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Selection rules for transverse-mode excitation in nonlinear ring and Fabry-Perot resonators

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We establish that the energy transfer between transverse modes in nonlinear Fabry-Perot resonators with curved mirrors, having degenerate or nearly degenerate modes, is governed by selection rules. Although these rules are derived in a perturbative limit, assuming a thin medium centered in the resonator, we have found numerically and experimentally that they still account for the otherwise unexpected enhancement or suppression of certain modes in more realistic cases. The existence of selection rules constitutes a fundamental difference between standing wave and traveling wave cavities and therefore prevents, in principle, the possibility of describing experiments on transverse effects performed in Fabry-Perot cavities with (simpler) ring resonator models. [S1050-2947(98)10605-4]

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I. INTRODUCTION

Transverse effects in nonlinear resonators have been studied in depth both experimentally and theoretically in the past decade [1–5]. Whereas the first investigations focused on the basic scenarios and on the physical mechanism of pattern formation and therefore stressed the universality of the observed phenomena, some more recent work has demonstrated the significance of those details of the experimental setup that one may be tempted to disregard in a first analysis. Mathis *et al.* recently demonstrated that the number of mirrors (odd or even) of a ring resonator affects the symmetry of the patterns observed in a laser [6]. Heckenberg *et al.* showed that in out-of-plane resonators, the degeneracy between vortex modes of opposite topological charge is broken, thereby fixing the direction of rotation of circling vortices [7]. Both results arise from the properties of the modal spectrum of the linear resonator. However, even in those cases where the configuration of the resonator does not affect its linear properties, its *nonlinear* behavior may change from case to case. Indeed, in a recent paper [8], we demonstrated that the nonlinear features of ring and standing wave resonators that have the *same linear* properties will in general be *different*. This is a consequence of the fact that in a standing wave cavity the medium interacts with a forward *and* a backward beam *simultaneously*, whose intensity distributions are in general different, due to the differing values of the Gouy phase shift accumulated by different families of excited modes during propagation. This carries the far-reaching consequence that models written for traveling-wave cavities cannot predict the results of experiments conducted in standing-wave resonators, even if the longitudinal grating formed in the medium by the standing waves can be neglected. Hence, one must strongly resist the temptation to apply the same ''near equivalence'' of standing wave and traveling wave resonators that generally holds for the single-transverse-mode case [9] to the multi-transverse-mode situation.

In this paper, we show that even in the case where essentially only *one* mode is injected into a passive nonlinear cavity, the equivalence between standing and traveling wave cavities does not hold as soon as an energy transfer towards other resonator modes is permitted by the nonlinearity. One might expect that the first higher-order mode to receive energy from the fundamental mode is the Gauss-Laguerre mode TEM_{10} — which is indeed true in ring resonator models (e.g., [10,11]), but may not be true in general. In particular, we demonstrate, both theoretically and experimentally, that in a confocal (or close to confocal) Fabry-Perot resonator, the generation of the TEM_{20} has a higher efficiency, independent of the details of the nonlinearity. We will further show that this is just a special case of the general rules governing mode conversion in resonators with degenerate transverse modes.

The case of mode degeneracy is of particular interest because it allows the formation of nontrivial patterns. In fact, numerous experiments on transverse effects have been performed using such degeneracies in Fabry-Perot resonators, e.g., [12,13], and in particular, in the confocal or close to confocal configuration [4,5,14–16].

The paper is organized as follows: After the introduction of the model in Sec. II, we give an analytical treatment based on a perturbation analysis in Sec. III and we interpret the results heuristically. In Sec. IV, we relax some of the restrictions imposed in the analytical treatment and numerically investigate a specific example. We show that, in spite of the less stringent conditions that a realistic experimental system imposes, the selection rules are still very well satisfied. Finally, in Sec. V, we give experimental evidence of the selection rules in a resonator containing sodium vapor as the nonlinear medium.

II. DEFINITION OF THE MODEL

Figure 1 shows schematically the passive system to be considered. A Fabry-Perot resonator of length $2d$, with two

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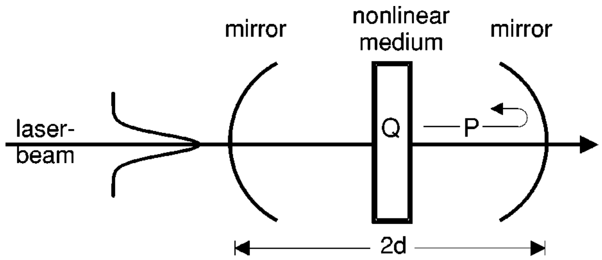

FIG. 1. Schematic diagram of the system considered.

identical curved mirrors of focal length f_m and reflectivity R, is set up in a close to confocal configuration ($d-f_m \ll f_m$). A thin layer of nonlinear medium of length $L \ll f_m$ is placed in the center of this resonator. A Gaussian laser beam is injected into the cavity. The transverse extent of the beam is assumed to be considerably smaller than the transverse size of the nonlinear medium (and of the mirrors).

We introduce the slowly varying amplitude Ψ of the monochromatic, scalar optical field E by

$$E=\tfrac{1}{2}\,(\Psi\ \exp\{-i(kz-\omega t)\}+\text{c.c.}), \tag{1}$$

where ω is the frequency and k the wave number. The field induces a polarization in the nonlinear medium. We assume that the slowly varying amplitude $\mathcal{P}$ of the polarization can be expressed as

$$\mathcal{P}=\epsilon_0\chi\Psi, \tag{2}$$

where the susceptibility χ is assumed to depend on the modulus squared of the field and may also be affected by the presence of nonlocal coupling mechanisms (e.g., transverse diffusion). This is typically the case in optical pattern formation.

The propagation of Ψ is described by the paraxial wave equation,

$$\frac{\partial}{\partial z}\Psi=-i\frac{k}{2}\left(\frac{1}{k^2}\nabla_\perp^2+\chi\right)\Psi, \tag{3}$$

where $\nabla_\perp^2$ is the transverse part of the Laplacian. Formal integration of Eq. (3) over a small distance δz yields

$$\Psi(z+\delta z)=\exp\left\{-i\frac{k\,\delta z}{2}\left(\frac{1}{k^2}\nabla_\perp^2+\chi\right)\right\}\Psi(z). \tag{4}$$

In general, propagation through media is treated numerically—using Eq. (4)—as a sequence of diffraction and refraction steps that take place in the thin layers into which the medium is decomposed (split-step operator method, e.g., [17]). In our case, we will assume the medium to be thin enough to be described by only one application of the refraction operator,

$$Q=\exp\left\{-i\frac{kL}{2}\chi\right\}, \tag{5}$$

where χ is the transverse susceptibility distribution at the position of the medium. The propagation of the beam from the medium to the mirror, the focusing of the beam by the curvature of the mirror, and the propagation back to the medium is described by the propagation operator,

$$P=\sqrt{R}\ \exp\left\{-i\frac{d}{2k}\nabla_\perp^2\right\}\exp\left\{ik\frac{r^2}{2f_m}\right\}\exp\left\{-i\frac{d}{2k}\nabla_\perp^2\right\}. \tag{6}$$

A complete round trip is thus described by the operator product $QPQP$.

The field inside the empty resonator is analyzed in terms of the complete set of orthonormal Gauss-Laguerre eigenmodes (see, e.g., [18,19]),

$$\begin{aligned}|pl\rangle=&\sqrt{\frac{2p!}{(|l|+p)!\,\pi w^2(z)}}\\ &\times\exp\{i(2p+|l|+1)\Delta\Phi(z)\}\cdot\rho^{|l|}(z)\cdot L_p^{|l|}[\rho^2(z)]\\ &\times\exp\left\{-ik\frac{r^2}{2q(z)}-il\phi\right\},\end{aligned} \tag{7}$$

where L_p^l is the generalized Laguerre polynomial, p represents the radial mode index, l the azimuthal mode index, $q(z)$ is the complex beam parameter and $\rho(z):=\sqrt{2}r/w(z)$ is the normalized cylindrical coordinate. The Gouy phase shift $\Delta\Phi$,

$$\Delta\Phi(z)=\arctan\left(\frac{z-z_0}{z_R}\right), \tag{8}$$

is measured with respect to the position z_0 of the beam waist [19,20]. The total Gouy phase shift has to be evaluated by summing up all the individual phase shifts gained in each cavity section (computed with the value of the beam parameter in that section). The intracavity field amplitude Ψ can be decomposed into a superposition of Gauss-Laguerre modes $|pl\rangle$,

$$\Psi=\sum_{p,l}a_{pl}|pl\rangle, \tag{9}$$

that can be grouped in families of modes having a degenerate resonance (cf., e.g., [21]) identified by the index

$$s=2p+|l|. \tag{10}$$

III. PERTURBATIVE CALCULATION OF MODE CONVERSION

A. Contributions to the refraction operator

We describe the radial dependence of the susceptibility profile $\chi(r)$ by a Taylor series in the coordinate r. Since we inject a Gaussian beam into the cavity, and neglect the possibility of spontaneous symmetry breaking, only the even terms will appear:

$$\chi(r)=\chi_0-\chi_2r^2+\chi_4r^4+\cdots\ . \tag{11}$$

In systems with strong transverse coupling by diffusionlike processes, a truncation of Eq. (11) after the fourth-order term represents a very satisfactory approximation for the susceptibility profile *within* the laser beam ([5,14]; cf. [22] for a quantitative investigation).

The explicit form of the operator Q is given by

$$Q=\exp\left\{-i\frac{kL}{2}\chi_0\right\}\exp\left\{i\frac{kL}{2}\chi_2 r^2\right\}\exp\left\{-i\frac{kL}{2}\chi_4 r^4\right\}\cdots. \tag{12}$$

The exponentials in Eq. (12) each have a different physical meaning:

In the following we assume a lossless medium, hence, the first term describes an additional (uniform) phase shift of the field by the medium, which is independent of the transverse coordinate and is therefore the same for all modes.

The second exponential can be identified as the action of a *thin lens*, with a focal length $f=1/(\chi_2 L)$. Therefore, up to second order, the nonlinear resonator can be regarded as a resonator with an additional, parameter-dependent intracavity lens, whose eigenmodes will still be Gauss-Laguerre modes, though with a different beam parameter. A lengthy but straightforward calculation yields that in the close to confocal configuration ($d-f_m \ll f_m$) and low nonlinearity ($f_m \ll f$, ensuring that the resonator remains close to confocal) the Gouy phase shift per resonator round trip is given by

$$\Phi_s=(s+1)\left(\pi+\frac{f_m}{f}+\frac{2(d-f_m)}{d}\right). \tag{13}$$

Disregarding uniform phase shifts, which are the same for all modes, Eq. (13) also describes the modal spectrum of a close-to-confocal resonator. A confocal resonator ($d=f_m$) is no longer confocal in the presence of a nonlinear medium ($1/f\neq 0$). However, by a suitable cavity length adjustment,

$$d=f_m\frac{1}{1+f_m/2f}, \tag{14}$$

it is possible to reestablish the highly degenerate modal spectrum of the confocal resonator.

The third exponential (and higher-order ones) in Eq. (12) cannot be absorbed into the definition of the mode, since there are no explicit terms of the form $\exp\{-ir^4\}$ in the expression for $|pl\rangle$, cf. Eq. (7). Therefore, such terms introduce a generic coupling between different modes. In order to discuss the influence of the nonparabolic terms, we eliminate the homogeneous and parabolic contributions to the refraction operator Q by considering a suitable set of eigenmodes of the resonator with a self-induced intracavity lens.

B. Introductory example

Let us first assume the injected beam to be mode matched to the fundamental mode of the resonator *with intracavity lens* ($a_{00}=1$). We then perform a perturbation calculation of the excitation of higher-order modes due to the nonparabolic component of the refractive index distribution, which we now characterize with the dimensionless parameter,

$$\gamma:=kL\chi_4\frac{w^4}{8}, \tag{15}$$

where w is the beam waist of the fundamental mode. We assume that the nonparabolic terms are small, $|\chi_4 r^4| \ll |\chi_2 r^2|$, near the optical axis. This is equivalent to setting

$$|\gamma|\ll\frac{f_m}{f}\ll 1, \tag{16}$$

and we can therefore approximate the fourth-order term of the operator Q,

$$Q_4=\exp\{-i\gamma\rho^4\} \tag{17}$$

$$\approx 1-i\gamma\rho^4 \tag{18}$$

$$=1+\gamma[-2iL_0^0(\rho^2)+4iL_1^0(\rho^2)-2iL_2^0(\rho^2)], \tag{19}$$

to first order in γ. The expansion of the perturbation operator in Laguerre polynomials shows that a cylindrically symmetric mode remains cylindrically symmetric. Furthermore, a mode with radial index p can only directly excite modes with $p\pm 1$ or $p\pm 2$. However, during successive round trips, all modes can be gradually excited. In order to keep this introductory example as simple as possible, we neglect this cascading excitation, which is possible in the case of

$$\gamma^2\ll T^2, \tag{20}$$

where $T=1-R$ is the transmissivity of the mirrors [23]. We can, in this case, confine the description to the three lowest-order cylindrically symmetric Gauss-Laguerre modes. For this purpose we adopt a matrix representation of Eq. (19) for Q_4, which operates on a state vector $(a_{00},a_{10},a_{20})^T$ with elements a_{j0} ($j=0,1,2$) representing the amplitudes of the three modes under consideration:

$$Q_4=1+\gamma\begin{pmatrix}-2i & 4i & -2i\\ 4i & -14i & 16i\\ -2i & 16i & -38i\end{pmatrix}. \tag{21}$$

The off-diagonal elements describe the coupling between different modes. Note that in a *single pass* the excitation of the $|10\rangle$ mode is twice as high as that of the $|20\rangle$ mode. However, we will see in the following that this result, valid for the single pass, does not apply to the stationary state of the intracavity field.

To calculate the steady-state solution, one has to take into account the phase changes of the modes due to the propagation in the empty regions of the resonator, over the round trip. Choosing the phase of the incoupled fundamental mode as a reference, the operator for propagation from the medium to the mirror and back is given by

$$P=\sqrt{R}e^{-i\Phi_{00}/2}\begin{pmatrix}e^{i\Phi_{00}/2} & 0 & 0\\ 0 & e^{i\Phi_{10}/2} & 0\\ 0 & 0 & e^{i\Phi_{20}/2}\end{pmatrix}, \tag{22}$$

where the phases Φ_{p0} are given by Eq. (13). In the confocal configuration—which can, if necessary, be established by a length correction [see Eq. (14)]—the operator becomes

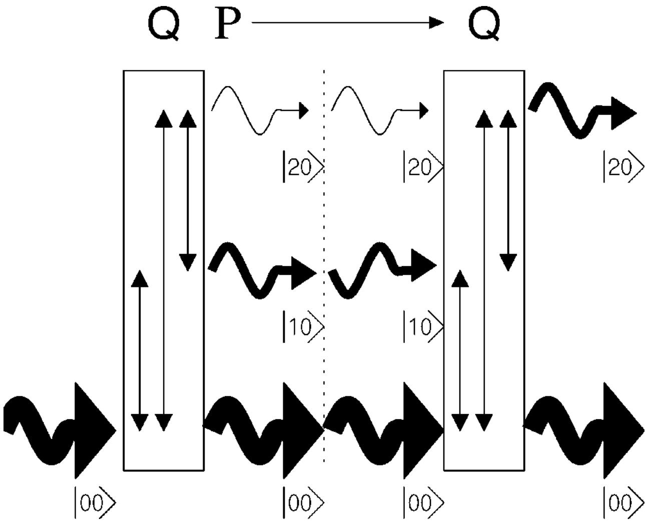

FIG. 2. Schematic diagram illustrating the energy transfer and interference conditions between different transverse modes in a confocal Fabry-Perot cavity. See text for further explanations.

$$P=\sqrt{R}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

The operator responsible for the coupling towards higher-order modes for a resonator round trip is given by

$$PQ_4PQ_4=R+\gamma R\begin{pmatrix} -4i & 0 & 4i \\ 0 & -28i & 0 \\ 4i & 0 & -76i \end{pmatrix} \tag{24}$$

to first order in γ. The fact that the off-diagonal element describing the coupling between the $|00\rangle$ and the $|10\rangle$ is zero implies that *no energy is transferred from the fundamental into the first higher-order mode*. Obviously a *selection rule* exists here, allowing only energy transfer to the second higher-order mode.

The physical origin of this surprising observation is illustrated in Fig. 2. The left-hand side of the figure shows that the fundamental mode traversing the nonlinear medium does excite mode $|10\rangle$, and in larger amounts than it does mode $|20\rangle$! The fundamental mode and the generated higher-order modes propagate in the resonator half a round trip and then return to the medium. At this encounter, the $|00\rangle$ creates a second contribution to the $|10\rangle$. However, during the propagation to the mirror and back [cf. Eq. (23)], mode $|10\rangle$ acquires *an additional* π *(relative) phase shift compared to the fundamental mode*, as indicated by the form of the inverted sinusoid reaching the medium in the right-hand side of the figure. Hence, there is *destructive interference* between the $|10\rangle$ mode components generated half a round trip apart. Therefore, no excitation of the $|10\rangle$ mode is possible for the full round trip at steady state. In contrast, the contributions to the $|20\rangle$ mode, which are generated half a round trip apart, *interfere constructively* because of a 2π phase shift [cf. Eq. (23)]. In this way, a significant amount of excitation can be accumulated in this mode, in spite of the fact that its excitation per single pass is much weaker than for the $|10\rangle$ mode.

We now take into account the interference of the intracavity field with the injected field, and obtain ''nonlinear'' Airy functions for the transmission of the investigated modes (transmitted power $P_{p0,t}$ of mode $|p0\rangle$ divided by the power $P_{00,i}$ of the incident fundamental mode) as a function of the resonator phase δ ($\delta=2kL+\Phi_0$, mod 2π in the empty resonator)

$$\frac{P_{00,t}(\delta)}{P_{00,i}}=\frac{1}{N(\delta)}[A(\delta)+4\gamma^2A(\delta)K(\delta)], \tag{25a}$$

$$\frac{P_{10,t}(\delta)}{P_{00,i}}=\frac{1}{N(\delta)}[0+16\gamma^2A(\delta)], \tag{25b}$$

$$\frac{P_{20,t}(\delta)}{P_{00,i}}=\frac{1}{N(\delta)}[0+4\gamma^2A(\delta)K(\delta)], \tag{25c}$$

where

$$A(\delta)=\frac{T^2}{T^2+4R\,\sin^2\left(\frac{\delta}{2}\right)} \tag{26}$$

is the usual Airy function. The function

$$K(\delta)=\frac{T^2+4R\,\cos^2\left(\frac{\delta}{2}\right)}{T^2+4R\,\sin^2\left(\frac{\delta}{2}\right)} \tag{27}$$

describes the mode conversion and

$$N(\delta)=1+\gamma^2A(\delta)\left(32\frac{R}{T^2}A(\delta)+8K(\delta)+16\right) \tag{28}$$

ensures the normalization $a_{00}^2+a_{10}^2+a_{20}^2=1$. The calculation is valid for the conditions stated in Eqs. (16) and (20).

Figure 3(a) shows the result of a numerical evaluation of Eq. (25). It is evident that the mode conversion is maximum close to resonance, $\delta\approx 0$ [cf. Eq. (27)]. The excitation of mode $|20\rangle$ is much higher than that of mode $|10\rangle$. On resonance, this ratio is

$$\frac{P_{20,t}(0)}{P_{10,t}(0)}=\frac{1}{4}+\frac{R}{T^2}. \tag{29}$$

After a single-pass (which corresponds to the condition $R\to 0$) mode $|10\rangle$ contains four times the energy of mode $|20\rangle$. However, the feedback introduced by the resonator can render the excitation of mode $|20\rangle$ orders of magnitude higher than that of mode $|10\rangle$. For example, for $R=0.93$ this ratio is about 200:1. In the stationary state, the energy in mode $|10\rangle$ is essentially that transferred in a single pass. This can be inferred very nicely from a numerical simulation of the evolution of the transmitted power after switching on the incident beam [Fig. 3(b)]. The energy in mode $|10\rangle$ saturates or even decreases after just one round trip, whereas the energy in mode $|20\rangle$ shows a monotonic increase toward its stationary value.

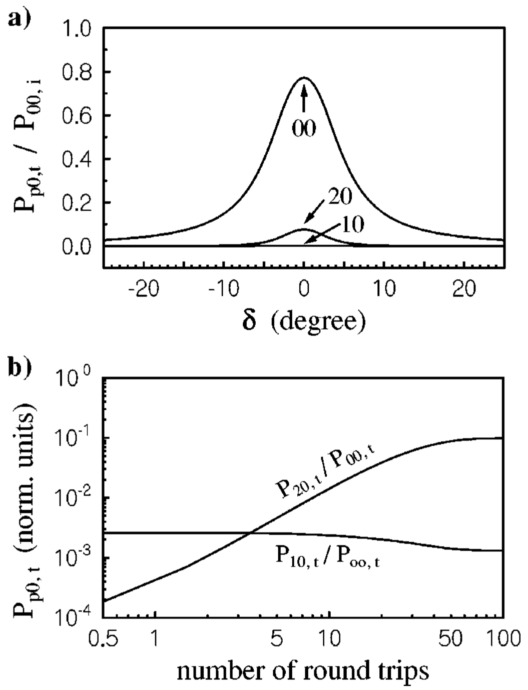

FIG. 3. (a) Transmitted power in the considered transverse modes as a function of the resonator phase [result of the perturbation analysis, Eqs. (25)]. Parameters: $R=0.93$ (our experimental situation) and $|\gamma|=0.006$. (b) Transient behavior of the transmitted power (result of a numerical simulation with a diffusive susceptibility profile).

C. General considerations

We also performed numerical simulations for radial distributions of the susceptibility which are not necessarily of polynomial form, like the one assumed in Eq. (11). In fact, Fig. 3(b) itself was calculated for a profile resulting from the nonlinear interaction of a Gaussian beam with a medium (sodium vapor) in the presence of transverse diffusion (see next section). This figure proves that a strong selection in higher-order mode excitation survives for more general forms of the refractive index profile. In addition, the simulations allow us to determine what happens if the mode degeneracy is broken: the strength of the mode conversion is rather drastically reduced already at lens powers as low as $1/f=1$ m^{-1} in a 25-cm-long resonator, which is adjusted for exact confocality in the absence of the nonlinear medium.

We stress that the selection rule is an interference condition arising from the phase shifts due to propagation. It does not essentially depend on the structure of the operator Q, but on the structure of the propagation operator P. Let us now consider the mode conversion among all those modes that are degenerate in a confocal resonator, e.g., the modes of the families s with s even. Choosing the phase of the fundamental mode as a reference, the operator P is given by

$$P=\sqrt{R}\,\exp\left(i\pi\frac{s}{2}\right)\mathbf{1}, \tag{30}$$

where $\mathbf{1}$ denotes the identity matrix. The following results are valid for Gauss-Laguerre modes TEM_{pl}, where $s=2p+|l|$, as well as for Gauss-Hermite mode TEM_{mn}, where $s=m+n$, since P_{ij} depends only on the family index s.

The medium operator Q is decomposed into the homogeneous and parabolic contributions [cf. Eq. (12)], which are taken into account by using a suitable basis set of Gauss-Laguerre or Gauss-Hermite modes of the resonator with intracavity lens, and an operator Q', which induces mode coupling. Q' is a generalization of Q_4, since it is no longer assumed that the operator stems from a truncation at fourth order of the Taylor expansion of the susceptibility, but only that the deviation of Q' from the identity is small,

$$Q'=\mathbf{1}+\gamma'\tilde{Q}, \tag{31}$$

and is characterized by a parameter γ' ($\gamma'\ll 1$).

Then, to first order in γ', all those round-trip operator elements $(PQ'PQ')_{ij}$ for which the condition

$$s(i)-s(j)=2+4\kappa, \quad \kappa\in\mathbb{Z} \tag{32}$$

is met, vanish. This means that after passing through the medium twice during the intracavity round trip, coupling between modes of these families $s(i)$ and $s(j)$ is destructive. The coupling of modes with family indices s and s' is, instead, constructive for

$$s'-s=4\kappa, \quad \kappa\in\mathbb{Z}. \tag{33}$$

These selection rules are accurate (to first order in γ') if the modes are degenerate and if the mirror transmissivity T goes to zero. However, for nonvanishing but low mirror transmissivity, the energy transfer into the ‘‘allowed’’ modes can occur via a large number of cascading stages, while the energy transferred into the ‘‘forbidden’’ modes is essentially the same as in a single pass. Therefore, one can expect the selections rules to be rather well satisfied even in the case of low mirror transmissivity [cf. Eq. (29)].

We recall that we assume the nonlinear medium to be a thin slice placed at the center of the resonator. If the slice is shifted away from the resonator center towards a mirror, the interference between components of the modes generated at successive encounters will not be completely destructive or constructive, and the selection rules will be only partially satisfied. If the medium is placed against one of the mirrors, the propagation effects disappear and therefore there are no selection rules.

We stress that the existence of selection rules is not limited to the close-to-confocal situation, though this is certainly the one that is most relevant. Since energy transfer will not occur for any two modes with a relative half-round-trip Gouy-phase shift equal to π, similar considerations apply to all Fabry-Perot resonators with ‘‘accidental’’ mode degeneracies, i.e., those for which the condition

$$(s'-s)\Phi_0=2\pi+4\pi\kappa, \quad \kappa\in\mathbb{Z} \tag{34}$$

is met for modes of families s and s'. Energy transfer can, however, occur for modes with

$$(s'-s)\Phi_0=4\pi\kappa, \quad \kappa\in\mathbb{Z}. \tag{35}$$

Note that there are no selection rules in the plano-planar configuration, because the Gouy-phase shift vanishes. In the opposite limit, for the concentric resonator configuration, a selection rule exists which suppresses energy transfer between modes that satisfy the following condition:

$$(s'-s)=1+2\kappa, \quad \kappa\in\mathbb{Z}. \tag{36}$$

For example, mode conversion from the fundamental mode $(s=0)$ to the doughnut mode $(s'=1)$ is forbidden in a concentric Fabry-Perot resonator although these modes are degenerate.

It is important to observe that such selection rules do *not* exist in *ring* resonators, since in a traveling wave cavity the medium is encountered only once during a round trip and therefore no interference conditions arise after a half round trip. This constitutes a *qualitative* difference between standing wave and ring resonators and requires considerable care in any attempt to describe experiments on mode conversion in Fabry-Perot resonators with ring resonator models—or vice versa.

IV. NUMERICAL INVESTIGATIONS

In the preceding analytical treatment, we have considered the index of refraction profile to be ''fixed.'' In a more general situation, the generated modes will in turn influence the susceptibility profile. Therefore, with the help of a numerical simulation, we check whether the selection rules are still observable in a more realistic situation, where we include finite absorption and outcoupling losses. As in the experiment (cf. Sec. V), we consider sodium as the nonlinear medium. Its nonlinearity is saturable and is due to optical pumping between the Zeeman sublevels of the ground state of the D_1 line in interaction with circularly polarized light (see, e.g., [24]). Transverse coupling is provided by the thermal atomic diffusion and by the reabsorption of trapped resonance fluorescence (e.g., [25]), also treated in a diffusive limit [23]. Details about the model and numerical treatment can be found in [23]. However, the main results are independent of the details of the nonlinearity.

In order to model a realistic case, we consider the empty resonator to be exactly confocal and choose incoupling conditions such that only 78% of the energy is coupled into the fundamental mode of the empty cavity while the remaining 22% goes into the first higher order mode $|10\rangle$ $(s=2)$ [26]. Figure 4 shows the sum of the transmitted power for the various families of modes with $s=\text{const.}$, normalized to the total input power, as a function of the resonator phase δ ($\delta=0$ is the resonance for the even modes in the empty resonator.). In contrast to the analytical treatment, here we decompose the intracavity field into modes of the empty resonator, since the intracavity lens power is parameter dependent and a real-time-compensation of the resonance shift by an appropriate change in the macroscopic cavity length [cf. Eqs. (13), (14)]—although possible in principle—is not realizable in an experiment. Nevertheless, the power contained in the $s=0,2,4,6$ families is still larger than 93%, so that this basis set still appears to be suitable to characterize the intracavity field.

We first concentrate on the two resonance curves labeled $s=0$ and $s=2$ (these modes are directly excited by the incoupled beam). Due to the nonlinear lifting of the mode degeneracy (cf. [22]), in the upper range of phase angles shown $(\approx 80°$ to $\approx 100°)$, the energy carried by the $|10\rangle$ mode is comparable to, or even higher than, that carried by the fundamental mode. On the other hand, for lower values of δ the $|00\rangle$ mode takes a larger part of the energy ($P_t/P_i|_{00}=0.3$ at $\delta=40°$ and $P_t/P_i|_{00}=0.7$ at $\delta=15°$, off scale in the figure).

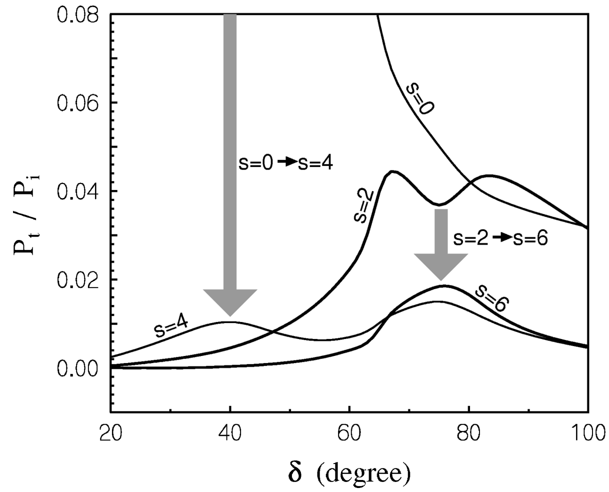

FIG. 4. Transmitted power in the transverse modes with $s=2p+|l|=\text{const}$ as a function of the resonator phase. Parameters of the simulation, laser power: 2.5 mW; detuning with respect to sodium D_1 line, 65 GHz; sodium particle density, $1.9\times10^{13}\ \text{cm}^{-3}$; confocal resonator with length: 250 mm; $R=0.93$; ensemble lifetime of population of excited state, 160 ns, i.e., 10 times the natural lifetime. This parameter characterizes the strength of the radiation trapping effects [25].

The splitting of these resonances (degenerate in the empty cavity) is due to the self-induced intracavity lens [cf. Eq. (13)] and their relative amplitude is determined by the filtering action—due to the quadratic terms in Eq. (12)—of the resonator with internal lens [5,27]. The fact that the power contained in the $s=2$ family is negligible where the $s=0$ mode is strongest indicates a lack of energy transfer from one to the other.

We now consider those effects that do not result from self-lensing but from mode conversion. For values of $\delta\approx 40°$, there is a noticeable amount of energy transferred from the fundamental mode towards modes of the family $s=4$ (marked $s=0\rightarrow s=4$ in Fig. 4), which is now more strongly excited than the $s=2$ family. In this region of δ values, the modes belonging to the family $s=6$ contain a negligible amount of energy, thus showing that there is no appreciable transfer from $s=0\rightarrow s=6$, in agreement with the selection rule.

For $\delta\approx 75°$, i.e., when the modes of the $s=2$ family are strongly excited, it is the $s=6$ family of modes that receives a significant amount of energy through mode conversion. Indeed, this transfer appears to be strong enough to cause a dip in the resonance of the $s=2$ family (marked by $s=2\rightarrow s=6$). Though the selections rules do not appear to be strictly valid in this resonator phase range, it is remarkable that, as predicted, the excitation of the $s=6$ family is stronger than that of the $s=4$ family.

V. EXPERIMENT

We check the validity of the above treatment by measuring the energy transfer between transverse modes in a con-

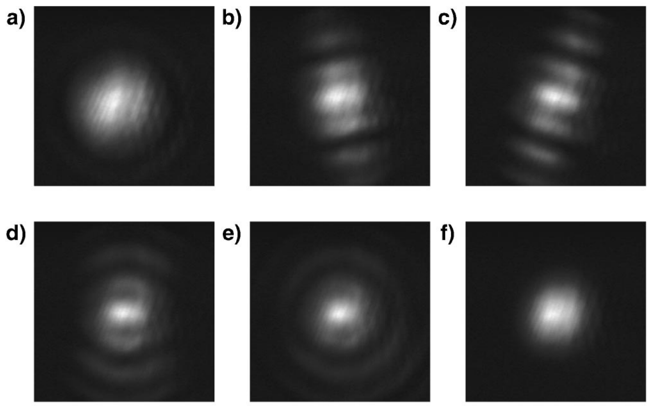

FIG. 5. Sequence of patterns observed for decreasing resonator phase. Parameters, laser power about 100 mW; detuning with respect to sodium D_1 line, 50 GHz; sodium particle density, 10^{13} cm^{-3}, resonator length, 240 mm (10 mm shorter than the confocal length).

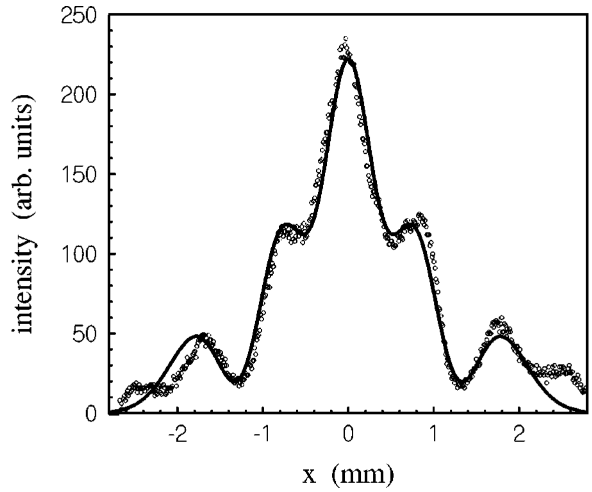

FIG. 6. Cross section along the major principal axis of the pattern in Fig. 5(b). The measured profile is denoted by circles, the solid line is the result of a least-squares fit to a superposition of Gauss-Hermite modes (see text).

focal (or nearly confocal) Fabry-Perot resonator filled with sodium vapor as a nonlinear medium. In the experiment, a spatially filtered Gaussian beam of a cw dye laser is injected into a sodium vapor cell that is placed in the center of a Fabry-Perot resonator (focal length of mirrors $f_m = 125$ mm, reflectivity $R = 0.93$). The parameters of the input beam are controlled by a system of two lenses. The fraction of energy coupled into the higher-order modes of the empty resonator is estimated to be smaller than about 10%. The transverse section of the transmitted beam is monitored by a coupled charge density (CCD) camera, placed 800 mm behind the resonator. Details of the setup can be found in [5].

As explained in Sec. IV, it is not possible to maintain the mode degeneracy during a scan of the resonator phase in an actual experimental setup. To circumvent this problem, we performed systematic studies where we varied the macroscopic resonator length around its confocal value and have operated the system on both sides of the atomic resonance, therefore allowing for the appearance of self-induced converging or diverging lenses.

When the resonator is shorter than its confocal length (by about 10 mm) we observe a rather complex sequence of patterns as a function of the resonator phase on the focusing side of the atomic resonance (Fig. 5). The patterns in Figs. 5(b) and 5(c) have at least four distinct intensity maxima (excluding the central one) thereby indicating that they include substantial contributions of the TEM_{40} mode—in Gauss-Hermite notation. For this resonator configuration, on the defocusing side of the resonance, the sequences are simpler and the patterns do not show more than two peaks off center—an indication that they can be well described by modes of up to order 2. In the complementary situation (resonator longer than its confocal length), the changes in patterns are less striking, but noticeable. In this case, as expected from symmetry considerations, the more complex patterns appear on the defocusing side of the resonance.

We interpret these observations as follows. On the focusing side of the resonance the focal power of the self-induced intracavity lens is positive. Therefore, the resulting increment in Gouy-phase shift that it causes can compensate for the frequency splitting introduced by the length offset $d - f_m < 0$ of the empty resonator [cf. Eqs. (13), (14)]. This compensation will occur more or less exactly for some specific resonator phase and strong mode conversion will ensue. On the defocusing side of the line, instead, the effect of the lens and of the length offset combine and enhance the splitting, thereby strongly disfavoring mode conversion. If the length offset of the resonator changes its sign, the role of the defocusing and focusing side of the resonance are interchanged.

To obtain more quantitative information, we extracted one-dimensional cuts from the patterns of Figs. 5(b) and 5(c) along their principal axes. These intensity profiles were fitted to a superposition of the TEM_{00}, TEM_{20}, and TEM_{40} modes in Gauss-Hermite notation (Fig. 6). Although the fitting procedure contains six or seven free parameters to be adjusted for the optimization (the amplitudes of the three modes, two mutual phases, a normalization constant, and, possibly, the beam radius) the estimated uncertainty is low enough to state that the excitation of the TEM_{40} is considerably larger than that of the TEM_{20} [about 30% versus 4% for the pattern in Fig. 5(b) and about 20% versus 10% in the case of Fig. 5(c)]. This is a strong experimental confirmation of the existence of energy transfer, governed by selection rules, among modes of a Fabry-Perot resonator.

VI. CONCLUSIONS

We have established the existence of selection rules that govern the energy transfer between different transverse modes in Fabry-Perot cavities containing a nonlinear medium. Their origin is the constructive (or destructive) interference between the complex amplitudes of the two contributions to a mode that are generated in the two interactions with the nonlinear medium within one roundtrip. As a consequence of these rules, the spatial patterns that may be expected in or predicted for such a Fabry-Perot cavity are different from those occurring in a ring resonator.

The discussion in this paper concentrates on passive systems in which an external beam is injected. Nevertheless, it is to be expected that the existence or prohibition of a direct, coherent energy transfer between different transverse modes also influences the asymptotic modal distribution in active systems.

ACKNOWLEDGMENTS

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