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Two-fluid hydrodynamics for a trapped weakly interacting Bose gas

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We derive coupled equations of motion for the condensate (superfluid) and noncondensate (normal fluid) degrees of freedom in a trapped Bose gas at finite temperatures. Our results are based on the Hartree-Fock-Popov approximation for the time-dependent condensate wave function, and thermodynamic local equilibrium for the noncondensate atoms. In the special case of a uniform weakly interacting gas, our hydrodynamic equations are shown to be consistent with the two-fluid equations of Landau. The collective modes in a parabolically trapped Bose gas include the analog of the out-of-phase second-sound mode in superfluid ^{4}He. [S1050-2947(98)02806-6]

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The low-frequency dynamics of superfluid ^{4}He is commonly described using the two-fluid phenomenology first developed by Tisza [1] and Landau [2]. This description, later shown to be a consequence of a Bose broken symmetry [3,4], accounts for the characteristic features associated with superfluidity in terms of the relative motion of normal fluid and superfluid degrees of freedom. In particular, it predicts the existence of second sound as an out-of-phase oscillation of the two components. In the present paper, we give a simple microscopic derivation of two-fluid equations for an *inhomogeneous* weakly interacting gas of trapped atoms. In this situation, the superfluid is identified with the condensate atoms as described by a macroscopic wave function, while the normal fluid density is associated with the noncondensate thermal cloud. In the uniform density limit, we show that our equations are consistent with the standard two-fluid equations [2,3]. However, the hydrodynamic behavior of Bose gases is quite different from that of superfluid ^{4}He.

Our analysis is based on the equation of motion of the macroscopic Bose wave function [3,4] $\Phi(\mathbf{r},t)$, as determined within the time-dependent Hartree-Fock-Popov (HFP) approximation, which is a generalization of Refs. [5] and [6]. This condensate wave function is coupled to the thermally excited atoms making up the noncondensate which is described in terms of a semiclassical phase space distribution function $f(\mathbf{r},\mathbf{p},t)$. The further assumption that collisions are sufficiently rapid to force local equilibrium within the thermal cloud leads to a set of hydrodynamic conservation laws for the noncondensate component. We are thus able to give a simple and direct derivation of a closed set of equations for the coupled motion of the condensate and noncondensate in a trapped Bose gas. This complements more formal derivations [4] of two-fluid equations which are very complicated and, moreover, still require some microscopic model for the evalution of thermodynamic quantities.

While the hydrodynamic regime requires the size of the trapped gas to be much larger than the collisional mean free path (see the discussion in Ref. [7]), this regime has recently been studied in traps used at MIT [8]. We give quantitative predictions for the hydrodynamic normal modes of a trapped Bose gas involving both condensate and noncondensate components.

We first consider the dynamics of the condensate. As usual, the Bose field operator is conveniently separated into condensate and noncondensate parts: $\hat{\psi}(\mathbf{r})=\Phi(\mathbf{r})+\tilde{\psi}(\mathbf{r})$. For an arbitrary nonequilibrium state, the spatially and time-varying macroscopic wave function $\Phi(\mathbf{r},t)\equiv\langle\hat{\psi}(\mathbf{r})\rangle_t$ is described within the time-dependent Hartree-Fock-Popov approximation by the equation of motion

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t}=\left[-\frac{\hbar^2\nabla^2}{2m}+U_{\mathrm{ext}}(\mathbf{r})+2g\tilde{n}(\mathbf{r},t)+gn_c(\mathbf{r},t)\right]\times\Phi(\mathbf{r},t)\equiv\hat{\mathcal{H}}(\mathbf{r},t)\Phi(\mathbf{r},t). \tag{1}$$

Here the nonequilibrium noncondensate density is given by $\tilde{n}(\mathbf{r},t)=\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle_t$, the condensate density is $n_c(\mathbf{r},t)=|\Phi(\mathbf{r},t)|^2$, and $g=4\pi a\hbar^2/m$ is the interaction strength. Equation (1) represents a natural extension of recent work [5,6] which approximated $\tilde{n}(\mathbf{r},t)$ in Eq. (1) by the equilibrium value $\tilde{n}_0(\mathbf{r})$, thereby ignoring the collective behavior of the noncondensate. In contrast, the time-dependent condensate wave function $\Phi(\mathbf{r},t)$ in Eq. (1) is coupled into the fluctuations of the noncondensate, and a dynamical equation for the latter is also required.

It is convenient to recast the condensate equation of motion into a pair of hydrodynamic equations using the amplitude and phase representation $\Phi(\mathbf{r},t)=|\Phi(\mathbf{r},t)|e^{i\theta(\mathbf{r},t)}$. Substituting this form into Eq. (1) and separating real and imaginary parts, one finds

$$\begin{gathered}\frac{\partial n_c}{\partial t}=-\nabla\cdot(n_c\mathbf{v}_c),\\ m\left[\frac{\partial\mathbf{v}_c}{\partial t}+\frac{1}{2}\nabla\mathbf{v}_c^2\right]=-\nabla\phi,\end{gathered} \tag{2}$$

where $\mathbf{v}_c(\mathbf{r},t)\equiv\hbar\nabla\theta(\mathbf{r},t)/m$. The potential $\phi(\mathbf{r},t)$ is defined by

$$\phi(\mathbf{r},t)\equiv\frac{1}{|\Phi(\mathbf{r},t)|}\hat{\mathcal{H}}(\mathbf{r},t)|\Phi(\mathbf{r},t)|, \tag{3}$$

where $\hat{\mathcal{H}}(\mathbf{r},t)$ is the HFP Hamiltonian given in Eq. (1). Anticipating the identification of $\mathbf{v}_c$ with the superfluid velocity $\mathbf{v}_S$, we see that ϕ plays the role of the chemical potential associated with the superfluid motion [3].

We next consider the dynamics of the noncondensate in the low-frequency collision-dominated hydrodynamic regime. In the semiclassical limit valid at finite temperatures [9] [with $k_BT \gg \hbar\omega_0$, $gn_0(\mathbf{r})$, where ω_0 is a characteristic trap frequency], the dynamics can be formulated in terms of a quantum kinetic equation for the distribution function $f(\mathbf{r},\mathbf{p},t)$ [10,11]. Since Eq. (2) implies that the number of particles in the condensate is conserved, we must for consistency exclude those processes which scatter atoms in and out of the condensate [11]. In this situation, only collisions between excited atoms are relevant, and we can use the kinetic equation [10]

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_r-\nabla U(\mathbf{r},t)\cdot\nabla_p\right]f(\mathbf{r},\mathbf{p},t)=\left.\frac{\partial f}{\partial t}\right|_{\mathrm{coll}}. \tag{4}$$

Here $U(\mathbf{r},t)\equiv U_{\mathrm{ext}}(\mathbf{r})+2g[\tilde{n}(\mathbf{r},t)+n_c(\mathbf{r},t)]$ includes the self-consistent Hartree-Fock dynamic mean field in which the condensate part $2gn_c(\mathbf{r},t)$ can be viewed as an additional external field acting on the noncondensate.

The required hydrodynamic equations are obtained from Eq. (4) by making the further assumption that collisions force the distribution function to take the local equilibrium form [10]

$$f_0(\mathbf{r},\mathbf{p},t)=\frac{1}{\exp\left[\beta\left\{\frac{1}{2m}[\mathbf{p}-m\mathbf{v}_n]^2+U-\mu\right\}\right]-1}, \tag{5}$$

where the thermodynamic variables β, $\mathbf{v}_n$, and μ, together with U, all depend on $\mathbf{r}$ and t. When Eq. (5) is substituted into Eq. (4), the collision integral vanishes by virtue of the local equilibrium form of the distribution function. Taking moments of Eq. (4) with respect to *1*, p_μ, and p^2, we obtain the closed set of equations [12]

$$\begin{gathered}\frac{\partial\tilde{n}}{\partial t}+\nabla\cdot(\tilde{n}\mathbf{v}_n)=0,\\ m\tilde{n}\left[\frac{\partial\mathbf{v}_n}{\partial t}+(\mathbf{v}_n\cdot\nabla)\mathbf{v}_n\right]=-\nabla\tilde{P}-\quad\tilde{n}\nabla U,\\ \frac{\partial\tilde{\epsilon}}{\partial t}+\frac{5}{3}\nabla\cdot(\tilde{\epsilon}\mathbf{v}_n)=\quad\mathbf{v}_n\cdot\nabla\tilde{P}.\end{gathered} \tag{6}$$

The quantity $\tilde{\epsilon}$ is the nonconvective part of the kinetic-energy density defined with $\mathbf{v}_n=0$ in Eq. (5). Similarly, $\tilde{P}=\frac{2}{3}\tilde{\epsilon}$ is the kinetic contribution to the local equilibrium pressure defined by

$$\tilde{P}(\mathbf{r},t)\equiv\int\frac{d\mathbf{p}}{h^3}\frac{p^2}{3m}f_0(\mathbf{r},\mathbf{p},t)|_{\mathbf{v}_n=0}=\frac{1}{\beta\Lambda^3}g_{5/2}(z(\mathbf{r},t)), \tag{7}$$

with

$$z(\mathbf{r},t)\equiv e^{\beta(\mathbf{r},t)[\mu(\mathbf{r},t)-U(\mathbf{r},t)]}$$

and

$$\Lambda(\mathbf{r},t)=(2\pi\hbar^2/mk_BT(\mathbf{r},t))^{1/2}.$$

$$\begin{gathered}z(\mathbf{r},t)\equiv e^{\beta(\mathbf{r},t)[\mu(\mathbf{r},t)-U(\mathbf{r},t)]}\\ \Lambda(\mathbf{r},t)=\frac{2?\hbar^2}{mk_BT(\mathbf{r},t)^{1/2}}.\end{gathered} \tag{8}$$

Equations (2) and (6) constitute our full set of nonlinear hydrodynamic equations for a trapped Bose gas at finite temperatures.

At the level of approximation we are considering, the condensate and noncondensate satisfy separate continuity equations. Combining these two equations gives the expected two-fluid continuity equation

$$\frac{\partial n}{\partial t}=-\nabla\cdot\mathbf{j}, \tag{9}$$

where $n\equiv\tilde{n}+n_c$ and $\mathbf{j}\equiv\tilde{n}\mathbf{v}_n+n_c\mathbf{v}_c$. We identify the normal fluid density with $\tilde{n}(\mathbf{r},t)$ and the superfluid density with $n_c(\mathbf{r},t)$. The former identification is supported by noting that Eq. (5) yields

$$\tilde{n}(\mathbf{r},t)=-\int\frac{d\mathbf{p}}{h^3}\frac{p^2}{3m}\frac{\partial f_0(\varepsilon_p)}{\partial\varepsilon_p}, \tag{10}$$

where $\varepsilon_p(\mathbf{r},t)\equiv(p^2/2m)+U(\mathbf{r},t)-\mu(\mathbf{r},t)$ is the excitation energy. This is the usual Landau formula for the normal fluid density [2,3].

The linearized version of Eqs. (2) and (6) allows one to consider small-amplitude oscillations about equilibrium. The equilibrium condensate wave function is determined by the solution of

$$\begin{aligned}\hat{\mathcal{H}}_0(\mathbf{r})\Phi_0(\mathbf{r})\equiv&\left[-\frac{\hbar^2\nabla^2}{2m}+U_{\mathrm{ext}}(\mathbf{r})+2g\tilde{n}_0(\mathbf{r})\right.\\ &\left.+gn_{c0}(\mathbf{r})\right]\Phi_0(\mathbf{r})\\ =&\mu_0\Phi_0(\mathbf{r}),\end{aligned} \tag{11}$$

with $n_{c0}(\mathbf{r})=|\Phi_0(\mathbf{r})|^2$. The equilibrium noncondensate density $\tilde{n}_0(\mathbf{r})$ is $g_{3/2}(z_0)/\Lambda^3$, the equilibrium fugacity is defined as $z_0=e^{\beta_0[\mu_0-U_0(\mathbf{r})]}$, with $U_0(\mathbf{r})=U_{\mathrm{ext}}(\mathbf{r})+2gn_0(\mathbf{r})$. Equation (11) must be solved self-consistently together with $\tilde{n}_0(\mathbf{r})$.

The linearization of Eq. (2) around equilibrium leads to the condensate equations

$$\begin{gathered}\frac{\partial\delta n_c}{\partial t}=-\nabla\cdot(n_{c0}\delta\mathbf{v}_c),\\ m\frac{\partial\delta\mathbf{v}_c}{\partial t}=-\nabla\delta\phi,\end{gathered} \tag{12}$$

where

$$\delta\phi(\mathbf{r},t)\equiv\frac{1}{|\Phi_0(\mathbf{r})|}[\hat{\mathcal{H}}_0(\mathbf{r})-\mu_0]\delta|\Phi(\mathbf{r},t)|+g\,\delta n_c(\mathbf{r},t)$$
$$+2g\,\delta\tilde{n}(\mathbf{r},t). \tag{13}$$

In arriving at this result, we have noted that $\nabla\phi_0(\mathbf{r})=0$ in equilibrium. Similarly, the linearization of Eq. (6) leads to the equations

$$\frac{\partial\delta\tilde{n}}{\partial t}=-\nabla\cdot(\tilde{n}_0\delta\mathbf{v}_n),$$

$$m\tilde{n}_0\frac{\partial\delta\mathbf{v}_n}{\partial t}=-\nabla\delta\tilde{P}-\delta\tilde{n}\nabla U_0-2g\tilde{n}_0\nabla(\delta\tilde{n}+\delta n_c), \tag{14}$$

$$\frac{\partial\delta\tilde{P}}{\partial t}=-\frac{5}{3}\nabla\cdot(\tilde{P}_0\delta\mathbf{v}_n)+\frac{2}{3}\delta\mathbf{v}_n\cdot\nabla\tilde{P}_0,$$

where $\tilde{P}_0(\mathbf{r})$ is the equilibrium kinetic pressure which satisfies $\nabla\tilde{P}_0=-\tilde{n}_0\nabla U_0$. Above T_{BEC}, these equations reduce to those of Ref. [7] if we ignore the effect of interactions ($g=0$).

For its intrinsic interest, and in order to better understand the implications of our two-fluid equations, we now consider the limit of a *homogeneous* system $[U_{\mathrm{ext}}(\mathbf{r})=0]$. In this special case, Eq. (11) yields a uniform condensate with the chemical potential having the Thomas-Fermi (TF) value $\mu_0=2g\tilde{n}_0+gn_{c0}$. Taking all equilibrium quantities to be spatially independent, and noting that the first term on the right-hand side of Eq. (13) can be neglected in the long-wavelength limit, the two velocity equations reduce to

$$m\frac{\partial\delta\mathbf{v}_c}{\partial t}=-2g\nabla\delta\tilde{n}-g\nabla\delta n_c, \tag{15}$$
$$m\tilde{n}_0\frac{\partial\delta\mathbf{v}_n}{\partial t}=-\nabla\delta\tilde{P}-2g\tilde{n}_0\nabla\delta\tilde{n}-2g\tilde{n}_0\nabla\delta n_c,$$

with

$$\frac{\partial\delta\tilde{P}}{\partial t}=-\frac{5}{3}\tilde{P}_0\nabla\cdot\delta\mathbf{v}_n. \tag{16}$$

The two equations in Eq. (15) can be combined to give

$$m\frac{\partial}{\partial t}\delta\mathbf{j}=-\nabla\delta\tilde{P}-2g(n_{c0}+\tilde{n}_0)\nabla\delta\tilde{n}-g(n_{c0}+2\tilde{n}_0)\nabla\delta n_c$$
$$\equiv-\nabla\delta P. \tag{17}$$

One can verify that Eq. (17) is consistent with the following expression for the *total* local thermodynamic pressure:

$$P=\tilde{P}+\tfrac{1}{2}g[n^2+2n\tilde{n}-\tilde{n}^2], \tag{18}$$

which is the equation of state at the level of approximation we are considering. Within the same approximation, the total internal energy density is given by

$$\epsilon(\mathbf{r},t)=\tilde{\epsilon}(\mathbf{r},t)+\tfrac{1}{2}g\langle\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle_t$$
$$\simeq\tilde{\epsilon}(\mathbf{r},t)+\tfrac{1}{2}g[n^2+2n\tilde{n}-\tilde{n}^2]. \tag{19}$$

Assuming the equilibrium thermodynamic relation

$$\epsilon+P=sT+\mu n \tag{20}$$

allows us to identify the entropy density s. Using $\mu_0=g(2\tilde{n}_0+n_{c0})$ together with Eqs. (18), (19), and (20) gives

$$s_0T_0=\tfrac{5}{2}\tilde{P}_0+gn_{c0}\tilde{n}_0. \tag{21}$$

Here $\tilde{P}_0$ is the equilibrium kinetic pressure defined in Eq. (7), with $z\equiv z_0=e^{\beta_0(\mu_0-2gn_0)}=e^{-\beta_0gn_{c0}}$. Using the local equilibrium expression for $\tilde{P}$ in Eq. (7), one finds that the fluctuation in the total pressure P is given by

$$\delta P=s_0\delta T+\tilde{n}_0\delta\mu+gn_{c0}(2\,\delta\tilde{n}+\delta n_c), \tag{22}$$

with s_0 defined by Eq. (21). Comparing this to the thermodynamic relation $\delta P=s_0\delta T+n_0\delta\mu$, we arrive at

$$\delta\mu=\delta(2g\tilde{n}+gn_c). \tag{23}$$

This result confirms (in the case of a uniform equilibrium density) that $\delta\phi(\mathbf{r},t)$ in Eqs. (12) and (13) is indeed the fluctuation in the local chemical potential. More generally, we have verified that Eq. (2) is equivalent to the key Landau equation for superfluid flow [2,3],

$$m\left[\frac{\partial\mathbf{v}_S}{\partial t}+\frac{1}{2}\nabla\mathbf{v}_S^2\right]=-\nabla\mu(\mathbf{r},t). \tag{24}$$

Finally, using Eq. (19), we find

$$\delta\epsilon=\tfrac{3}{2}\,\delta\tilde{P}+gn_{c0}(2\,\delta\tilde{n}+\delta n_c)+2g\tilde{n}_0\delta n. \tag{25}$$

Inserting this result into the thermodynamic relation $\delta\epsilon=T_0\delta s+\mu_0\delta n$, a simple calculation gives

$$T_0\delta s=\tfrac{3}{2}\,\delta\tilde{P}+gn_{c0}\delta\tilde{n}. \tag{26}$$

Taking the time derivative, and using Eq. (16) and the continuity equation for $\delta\tilde{n}(\mathbf{r},t)$, we obtain the linearized form of Landau's entropy conservation equation [2,3]

$$\frac{\partial\delta s}{\partial t}=-s_0\nabla\cdot\delta\mathbf{v}_n. \tag{27}$$

In the uniform case, Eqs. (15) and (16) are easily solved to give the expected first and second sound phonon modes [13]. In contrast to superfluid ^{4}He, the second sound mode in a gas involves a condensate oscillation largely uncoupled from the noncondensate, with a velocity given by $u_2=(gn_{c0}/m)^{1/2}$.

As a specific application to a trapped Bose gas, we consider the center-of-mass mode solution for an anisotropic parabolic potential $U_{\mathrm{ext}}(\mathbf{r})=\frac{1}{2}m(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)$. On the basis of the generalized Kohn theorem [14], one expects three modes in which the gas oscillates rigidly along each of the principal directions at the appropriate trap frequency ω_i.

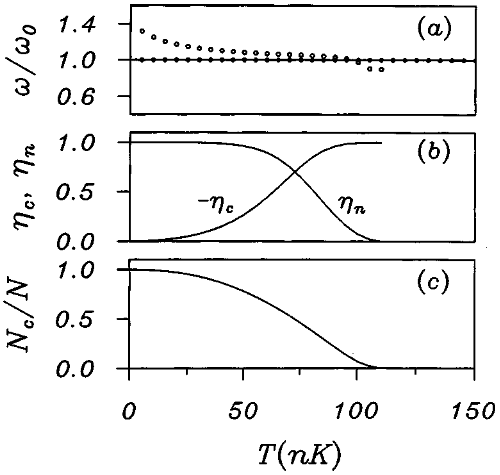

FIG. 1. (a) Mode frequencies for the in-phase (solid dots) and out-of-phase (open dots) dipole modes vs temperature for 2000 Rb atoms in an isotropic parabolic trap (see Ref. [5] for the physical parameters used). (b) Condensate (η_c) and noncondensate (η_n) amplitudes for the out-of-phase dipole mode. (c) Fraction of atoms in the condensate as a function of temperature.

Denoting the displacement along one of these directions by $\boldsymbol{\eta}(t)$, the condensate and noncondensate densities *both* behave as $n_0[\mathbf{r}-\boldsymbol{\eta}(t)]$, giving a density fluctuation $\delta n(\mathbf{r},t)=-\nabla n_0(\mathbf{r})\cdot\boldsymbol{\eta}(t)$, and a velocity field $\mathbf{v}(\mathbf{r},t)=\dot{\boldsymbol{\eta}}(t)$ which is spatially independent. Our linearized equations (12) and (14) admit a solution of this kind, with both the condensate and noncondensate having identical displacements $\boldsymbol{\eta}(t)=\boldsymbol{\eta}_0\cos\omega_i t$. Thus, in contrast to the static HFP used in Ref. [5], the dynamic HFP theory given by Eq. (1) is consistent with the generalized Kohn theorem.

Our equations also admit an out-of-phase dipole mode in which the condensate and noncondensate oscillate against each other. Although a rigorous estimate of the mode frequency can be obtained by recasting our hydrodynamic equations (12)–(14) into the form of a variational principle [15], the same result can be derived using the following picture. The condensate and noncondensate are represented by two particles with masses $M_c=mN_c$ and $M_n=m\tilde{N}$, each confined in the parabolic potential and coupled together by a spring with force constant k. Since the interaction energy between the two components is $E_{int}=2g\int d\mathbf{r}\, n_c(\mathbf{r})\tilde{n}(\mathbf{r})$, k can be determined by considering small displacements of the two components along the ith direction. The equations of motion for the two coupled masses can then be solved, giving the in-phase center-of-mass mode at the frequency ω_i discussed above, and an out-of-phase mode at the frequency [15]

$$\Omega_i^2=\omega_i^2-2g\frac{M_c+M_n}{M_cM_n}\int d\mathbf{r}\,\frac{\partial n_{c0}(\mathbf{r})}{\partial x_i}\,\frac{\partial \tilde{n}_0(\mathbf{r})}{\partial x_i}. \qquad (28)$$

In Fig. 1 we show the in- and out-of-phase mode frequencies as functions of temperature for a gas trapped in an isotropic parabolic potential. We also show, for the out-of-phase mode, the relative amplitudes of the condensate and noncondensate oscillations which satisfy the condition $M_c\eta_c+M_n\eta_n=0$, corresponding to the center of mass being stationary for any temperature. A measurement of these amplitudes would therefore determine directly the ratio $N_c/\tilde{N}$ of the two fluid components. This out-of-phase mode of the trapped Bose gas is the analog of the usual second sound mode [3] in bulk superfluid ^{4}He (for which $\rho_S\mathbf{v}_S+\rho_N\mathbf{v}_N=0$).

In summary, we have given a simple derivation of two-fluid hydrodynamics for a trapped weakly interacting Bose gas, as summarized by Eqs. (12)–(14). Our analysis illustrates how such a two-fluid description arises naturally from the existence of a macroscopic condensate wave function which is coupled to the noncondensate atoms. We have also verified how our two-fluid equations can lead to the well-known equations [Eqs. (2), (9), (17), and (27)] of Landau [2–4]. Finally, we have shown that our equations are consistent with the generalized Kohn theorem, and obtained the out-of-phase mode analog of second sound in a trapped Bose gas. A complete discussion of the normal-mode solutions of our two-fluid equations will be given elsewhere [15].

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