{2b_i^2}\right]-\frac{G}{b_1b_2b_3}\right\}, \tag{22}$$

where the notations

$$R_i=\left\langle\int d\mathbf{r}\,\Psi^*r_i^2\Psi\right\rangle,\quad P_i=\left\langle\int d\mathbf{r}\frac{\hbar^2}{m}\nabla_i\Psi^*\nabla_i\Psi\right\rangle,$$

$$G=\left\langle\frac{u_o}{2}\int d\mathbf{r}\,\Psi^*\Psi^*\Psi\Psi\right\rangle \tag{23}$$

are introduced. Taking into account that these quantities do not depend on b_i, one can vary S_b with respect to b_i, and obtain the classical equations of motion

$$\ddot{b}_i+\omega_i^2b_i-\frac{G}{mR_i}\frac{1}{b_ib_1b_2b_3}-\frac{P_i}{mR_i}\frac{1}{b_i^3}=0,\quad i=1,2,3. \tag{24}$$

We note that the procedure suggested above extends the variational approach [23] to the case of finite temperatures. Equations (24) [see also Eqs. (9) of Ref. [23]] reproduce correctly the $T=0$ low-energy spectrum obtained in Ref. [24] for the isotropic trap characterized by the relation $\omega_1=\omega_2=\omega_3=\omega_{\text{ho}}$. Indeed, in the case of zero temperature one should take the means (23) over the condensate state by setting $a_n=a_n^*=0$ in Eq. (13), and, accordingly, taking $\Psi=\Phi_c$ in Eq. (23). Then, the virial theorem yields

$$m\omega_i^2R_i^{(c)}-P_i^{(c)}-G^{(c)}=0, \tag{25}$$

with the superscript (c) indicating that the means in Eq. (23) are taken over the condensate state. Employing this relation in Eqs. (24), one finds

$$\ddot{b}_i+\omega_i^2b_i-\omega_i^2(1-\xi_i^{(c)})\frac{1}{b_ib_1b_2b_3}-\xi_i^{(c)}\omega_i^2\frac{1}{b_i^3}=0, \tag{26}$$

where the notation

$$\xi_i^{(c)}=\frac{P_i^{(c)}}{m\omega_i^2R_i^{(c)}} \tag{27}$$

is introduced. We note that $\xi_i^{(c)}$ determines the ratio of the averages of the kinetic energy to the harmonic potential energy both taken for the i direction. Accordingly, for the case of the isotropic trap considered in Ref. [24], one can find $\xi_i^{(c)}=E_{\text{kin}}/E_{\text{ho}}$, where E_{kin} and E_{ho} stand for the total kinetic and the total harmonic energies, respectively. Then, from the linearized version of Eqs. (26), we reproduce the frequencies of the quadrupolar mode $\omega_Q=\sqrt{2}\omega_{\text{ho}}(1+E_{\text{kin}}/E_{\text{ho}})^{1/2}$ and the breathing mode $\omega_M=\omega_{\text{ho}}(5-E_{\text{kin}}/E_{\text{ho}})^{1/2}$ derived in Ref. [24] by means of the sum rule technique.

Note also that for $T\neq 0$, Eqs. (24) coincide with those derived in Refs. [22] in the Thomas-Fermi limit. This can be seen by means of setting the kinetic-energy terms $P_i=0$ in Eqs. (24), and performing the scaling transformation

$$b_i=\kappa_i\tilde{b}_i, \tag{28}$$

with some κ_i chosen in such a way as to make the solution $\tilde{b}_i=1$ the equilibrium one. Furthermore, it can be seen that in the isotropic two-dimensional (2D) case, when only two scaling variables b_1 and b_2 should be taken into account, the dependence on P_i can be eliminated by the scaling transformation (28), so that the frequency of the breathing mode does not depend on P_i. This implies that no dephasing of the 2D breathing mode should occur in this case, in accordance with the result of exact calculations of Ref. [25].

In order to simplify the following analysis of the $T\neq 0$ case, let us consider a breathing mode in the 3D isotropic trap. Thus we set $b=b_1=b_2=b_3$ and $b=\kappa\tilde{b}$ in transformation (20) as well as in the action S_b [Eq. (21)]. Then, varying $\delta S_b/\delta b=0$, we obtain the nonlinear equation describing the low-energy adiabatic dynamics of the breathing mode in the presence of the normal component:

$$\ddot{\tilde{b}}+\omega_{\text{ho}}^2\tilde{b}-\frac{\omega_{\text{ho}}^2}{\tilde{b}^4}+\xi\omega_{\text{ho}}^2\frac{1-b^4}{\tilde{b}^4}=0,\quad \xi=\frac{P}{mR\omega_{\text{ho}}^2\kappa^4}, \tag{29}$$

where $P=P_1+P_2+P_3$, $R=R_1+R_2+R_3$, and κ obeys the relation

$$\omega_{\text{ho}}^2-\frac{P}{m\kappa^4R}-\frac{3G}{m\kappa^5R}=0. \tag{30}$$

An explicit form of ξ can be found in the limit $P\to 0$, which corresponds to neglecting the kinetic energy of the system if compared with the interaction energy. Setting $P=0$ here, one finds κ, and then, resorting back to Eq. (29), obtains

$$\xi=\frac{P}{\omega_{\mathrm{ho}}^{2/5}(mR)^{1/5}(3G)^{4/5}}. \tag{31}$$

In order to express P, R, and G in terms of the products $a_n^* a_n$ which, as was mentioned above, should then be identified with the population factors $N_n=a_n^* a_n$ of the quasiparticles in the second quantized picture, one should use representations (12), (13), and (23). However, in the Thomas-Fermi limit valid for large numbers N_c of atoms in the condensate, one may neglect corrections to R, G, and N_c due to the excitations. Indeed, in the condensate state $R\sim G\sim N_c r_c^2$, where r_c stands for the condensate radius $r_c\sim N_c^{1/5}$ [26] in such a limit. Therefore, $R\sim G\sim N_c^{7/5}$. The kinetic term $P\sim N_c/r_c^2\sim N_c^{3/5}$. High-energy excitations produce changes δR, δG, δN_c, and δP of R, G, N_c, and P, respectively. These changes can be estimated as $\delta R\sim\delta P\sim N' T/\omega_{\mathrm{ho}}$, $\delta G\sim u_o n_c N'$ and $\delta N_c=-N'$, where N' stands for a fluctuation of the normal component. Therefore, the relative contributions $\delta R/R$, $\delta G/G$, $\delta N_c/N_c$, and $\delta P/P$ to the value of ξ are very different. Specifically, one finds $(\delta P/P)(R/\delta R)\sim N_c^{4/5}\gg 1$ and $(\delta P/P)(G/\delta G)\sim(\delta P/P)(N_c/\delta N_c)\sim N^{2/5}T/\omega_{\mathrm{ho}}\gg 1$, as long as $N_c a/r_{\mathrm{ho}}\gg 1$ and $T>\omega_{\mathrm{ho}}$, where r_{ho} stands for the oscillator radius associated with ω_{ho}. Thus, in calculating ξ in Eq. (31), one should take into account only the contribution due to δP, and take the values R and G determined for the condensate state by Eq. (23) for some mean value of N_c.

Finally, employing representations (12) and (13) in Eq. (23), and taking the means so that $\langle a_m^\dagger a_n\rangle=N_n\delta_{mn}$, one finds

$$\xi=\xi^{(c)}+\sum_n g_n N_n,\quad R_c=\int d\mathbf{r}\, r^2|\Phi_c|^2,$$

$$g_n=-\frac{\hbar^2}{m^2 R_c\omega_{\mathrm{ho}}^2}\int d\mathbf{r}(U_n^*\Delta U_n+V_n^*\Delta U_n), \tag{32}$$

where the virial relation (25) has been utilized, and $\xi^{(c)}$ denotes the contribution due to the kinetic energy of the condensate. In what follows we will neglect this term which does not depend on N_n.

The solution $\tilde{b}$ of Eq. (29) should be averaged over ξ represented by Eq. (32). Such an averaging can be understood as a thermal ensemble averaging over possible Fock states of the quasiparticles. This interpretation closely resembles the case of destructive measurements [10], when the initial conditions determined by N_n for each newly created condensate can vary from one condensate to another. In the case of nondestructive measurements, the averaging of the solution $\tilde{b}$ should be performed over a single many-body state, which is a mixed state rather than a pure Fock state with respect to N_n. In accordance with the ergodic hypothesis, such an averaging should give the same result as that obtained by means of thermal averaging as long as the number of quasiparticles is sufficiently large. In what follows we will not distinguish these two cases, and will employ the thermal averaging $\langle\cdots\rangle_T$ with respect to N_n. This averaging can be performed explicitly for the linearized solution of Eq. (29). Specifically, representing $\tilde{b}=1+\eta$ in Eq. (29) and keeping the terms linear in η, one finds

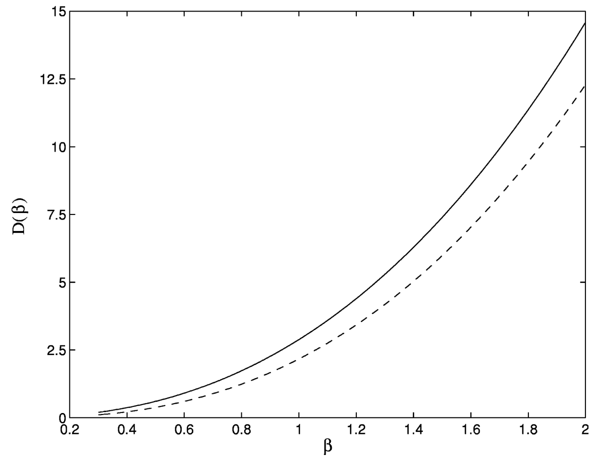

FIG. 1. The universal function $D(\beta)$: the solid line is the result of numerical calculations (A18) and (A19); the dashed line corresponds to the approximate formula (A28).

$$\langle\eta(t)\rangle_T=\eta_0\langle e^{i\omega_{\mathrm{ho}}(5-\xi)^{1/2}t}\rangle_T+\text{c.c.}\sim e^{i\overline{\omega}_M t-t^2/\tau_{dM}^2}+\text{c.c.},$$

$$\overline{\omega}_M=\sqrt{5}\,\omega_{\mathrm{ho}}-\frac{1}{2\sqrt{5}}\langle\xi\rangle_T, \tag{33}$$

where $\eta_0=\text{const}$ accounts for the initial condition $\tilde{b}(0)=1+\eta_0$, and the dephasing rate of the breathing mode is

$$\frac{1}{\tau_{dM}}=\omega_{\mathrm{ho}}\sqrt{\frac{1}{40}\sum_n g_n^2\overline{N}_n(1+\overline{N}_n)},\quad \overline{N}_n=\frac{1}{\exp\left(\frac{E_n}{T}\right)-1}, \tag{34}$$

with the coefficients g_n given by Eq. (32). Performing similar calculations for the quadrupolar mode, we find the relation $1/\tau_{dQ}=\sqrt{5/2}/\tau_{dM}$ for the dephasing rate of the quadrupolar mode. Taking into account Eqs. (32) and (34), we obtain an explicit expression for the rate $1/\tau_{dM}$ [Eq. (34)] in the WKB approximation [27] (see Appendix A) as

$$\frac{1}{\tau_{dM}}=\Gamma_M D(T/2\mu), \tag{35}$$

where the coefficient Γ_M is

$$\Gamma_M=\frac{35}{\sqrt{5\pi}}\left(\frac{r_{\mathrm{ho}}}{r_c}\right)^2\frac{a}{r_{\mathrm{ho}}}\omega_{ho},\quad r_{\mathrm{ho}}=\sqrt{\frac{\hbar}{m\omega_{\mathrm{ho}}}}, \tag{36}$$

and the universal dimensionless function $D(\beta)$ is defined in Appendix A (see Fig. 1), with the parameters r_c and μ given explicitly in Eq. (A2).

In the limits $\beta\gg 1$ $(T\gg 2\mu)$ and $\beta\ll 1$ $(T\ll 2\mu)$ the function $D(\beta)$ can be found explicitly [see Eqs. (A28) and (A32), respectively]. The current experimental situation is closer to the first case. It is convenient to express T in units of the transition temperature T_c of the Bose-Einstein condensation in the isotropic oscillator trap

$$T_c=\hbar\omega_{\rm ho}\left(\frac{N}{\zeta(3)}\right)^{1/3}, \tag{37}$$

where $\zeta(3)\approx 1.202$; N is the total number of the trapped atoms (for T not very close to T_c we set $N_c\approx N$). Then we find

$$\frac{1}{\tau_{dM}}=\frac{35\sqrt{0.3}}{15^{7/5}(\zeta(3))^{5/6}}\left(\frac{T}{T_c}\right)^{5/2}\left(\frac{r_{\rm ho}}{a}\right)^{2/5}N^{-17/30}\omega_{\rm ho} \tag{38}$$

from Eqs. (35)–(37), (A28) and (A29). Choosing the values $T/T_c=0.9$, $N=2\times(10^3–10^4)$ and $\omega_{\rm ho}=2\pi\times 200\ {\rm s}^{-1}$, $r_{\rm ho}=10^{-4}$ cm typical for the experiment [10], we obtain the rate $1/\tau_{dM}\approx 40–20\ {\rm s}^{-1}$. We note that these values are close to the damping rate observed in Ref. [10]. However, for the chosen parameters, $\beta=T/2\mu\approx 1.3$ which is, formally speaking, far from the requirement $\beta\gg 1$, insuring the validity of Eq. (38). Nevertheless, the above estimates remain valid. Indeed, evaluating the complete expressions (A18) and (A19) numerically (see Fig. 1) changes these estimates by only about 20%. Specifically, the rate becomes $\approx 50–25\ {\rm s}^{-1}$. For the lowest temperature achieved in the experiment [10], $T\approx 0.4T_c$, Eq. (38) becomes invalid because this temperature corresponds to $\beta\approx 0.6$. Accordingly, a numerical evaluation of $D(\beta)$ by means of Eqs. (A18) and (A19) and, then, a substitution of the result into Eq. (35) yields the rate $1/\tau_{dM}=8–4\ {\rm s}^{-1}$, which is also in the range obtained in Ref. [10].

We emphasize that in the anisotropic trap employed in Ref. [10] the damping is most likely to be caused by the LD [12–16,18], and not by the mechanism discussed above. The discussed mechanism in its pure form can be realized in the isotropic trap only. Therefore, a correspondence between the rates calculated above for the isotropic case and those measured in Ref. [10] for the anisotropic trap indicates that, while decreasing a degree of the trap anisotropy, the damping rate should practically stay unchanged despite the fact that the nature of the damping changes.

In the case $T\ll 2\mu$, one could use an explicit form [Eq. (A32)] for $D(\beta)$ and, correspondingly, find an explicit T dependence of rate (35). However, in this case the rate becomes so small that the mechanism of the quantum self-dephasing [8,9] comes into play.

It is interesting to investigate the dependence of the dephasing rate on the amplitude of the oscillations. It is worth noting that in the case $T=0$ such a dependence is very pronounced [8,9]. As will be seen below, the amplitude dependence at finite T is weak. Indeed, this dependence is due to the nonlinearity of the term $\sim\xi$ in Eq. (29). In the lowest order with respect to the initial value η_0 in Eq. (33), this dependence can be obtained by expanding Eq. (29) up to the terms $\sim\eta^2$ and η^3, and finding the correction to the frequency of the lowest harmonic in the order $\sim|\eta_0|^2$. Performing straightforward calulations (see Appendix B), and then averaging over the ensemble, we obtain

$$\frac{\tau_{dM}}{\tau_{dM}(A_1)}=1-\frac{7}{3}|A_1|^2, \tag{39}$$

the ratio of the rate $1/\tau_{dM}(A_1)$ determined in first order with respect to the amplitude $A_1=2\eta_0$ of the collective mode to the rate $1/\tau_{dM}$ in the zeroth order given by Eqs. (34) and (35). As one can see, the rate demonstrates a slow decrease as a function of the amplitude $A_1\ll 1$.

Here we have shown that the collective excitations of the confined Bose-Einstein condensate should demonstrate a dephasing caused by thermal fluctuations of the normal component. In Sec. IV we will discuss how this dephasing effect can be distinguished from irreversible dissipation experimentally.

IV. ECHO EFFECT IN A CONFINED BOSE-EINSTEIN CONDENSATE

The reversible nature of the damping can be tested in an echo experiment similar to the spin echo, photon echo, and phonon echo effects (see Ref. [20]). The nature of this effect can be briefly outlined as follows [20]. A short external pulse imposed on the system at the time $t=0$ excites a collective mode. The collective-mode amplitude decays due to dephasing as well as due to irreversible dissipation. Both processes are characterized by their typical rates $1/\tau_d$ and γ, respectively. The second pulse imposed at the time $t=\tau$ partly reverses in time the evolution of the system initiated by the first pulse. This implies a partial revival of the dephased amplitude at the time $t\approx 2\tau$. We note that the occurrence of the echo is a general property of the system, where irreversible damping is weaker than the dephasing. Thus a necessary condition for observing a distinct echo is $\tau_d<1/\gamma$ and $\tau_d<\tau<1/\gamma$.

Specific features of the echo depend on the details of the system. The time profiles of the responses, as Eq. (33) indicates, should be Gaussian in the case of the thermal dephasing discussed above. In the case of the LD these responses should be characterized by exponential relaxation. Presently available experimental data [10,11] do not allow the distinguishing of the Gaussian type damping from the exponential one [28]. In the next paper we will analyze the echo in the anisotropic confined condensate, where the main cause of the damping is the LD. Below we will study the situation in the isotropic trap, where the dephasing is caused by the thermal mechanism described above.

A relevant description for the case under consideration relies on Eq. (29) modified to incorporate the external drive as well as some possible irreversible dissipation. As discussed in Ref. [22], the external drive $\delta\omega^2(t)$, which changes the curvature of the trapping potential should be included in the linear part of the equation for the scaling variable $\tilde b$. Accordingly, Eq. (29) is rewritten as

$$\ddot{\tilde b}+[\omega_{\rm ho}^2+\delta\omega^2(t)]\tilde b-\frac{\omega_{\rm ho}^2}{\tilde b^4}+2\gamma\dot{\tilde b}+\xi\omega_{\rm ho}^2\frac{1-\tilde b}{\tilde b^4}=0. \tag{40}$$

For $\xi=\gamma=0$, one obtains the equation derived in Refs. [22] for the case $T=0$. The term $\sim\gamma$ describes the irreversible dissipation at $T\neq 0$. The term $\sim\xi$, already introduced in Eq. (29), with ξ given by Eq. (32), accounts for the dephasing effect discussed above.

The time-dependent part $\delta\omega^2(t)$ of the frequency should be driven so as to be in resonance with the collective mode, that is, in the form

$$\delta\omega^2(t) = -f(t)\exp(i\omega_0 t) - f^*(t)\exp(-i\omega_0 t),$$
$$\omega_0 = \sqrt{5}\omega_{\rm ho}, \tag{41}$$

where $f(t)$ stands for the complex amplitude of the external drive. This amplitude should be considered as a slow envelope of the resonant drive with a typical time $\tau_f \gg \omega_0^{-1}$ in order to avoid exciting other modes of the system. The echo, then, can be produced by making $f(t)$ reach a maximum at $t=0$ and then become zero until the time $t=\tau$, when $f(t)$ peaks again. For the sake of simplicity, we will ignore other modes, and will analyze the simplest situation when the external drive produces two δ pulses

$$\delta\omega^2(t) = -f_1\delta(t) - f_2\delta(t-\tau) \tag{42}$$

at $t=0$ and $t=\tau$, having amplitudes f_1 and f_2, respectively.

For the case of small amplitudes f_1 and f_2 of the drive, one should look for an evolution of the small perturbation around the equlibrium value $\tilde{b}=1$. We note that, in contrast to the conventional situation [20], the echo response in our model does not require nonlinearity of the dynamical equation. This is due to the fact that the external drive plays a twofold role. Specifically, on one hand, it gives rise to an effective external force $-\delta\omega^2(t)$, and, on the other hand, it excites the system parametrically. Indeed, linearizing Eq. (40) by the substitute $\tilde{b}=1+\eta$, with $\eta \ll 1$, one obtains

$$\ddot{\eta} + [\omega_{\rm ho}^2(5-\xi) + \delta\omega^2(t)]\eta + 2\gamma\dot{\eta} = -\delta\omega^2(t), \tag{43}$$

where the higher-order terms of η are neglected.

We assume that initially at $t=-\infty$ the mode was not excited $[\eta(-\infty)=\dot{\eta}(-\infty)=0]$. Then, taking into account Eq. (42), one finds, from Eq. (43),

$$\eta(0)=0, \quad \dot{\eta}(0)=f_1 \tag{44}$$

after the first pulse. The second pulse at $t=\tau$ results in a jump of $\dot{\eta}$, so that

$$\dot{\eta}(\tau+\varepsilon) = \dot{\eta}(\tau-\varepsilon) + f_2[1+\eta(\tau)],$$
$$\eta(\tau+\varepsilon) = \eta(\tau-\varepsilon) = \eta(\tau), \tag{45}$$

where $\varepsilon \to +0$.

We are looking for a solution at $t>\tau$. It has the forms

$$\eta(t) = Ae^{(iQ-\gamma)(t-\tau)} + A^* e^{(-iQ-\gamma)(t-\tau)},$$
$$Q = \omega_0(5-\xi)^{1/2} \approx \sqrt{5}\omega_0\left(1-\frac{\xi}{10}\right), \tag{46}$$

where we have taken into account that $\gamma \ll \omega_{\rm ho}$ and $\xi \ll \omega_{\rm ho}$. An explicit expression for the coefficient A can be obtained if one employs conditions (44) and (45). Finally, we find solution (46) for $t>\tau$ expressed as

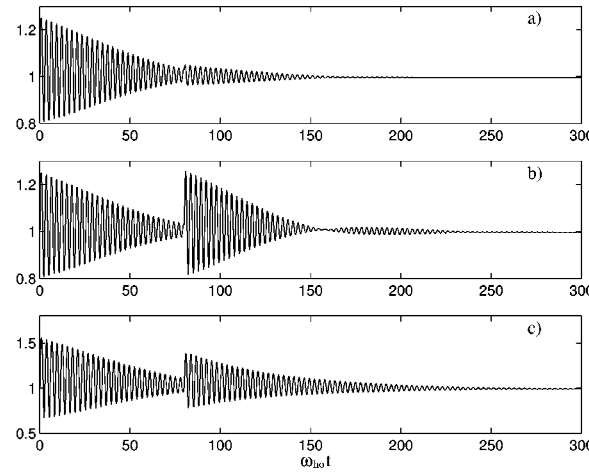

FIG. 2. The echo effect in the breathing mode oscillations of the isotropic confined Bose-Einstein condensate: the numerical solution $\tilde{b}(t)$ of Eqs. (40) and (42) ($\gamma=0.01\omega_{\rm ho}$, $\theta^2=0.02$, and $\tau=80\omega_{\rm ho}^{-1}$). Cases (a), (b), and (c) are different in amplitudes f_1 and f_2: (a) $f_1=0.5\omega_{\rm ho}$, $f_2=0.1\omega_{\rm ho}$; (b) $f_1=f_2=0.5\omega_{\rm ho}$; (c) $f_1=1\omega_{\rm ho}$, $f_2=0.5\omega_{\rm ho}$.

$$\eta(t) = \frac{f_1}{2iQ}\left(1+\frac{f_2}{2iQ}\right)e^{(iQ-\gamma)t} + \frac{f_2}{2iQ}e^{(iQ-\gamma)(t-\tau)} + \eta_e(t)$$
$$+\text{c.c.}, \tag{47}$$

where

$$\eta_e(t) = \frac{f_2 f_1}{4Q^2}e^{iQ(t-2\tau)-\gamma t} \tag{48}$$

represents the echo occurring at the time moment $t=2\tau$. After the thermal averaging over N_n, one finds

$$\langle \eta_e(t)\rangle_T = \frac{f_2 f_1}{10\omega_{\rm ho}^2}\cos[\omega(t-2\tau)]e^{-\gamma t-(t-2\tau)^2/\tau_{dM}^2} \tag{49}$$

for times $t>\tau>\tau_{dM}$. Then, the echo amplitude can be found as

$$A_e = \frac{f_2 f_1}{10\omega_{\rm ho}^2}\mathrm{e}^{-2\gamma\tau}. \tag{50}$$

In deriving Eqs. (49) and (50) in the limit under consideration, we have made the replacement $Q=\sqrt{5}\omega_{\rm ho}$ everywhere in Eqs. (47) and (48) except in the exponents, where the form of Q linearized in ξ and given by Eq. (46) has been employed. Then the averaging procedure results in the decay of all terms but $\eta_e(t)$ in Eq. (47) at the times $t\approx 2\tau$. Thus we obtained the echo effect in the linear approximation.

We have also analyzed the nonlinear echo problem for Eq. (40) numerically. This equation was solved for a given value of ξ, and then the final solution was averaged over the values of ξ distributed in accordance with the Gaussian $G(\xi)=\exp(-\xi^2/\theta^2)/\sqrt{\pi}\theta$, where θ determines the effective width of the distribution in such a way that the averaging of the linearized solution reproduces results (33) and (34). Specifically, we set $\theta=\sqrt{80}/\omega_{\rm ho}\tau_{dM}$. The results of the calculations are shown in Fig. 2. In case (a) the amplitude of the second pulse is too small to make the echo observable. In case (b), the second amplitude is five times stronger, so that the echo is distinct. In case (c), while the second pulse amplitude f_2 is the same as in the case (b), the amplitude of the first pulse f_1 is two times larger than that in cases (a) and (b) [note the different scale of the vertical axis in the case (c)]. As one can see, the echo in this case merges with the tail of the second pulse, which creates an impression that the decay time of the second pulse increases by several times. In order to produce the echo in the case of the large amplitudes f_1 and f_2, the time separation τ between the pulses should be increased. However, in this case the irreversible dissipation may strongly suppress the echo, in accordance with Eq. (50).

The echo effect analyzed above is a classical mechanics effect. Below a certain temperature T_Q, the rate of the quantum dephasing [8,9] should become faster than the damping induced by the normal component. Accordingly, the classical treatment employed above becomes no longer valid. The problem should be reformulated in terms of the quantum dynamics of the variable $\tilde{b}$ in a sense of the approach [9], with the external drive (42) taken into account. It can be shown that the echo still exists at $t=2\tau$. Therefore, the spontaneous quantum revival, determined by the interaction constant and thereby occurring at very long times [9], can be induced to occur at much shorter times comparable with the time of the quantum collapse [9]. This problem will be considered in a separate publication.

V. DISCUSSION

We have suggested a mechanism for the apparent damping of a Bose-Einstein condensate confined in the isotropic oscillator trapping potential. This damping is a reversible dephasing of the collective modes caused by thermal fluctuations of the population factors of the normal component. The calculation of the dephasing rate gives a value which is comparable with the experimentally observed rate of the damping of the low-energy collective modes in the atomic traps.

This mechanism of dephasing relies on the ensemble averaging of the collective mode over the initial population of the normal component. Thus an assumption is made that for any given initial distribution of the population factors of the ''hot'' quasiparticles, this distribution does not relax to equilibrium during the time of the dephasing τ_d. Accordingly, processes of relaxation due to the LD or collisions may suppress the discussed mechanism, if their relaxation times are comparable with τ_d. As long as the collisional damping introducing irreversibility is unlikely to be relevant for such small temperatures and densities, the LD is the only competing mechanism. However, the LD is expected to be significant for substantially anisotropic traps only. Therefore, in traps characterized by small anisotropy, our mechanism should dominate.

Both mechanisms of damping—Landau damping and that considered above—are reversible in nature, and therefore the evolution of the system can be partly reversed in time. We suggest testing this in the atomic traps by employing the echo effect. As our analytical and numerical calculations indicate, the echo amplitude as well as its position depend on the parameters of the external drive which can be varied over a wide range.

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APPENDIX A: WKB CALCULATION OF THE DEPHASING RATE

The WKB calculation of the dephasing rate presented here is essentially based on the results of Ref. [27]. Employing Eqs. (14), as well as the normalization condition $\int d\mathbf{r}(|U_n|^2-|V_n|^2)=1$ in Eq. (32), one finds

$$g_n=\frac{2}{mR_c\omega_{\rm ho}^2}\left\{E_n-\int d\mathbf{r}\left[\left(\frac{m\omega_{\rm ho}^2r^2}{2}+2|K|^2-\mu\right)\times(|U_n|^2+|V_n|^2)+(KV_nU_n^*+\text{c.c.})\right]\right\}, \tag{A1}$$

where the notation $K=u_o\Phi_c^2$ is introduced, and for the condensate wave function $\Phi_c=\sqrt{n_c}$ we employ the Thomas-Fermi solution [26]

$$n_c=\frac{m\omega_{\rm ho}^2}{2u_0}(r_c^2-r^2)\Theta(r_c-r),\quad r_c=r_{\rm ho}\left(\frac{15N_ca}{r_{\rm ho}}\right)^{1/5},$$

$$\mu=\frac{m\omega_{\rm ho}^2r_c^2}{2}, \tag{A2}$$

where $\Theta(z)$ is the step function; u_o and $r_{\rm ho}$ are defined in Eqs. (11) and (36), respectively. Accordingly, one finds the value of R_c in Eq. (32) as

$$R_c=\frac{r_c^7}{35ar_{\rm ho}^4}. \tag{A3}$$

States in the isotropic trap can be classified in terms of the angular momentum L, its z component L_z, and the radial quantum number n_r. Thus the index in Eq. (A1) as well as in the sum (34) should be understood as consisting of these three quantum numbers. This implies that the summation $\Sigma_n\cdots$ in Eq. (34) runs over three quantum numbers $n=(n_r,L,L_z)$. Because of the spherical symmetry, the summation over L_z can be performed trivially, which gives $\Sigma_n\cdots=\Sigma_{n_r}\Sigma_L(2L+1)\ldots$. As will be seen below, the large values $n_r\gg1, L\gg1$ dominate this sum. Therefore, we replace the summation by the integration over n_r, L

$$\sum_n\cdots\approx\int_0^\infty dn_r\int_0^{L_0}dL2L\ldots, \tag{A4}$$

where the upper limit L_0 is to be determined, and we made the replacement $2L+1\approx2L$. It is convenient to change the variable n_r to E by employing the quantization condition [27]

$$n_r+\frac{1}{2}=\frac{1}{\pi\hbar}\int_{r_1}^{r_2}dr\ p_r,\quad p_r=\sqrt{2m[\sqrt{E^2+|K|^2}-U_{\rm eff}(r)]},\tag{A5}$$

where

$$U_{\rm eff}(r)=\frac{1}{2}m\omega_{\rm ho}^2r^2+\frac{\hbar^2(L+1/2)^2}{2mr^2}+2|K|-\mu\tag{A6}$$

denotes the effective WKB potential [27], and the turning points r_1 and r_2 obey the equation $p_r=0$ or

$$\sqrt{E^2+K^2}-U_{\rm eff}(r)=0.\tag{A7}$$

Then integral (A4) acquires the form

$$\frac{2}{\pi\hbar}\int_0^\infty dE\int_0^{L_0}dL\ L\int_{r_1}^{r_2}\frac{dr}{v_r}\cdots,\tag{A8}$$

where Eq. (A5) has been employed, and v_r stands for the WKB radial velocity [27].

Before we proceed, it is convenient to employ dimensionless variables of length, energy, and angular momentum as

$$x=\frac{r}{r_c},\quad \epsilon=\frac{E}{\hbar\omega_{\rm ho}}\frac{r_{\rm ho}^2}{r_c^2},\quad J=L\frac{r_{\rm ho}^2}{r_c^2},\tag{A9}$$

respectively. Note that in these units the condensate radius r_c equals 1, and the chemical potential μ and the quantity K become

$$\mu'=\frac{\mu}{\hbar\omega_{\rm ho}}\frac{r_{\rm ho}^2}{r_c^2}=\frac{1}{2},\quad k=\frac{1}{2}(1-x^2)\Theta(1-x),\tag{A10}$$

respectively. Accordingly, Eq. (A7) yields two sets of solutions for the dimensionless turning points $x_{1,2}=r_{1,2}/r_c$,

$$x_1=\sqrt{y_0},\quad x_2=\sqrt{b_+},\quad J<\sqrt{2\epsilon}\tag{A11}$$

where

$$y_0=\frac{J^2(1+\sqrt{1+4\epsilon^2+2J^2})}{4\epsilon^2+2J^2},$$

$$b_\pm=\epsilon+\frac{1}{2}\pm\sqrt{\left(\epsilon+\frac{1}{2}\right)^2-J^2},\tag{A12}$$

and

$$x_1=\sqrt{b_-},\quad x_2=\sqrt{b_+},\quad \sqrt{2\epsilon}<J<\epsilon+\tfrac{1}{2},\quad \epsilon>\tfrac{1}{2}.\tag{A13}$$

As has been discussed in Ref. [27], solutions (A11) and (A13) correspond to the case when the classically allowed region extends into the condensate, and to the case when it is totally outside the condensate, respectively.

The U and V amplitudes inside the classically allowed region are [27]

$$U=\frac{C_0(\epsilon,J)}{2r_c^{3/2}}\left(\sqrt{\sqrt{1+(k/\epsilon)^2}+(k/\epsilon)}+\sqrt{\sqrt{1+(k/\epsilon)^2}-(k/\epsilon)}\right)\frac{\sin\phi}{x\sqrt{v_x}}Y_{L,L_z},$$

$$V=\frac{C_0(\epsilon,J)}{2r_c^{3/2}}\left(\sqrt{\sqrt{1+(k/\epsilon)^2}+(k/\epsilon)}-\sqrt{\sqrt{1+(k/\epsilon)^2}-(k/\epsilon)}\right)\frac{\sin\phi}{x\sqrt{v_x}}Y_{L,L_z}.\tag{A14}$$

Here Y_{L,L_z} is the spherical harmonic; the normalization constant is

$$C_0^{-2}(\epsilon,J)=\frac{1}{2}\int_{x_1}^{x_2}\frac{dx}{v_x},\tag{A15}$$

and the dimensionless radial velocity $v_x=\sqrt{mr_{\rm ho}^2/\hbar\omega_{\rm ho}r_c^2}v_r$ is given by

$$v_x=\sqrt{2\frac{\epsilon^2+k^2}{\epsilon^2}\left(\sqrt{\epsilon^2+k^2}-k-\frac{J^2}{2x^2}\right)},\quad x_1<x\leqslant 1\tag{A16}$$

inside the condensate, and by

$$v_x=\sqrt{2\epsilon+1-x^2-J^2/x^2},\quad 1<x<x_2\tag{A17}$$

outside the condensate. In calculation of $C_0(\epsilon,J)$ and in what follows, we replace $\sin^2\phi$ by $\frac{1}{2}$ because the WKB phase ϕ [27] varies rapidly inside the classically allowed region. The integrals outside this region are exponentially small, and we neglect them.

Substituting Eq. (A14) into Eq. (A1), and employing the units (A9) in Eq. (34), we find expressions (35) and (36), where the dimensionless function $D(\beta)$ is defined as

$$D(\beta)=\left[\int_0^\infty d\epsilon\frac{e^{\epsilon/\beta}}{(e^{\epsilon/\beta}-1)^2}\rho(\epsilon)\right]^{1/2},\tag{A18}$$

with the notation

$$\rho(\epsilon)=\int_0^{J_0(\epsilon)}dJ\ J\frac{C_0^2(\epsilon,J)}{2}\times\left\{\int_{x_1}^{x_2}\frac{dx}{v_x}\left[\epsilon-\left(\frac{1}{2}x^2-\frac{1}{2}+2k\right)\sqrt{1+\frac{k^2}{\epsilon^2}+\frac{k^2}{\epsilon}}\right]\right\}^2\tag{A19}$$

introduced, and k determined in Eq. (A10). The value of the limit $J_0(\epsilon)$ can be found from Eqs. (A11) and (A13). Specifically, for $\epsilon\leqslant 1/2$, only case (A11) can be realized. This implies that

$$J_0(\epsilon)=\sqrt{2\epsilon},\quad \epsilon\leqslant\tfrac{1}{2}.\tag{A20}$$

For $\epsilon > \frac{1}{2}$, Eq. (A13) yields

$$J_0(\epsilon)=\epsilon+\tfrac{1}{2}, \quad \epsilon>\tfrac{1}{2}. \tag{A21}$$

Consequently, integral (A19) can be expressed as $\rho(\epsilon)=\rho_1(\epsilon)+\rho_2(\epsilon)$,

$$\rho_1(\epsilon)=\int_0^{\sqrt{2\epsilon}} dJ\, J\frac{C_0^2(\epsilon,J)}{2}[\mathrm{In}_1(\epsilon,J)+\mathrm{In}_2(1,\epsilon,J)]^2,$$

$$\rho_2(\epsilon)=\Theta\left(\epsilon-\frac{1}{2}\right)\int_{\sqrt{2\epsilon}}^{\epsilon+1/2} dJ\, J\frac{C_0^2(\epsilon,J)}{2}[\mathrm{In}_2(\sqrt{b_-},\epsilon,J)]^2, \tag{A22}$$

where the notations

$$\mathrm{In}_1(\epsilon,J)=\int_{x_1}^{x_2}\frac{dx}{v_x}\frac{\epsilon\sqrt{\epsilon^2+k^2}}{k+\sqrt{\epsilon^2+k^2}}, \quad x_1<1$$

$$\mathrm{In}_2(\alpha,\epsilon,J)=\int_{\alpha}^{\sqrt{b_+}}\frac{dx}{v_x}\left(\epsilon+\frac{1}{2}-\frac{1}{2}x^2\right), \quad \alpha\geqslant 1 \tag{A23}$$

are introduced; $C_0^2(\epsilon,J)$ is given by Eq. (A15), and v_x is determined by Eqs. (A16) and (A17). Note that here we have employed the explicit expressions (A11)–(A13) for the turning points.

The value of the normalization constant C_0 can be found explicitly [27]. The integrals (A23) can also be calculated explicitly. We find

$$\frac{C_0^2(\epsilon,J)}{2}=\frac{2}{\pi}, \quad \mathrm{In}_2(\sqrt{b_-},\epsilon,J)=\frac{\pi}{4}\left(\epsilon+\frac{1}{2}\right) \tag{A24}$$

for $\epsilon>\frac{1}{2}$, $\sqrt{2\epsilon}<J<\epsilon+\frac{1}{2}$, and

$$\frac{C_0^2(\epsilon,J)}{2}=\left(\frac{2\epsilon\arccos\alpha_1}{\sqrt{2\epsilon^2+J^2}}+\arccos\alpha_2\right)^{-1},$$

$$\mathrm{In}_2(1,\epsilon,J)=\frac{1}{2}\left(\epsilon+\frac{1}{2}\right)\arccos\alpha_2-\frac{1}{4}\sqrt{2\epsilon-J^2},$$

$$\mathrm{In}_1(\epsilon,J)=\frac{\epsilon^2\arccos\alpha_1}{\sqrt{2\epsilon^2+J^2}}+\frac{1}{4}\sqrt{2\epsilon-J^2}-\frac{1}{2\sqrt{2}}\ln\alpha_3 \tag{A25}$$

for $J<\sqrt{2\epsilon}$, where we have introduced the notations

$$\alpha_1=\sqrt{\frac{2\epsilon^2+J^2-\epsilon+\epsilon\sqrt{1+4\epsilon^2+2J^2}}{2\epsilon\sqrt{1+4\epsilon^2+2J^2}}},$$

$$\alpha_2=\sqrt{\frac{\frac{1}{2}-\epsilon+\sqrt{(\epsilon+\frac{1}{2})^2-J^2}}{2\sqrt{(\epsilon+\frac{1}{2})^2-J^2}}},$$

$$\alpha_3=\frac{\sqrt{1+2\epsilon-\sqrt{1+4\epsilon^2+2J^2}}+\sqrt{1+2\epsilon+\sqrt{1+4\epsilon^2+2J^2}}}{\sqrt{2}[1+4\epsilon^2+2J^2]^{1/4}}. \tag{A26}$$

We note that in the formal limit $\beta\gg 1$, the function $D(\beta)$ given by Eq. (A18) can be found explicitly. Indeed, in this case the main contribution to Eq. (A19) comes from $\epsilon\gg 1$. This implies that only the term $\rho_2(\epsilon)$ in Eq. (A22) should be taken into account because it gives the highest power of ϵ as $\rho_2(\epsilon)\sim\epsilon^4$. As simple analysis of Eq. (A22) shows, the term $\rho_1(\epsilon)\sim\epsilon^3$. Thus taking $\rho(\epsilon)\approx\rho_2(\epsilon)$, and combining Eqs. (A24), (A22), (A25), and (A12), we find

$$\rho(\epsilon)=\frac{\pi}{16}\left(\epsilon^2-\frac{1}{4}\right)^2\Theta\left(\epsilon-\frac{1}{2}\right)\approx\frac{\pi}{16}\epsilon^4 \tag{A27}$$

for $\epsilon\gg 1$. Substituting this into Eq. (A18) and taking the limit $\beta\gg 1$, we obtain

$$D(\beta)\approx\sqrt{\frac{3\pi}{2}}\beta^{5/2}, \tag{A28}$$

which yields Eq. (38). This expression is shown in Fig. 1 by the dashed line. As one can see, in the range of β of the order of 1 the approximation (A28) underestimates the rate by approximately 20%.

We note that actual values of $\beta=T/2\mu$ are far from being $\beta\gg 1$. Indeed, employing Eqs. (A2) and (37), we find

$$\beta=\frac{T}{2\mu}=\left(\frac{r_{\mathrm{ho}}}{15a}\right)^{2/5}\zeta^{-1/3}(3)N^{-1/15}\frac{T}{T_c}, \tag{A29}$$

which yields values $\beta\approx 1$ for the experiment [10] for $T\approx T_c$. Therefore, for these values the function $D(\beta)$ should be found numerically (see the solid line in Fig. 1).

In the opposite limit $\beta\to 0$, which corresponds to $T\ll 2\mu$ or large N, the contribution to $D(\beta)$ due to $\rho_2(\epsilon)$ becomes exponentially small. Thus the term $\sim\rho_1(\epsilon)$ [Eq. (A22)] dominates in Eq. (A18). Taking into account that the effective values of $\epsilon \sim \beta$, one may perform an expansion in terms of the small parameter ϵ in Eqs. (A26) and (A25), and obtain

$$\rho(\epsilon) \approx \rho_1(\epsilon) = \rho_{01}\epsilon^{7/2}, \quad \epsilon \to 0, \tag{A30}$$

where the notation

$$\rho_{01} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} dx \sin x \frac{[x + \frac{1}{12}\sin 2x(7 + 11\cos^2)]^2}{x + \frac{1}{2}\sin 2x} \tag{A31}$$

has been introduced. A numerical evaluation of this integral gives $\rho_{01} \approx 1.5$. This yields, for Eq. (A18),

$$D(\beta) \approx D_0 \beta^{9/4}, \quad D_0 = \left[\rho_{01} \int_0^\infty dx \frac{e^x}{(e^x - 1)^2} x^{7/2} \right]^{1/2} \approx 4.4 \tag{A32}$$

in the limit $\beta \ll 1$.

APPENDIX B: CALCULATION OF THE AMPLITUDE DEPENDENCE OF THE DEPHASING RATE

Expanding Eq. (29) up to the third order with respect to η, one obtains

$$\ddot{\eta} + \omega_M^2 \eta - \alpha_M \eta^2 + \beta_M \eta^3 = 0, \tag{B1}$$

where the notations are

$$\omega_M^2 = (5 - \xi)\omega_{\rm ho}^2, \quad \alpha_M = 10\left(1 - \frac{2}{5}\xi\right)\omega_{\rm ho}^2,$$

$$\beta_M = 20\left(1 - \frac{\xi}{2}\right)\omega_{\rm ho}^2. \tag{B2}$$

The solution of Eq. (B1) up to the second order with respect to η_0 has a form

$$\eta = \frac{2\alpha_M}{\omega_M^2}|\eta_0|^2 + (\eta_0 e^{i\omega t} + \text{c.c.}) - \frac{\alpha_M}{3\omega_M^2}(\eta_0^2 e^{i2\omega t} + \text{c.c.}), \tag{B3}$$

where the effective frequency ω in the same order is

$$\omega = \omega_M + \omega' |\eta_0|^2, \quad \omega' = -\frac{5\alpha_M^2}{3\omega_M^3} + \frac{3\beta_M}{2\omega_M}. \tag{B4}$$

Now employing Eq. (B2) in Eq. (B4), and performing the thermal averaging of Eq. (B1) over ξ in the limit $\xi \ll 1$, we obtain

$$\langle \eta \rangle_T = \eta_0 e^{i\langle\omega\rangle_T t - t^2/\tau_{dM}^2(A_1)} + \text{c.c.}, \tag{B5}$$

where the constant as well as the second harmonic have been omitted; the dephasing rate $1/\tau_{dM}(A_1)$ as a function of the amplitude $A_1 = 2\eta_0$ of the first harmonic is given by Eq. (39).

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