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Fractional revivals in systems with two time scales

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We examine the dynamical evolution of wave packets in a large class of quantum systems where two quantum numbers determine the energy spectrum and consequently the dynamical behavior. Using a generic Hamiltonian, we study the formation of coherent structures. The ratio of two time scales as well as the symmetry of the initial wave packet decisively determines the revival and fractional revivals of the system for which we give analytical results. The theory is applicable to a wide class of systems from diatomic molecules to ions in two-dimensional traps and two interacting Bose condensates. [S1050-2947(98)00505-8]

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The emergence of coherent structures such as Schrödinger cats [1] and the occurrence of full, fractional, or super revivals [2–7] in the nonlinear evolution of a system have been recognized to be closely connected to the energy spectrum of the underlying Hamiltonian and the periodicity of the time evolution operator. Thus there are extensive calculations on Rydberg atoms [2], optical parametric oscillators [3], the Jaynes-Cummings model [4], transient signals from multilevel quantum systems [5], potential wells [6], molecular vibrational states [7], and light propagation in Kerr media [1]. It became apparent after all these explicit calculations that the general features of the nonlinear quantum dynamics are quite generic and in most cases can be understood by the first anharmonic contribution to the energy spectrum.

Almost all explicit calculations reported in the literature deal with systems whose energy spectrum depends on a *single* quantum number [8]. In reality, many systems exist whose energy levels depend nonlinearly on at least two quantum numbers [9,10]. The rotational-vibrational spectrum of diatomic molecules, particles in Morse-like potentials or in two-dimensional boxes, Stark wave packets, ions in a trap interacting with a two-dimensional field are but a few examples. Similar situations also arise when elliptically polarized light passes through a fiber. Recent work by Wright *et al.* [11] shows that, to a good approximation, a Bose condensate can be modeled as an anharmonic oscillator and thus the results that we present here should also be applicable to binary condensates.

Clearly, we can either study each of these systems individually or develop a general theory and uncover the generic features of the quantum dynamics of systems whose energy levels depend on two quantum numbers. We take the latter approach in this paper. We explain the complex revival structure of these systems by (i) presenting explicit analytical results under the framework of a general theory that can be modified to suit specific cases and (ii) studying the evolution of autocorrelation functions, phase distributions, and probability densities. We specifically examine the effects that arise from the existence of two time scales, which themselves are determined from the dependence of the energy spectrum on two quantum numbers. Thus the time scales that we consider arise from the anharmonicities in the energy spectrum and *not* from some elementary consideration such as the different frequencies of two oscillators.

In the special case in which one of the revival time scales, say T_+, goes to infinity, the system dynamics is effectively scaled by T_- alone. Otherwise, we show that the ratio of these time scales will play a significant role in the revival characteristics of the system. Furthermore, we demonstrate the crucial dependence of fractional revivals on the symmetry property of the wave packets. Our major analytical results are given in Eqs. (6)–(8) and (11).

We consider the following generic Hamiltonian, which would provide the *leading* anharmonic terms such as m^2, n^2, and mn in the energy spectrum (we use $\hbar=1$):

$$H=c_1[(a^\dagger a)^2+(b^\dagger b)^2]-c_2 a^\dagger a b^\dagger b. \tag{1}$$

An even more general Hamiltonian would have different coefficients of $(a^\dagger a)^2$ and $(b^\dagger b)^2$.

Our method of approach differs from that of Bluhm *et al.* [9] as, for an interacting system, we consider it to be more appropriate to introduce time scales based on a type of diagonalization rather than the ones based on bare parameters in the Hamiltonian. Thus we work with time scales $T_\pm=2\pi/(c_1\pm c_2/2)$ rather than $2\pi/c_1$ and $2\pi/c_2$. As for initial states, we can use a variety of wave packets involving a superposition of the eigenstates of the system provided the weight factors in the superposition are peaked with a small dispersion. In our calculation, the initial wave packet is a two-mode coherent state $|\alpha,\beta\rangle=|\alpha\rangle\otimes|\beta\rangle$. As is well known, the coherent state $|\alpha\rangle$ is a Gaussian wave packet in configuration space involving a superposition of all the number states with a weight factor given by the Poisson distribution.

The time evolution operator has the expansion

$$U(t)=\sum_{p,q}\exp\{-it[c_1(p^2+q^2)-c_2pq]\}|p,q\rangle\langle p,q|. \tag{2}$$

The double sum in Eq. (2) is split into two parts: $U_+(t)$, for which $p=q \pmod 2$, and $U_-(t)$, for which $p\neq q \pmod 2$. Setting $c_1\pm c_2/2=d_\pm$, we write

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$$U_+(t)=\sum_{j,k=0}^{\infty} e^{-2itd_+(j-k)^2}[e^{-2itd_-(j+k)^2}|2j,2k\rangle\langle 2j,2k| + e^{-2itd_-(j+k+1)^2}|2j+1,2k+1\rangle\langle 2j+1,2k+1|], \tag{3}$$

$$U_-(t)=\sum_{j,k=0}^{\infty} e^{-itd_-(2j+2k+1)^2/2}[e^{-itd_+(2j-2k-1)^2/2} \times|2j,2k+1\rangle\langle 2j,2k+1|+e^{-itd_+(2j-2k+1)^2/2} \times|2j+1,2k\rangle\langle 2j+1,2k|]. \tag{4}$$

This prompts us to express the initial two-mode coherent state $|\alpha,\beta\rangle$ in terms of the (normalized) ‘‘even’’ and ‘‘odd’’ states

$$|\alpha,\beta\rangle_\pm=(|\alpha,\beta\rangle\pm|-\alpha,-\beta\rangle)/\sqrt{2P_\pm},$$

$$P_\pm=1\pm\exp[-2(|\alpha|^2+|\beta|^2)]. \tag{5}$$

It is clear that $U_\pm(0)|\alpha,\beta\rangle_\pm=|\alpha,\beta\rangle_\pm$ and $U_\pm(t)|\alpha,\beta\rangle_\mp=0$. The corresponding autocorrelation functions $A=\langle\alpha,\beta|U(t)|\alpha,\beta\rangle$ and $A_\pm=\langle\alpha,\beta|_\pm U_\pm(t)|\alpha,\beta\rangle_\pm$ satisfy the simple condition $A=(P_+A_++P_-A_-)/2$. The two-dimensional problem is thus effectively diagonalized.

We first determine the collapse and revival of the system. The revival time for $|\alpha,\beta\rangle$ and $|\alpha,\beta\rangle_\pm$ will be some multiple of $T=sT_-=rT_+$, where r and s are mutually prime and $T_\pm=\pi/2d_\pm$. For integer values of N we get

$$U_+(NT)|\alpha,\beta\rangle_+=|\alpha\exp(i\pi\phi_+),\beta\exp(i\pi\phi_-)\rangle_+,$$

$$U_-(NT)|\alpha,\beta\rangle_-=\exp(-i\pi\phi_+/2)|\alpha,\beta\rangle_-,$$

$$\phi_\pm=N(s\pm r)/2. \tag{6}$$

Clearly, $|\alpha,\beta\rangle_-$ revives (but for an overall phase) for all values of N, whereas $|\alpha,\beta\rangle_+$ revives for even values of N only. For $|\alpha,\beta\rangle$, on the other hand, N depends crucially on s and r. If $s\neq r$ (mod 2), then $N=0$ (mod 8), i.e., $|\alpha,\beta\rangle$ revives at $8T$, $16T,\ldots$. If $s=r=1$ (mod 2), let $s=2p+1$ and $r=2q+1$. In this case, we find $N=0$ (mod 4) if $p=q$ (mod 2) and $N=0$ (mod 2) if $p\neq q$ (mod 2).

A few examples are in order. Suppose $s=2$ and $r=3$, so that $T_+/T_-=2/3=0.666,\ldots$, or, equivalently, $c_2/2c_1=0.2$. We set $c_1=\pi/9.6$ in some appropriate unit so that $T_-=6$, $T_+=4$, and $T=12$. The earliest revival times for $|\alpha,\beta\rangle_-$, $|\alpha,\beta\rangle_+$, and $|\alpha,\beta\rangle$ will be at $t/T_-=2$, 4, and 16, respectively (Fig. 1). Note that the graphs in Fig. 1 are symmetric about half the revival time.

If the ratio of the time scales is changed to 0.6 instead, then $s=3$ and $r=5$. Furthermore, setting $c_1=\pi/9$, we can keep T_- the same as before. However, the revival times will now change to $t/T_-=3$, 6, and 6 for $|\alpha,\beta\rangle_-$, $|\alpha,\beta\rangle_+$, and $|\alpha,\beta\rangle$, respectively.

If $T_-=T_+$, i.e., $c_2=0$ and $s=r=1$, then the revival periods for $|\alpha,\beta\rangle_-$, $|\alpha,\beta\rangle_+$, and $|\alpha,\beta\rangle$ will be at $t/T_-=1$, 2, and 4, respectively. Finally, if one of the time scales, say T_+, goes to ∞ (in which case $c_2=-2c_1$, $s\rightarrow 1$, and $r\rightarrow 0$), then the shortest times for the reproduction of $|\alpha,\beta\rangle_-$, $|\alpha,\beta\rangle_+$, and $|\alpha,\beta\rangle$ are, T_-, $2T_-$, and $8T_-$, respectively. This situation has an analog in the regeneration characteristics of a field $E(x)$ of wavelength λ propagating through a multimode *planar* waveguide of width b. If the field is symmetric in the transverse dimension $E(-x)=E(x)$, its regeneration length is $L=b^2/\lambda$; an antisymmetric field $E(-x)=-E(x)$ regenerates at a distance $2L$, while an arbitrary field is reproduced after a guided propagation of length $8L$ [12]. We mention in passing that this behavior in planar (and rectangular) waveguides can be traced to the quadratic dependence (in the paraxial approximation) of the propagation constant on the mode numbers.

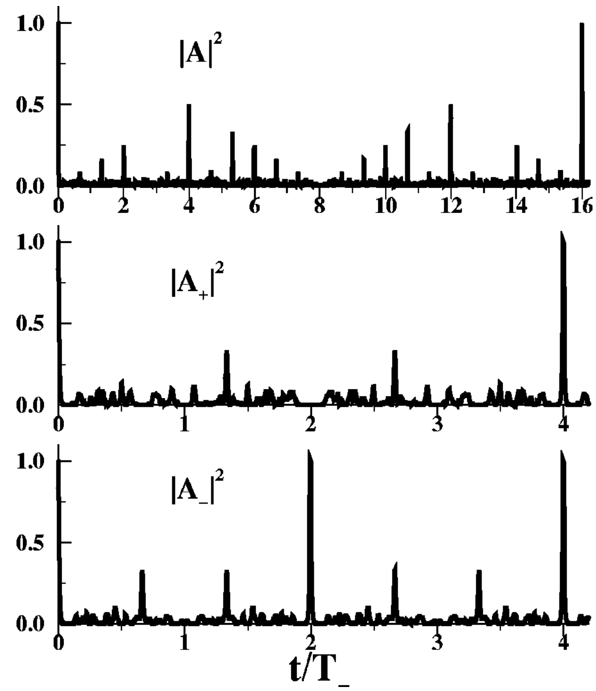

FIG. 1. Absolute square of the autocorrelation function as a function of time for $T_+/T_-=2/3$ when $\alpha=2$ and $\beta=3$.

Between revivals, the system undergoes collapse and fractional revivals. In the case of collapse, the autocorrelation functions go to zero typically as $\exp[-(|\alpha|^2+|\beta|^2)]$. However, the two-mode case also presents possibilities (which may be relevant for mesoscopically occupied states) when A_+ or A_- can be identically zero. Thus, for example, when $T_+/T_-=2/3$ and N is odd, $U_+(2NT_-)|\alpha,\beta\rangle_+=|i\alpha,-i\beta\rangle_+$, for which $A_+=0$ if $|\alpha|^2-|\beta|^2$ is an odd multiple of $\pi/2$. Since $|\alpha,\beta\rangle_-$ and $|\alpha,\beta\rangle_+$ have different collapse (or revival) time scales, it may even be possible to produce them by propagating a two-mode coherent state $|\alpha,\beta\rangle$ through a Mach-Zehnder type of interferometer with two different Kerr fibers.

Turning to fractional revivals of the odd and even states, we set $t=(m_1/n_1)T_-=(m_2/n_2)T_+$, where (m_1,n_1) and (m_2,n_2) are pairs of mutually coprime numbers. The nonlinear phase shifts produced by terms quadratic in the summation indices are written in terms of linear phase shifts by introducing their discrete Fourier transforms. We thus obtain

$$U_+(t)|\alpha,\beta\rangle_+ = \sum_{p_1=0}^{l_1-1}\sum_{p_2=0}^{l_2-1} a_{p_1}^{(m_1,n_1)} a_{p_2}^{(m_2,n_2)} \times |\alpha\ \exp(-i\pi\phi_+^{(e)}),\beta\ \exp(-i\pi\phi_-^{(e)})\rangle_+ , \tag{7}$$

$$U_-(t)|\alpha,\beta\rangle_- = e^{-i\pi(m_1/n_1+m_2/n_2)/4}\sum_{p_1=0}^{l_1-1}\sum_{p_2=0}^{l_2-1} a_{p_1}^{(m_1,n_1)} a_{p_2}^{(m_2,n_2)} \times \exp(i\pi\phi_+^{(o)})|\alpha\ \exp(-i\pi\phi_+^{(o)}),\beta\ \exp(-i\pi\phi_-^{(o)})\rangle_- , \tag{8}$$

where $\phi_\pm^{(e)} = p_1/l_1 \pm p_2/l_2$ and $\phi_\pm^{(o)} = \phi_\pm^{(e)} + (m_1/n_1 \pm m_2/n_2)/2$. The period l_j depends on the parity of m_j and n_j. It can be shown (dropping the subscript j for clarity) that

$$l = \begin{cases} n & \text{for } m \neq n \ (\text{mod } 2) \\ 2n & \text{for } m = n = 1 \ (\text{mod } 2). \end{cases} \tag{9}$$

The coefficients $a_p^{(m,n)}$ are given by

$$a_p^{(m,n)} = \frac{1}{l}\sum_{k=0}^{l-1} \exp(-i\pi m k^2/n + 2\pi i p k/l) \tag{10}$$

and can be evaluated analytically [13].

Note that the expressions for $U_-(t)|\alpha,\beta\rangle_-$ can be rewritten such that the phase factors associated with α and β are independent of m_1 and m_2 as in the case of $U_+(t)|\alpha,\beta\rangle_+$. Thus we get

$$U_-(t)|\alpha,\beta\rangle_- = e^{-i\pi(m_1/n_1+m_2/n_2)/4} \times \sum_{p_1=1}^{n_1}\sum_{p_2=1}^{n_2} b_{p_1}^{(m_1,n_1)} b_{p_2}^{(m_2,n_2)} \times e^{i\pi\theta_+}|\alpha e^{-i\pi\theta_+},\beta e^{-i\pi\theta_-}\rangle_- , \tag{11}$$

where $\theta_\pm = (n_1+1-2p_1)/(2n_1) \pm (n_2+1-2p_2)/(2n_2)$ and

$$b_p^{(m,n)} = \begin{cases} a_{(n-m+1)/2-p}^{(m,n,1)} & \text{if } m \neq n \ (\text{mod } 2) \\ a_{(n-m+1-2p)}^{(m,n,1)} & \text{if } m = n = 1 \ (\text{mod } 2). \end{cases} \tag{12}$$

We will now study fractional revivals by looking at the wave-packet dynamics of the system. At $t=0$, the even and odd states are represented by pairs of Gaussians centered at $x = \pm\alpha\sqrt{2}$ and $y = \pm\beta\sqrt{2}$ (we use real values of α and β):

$$\rho_\pm(x,y) = \pi^{-1/2} e^{-[(x\pm\alpha\sqrt{2})^2 + (y\pm\beta\sqrt{2})^2]/2},$$

$$\psi_\pm(x,y,0) = [\rho_-(x,y) \pm \rho_+(x,y)]/\sqrt{2P_\pm}, \tag{13}$$

whereas the initial state is given by $\psi(x,y,0) = \rho_-(x,y)$.

These are smooth functions of x and y. However, as the system evolves, the probability densities can develop ridged structures as well due to the occurrence of sinusoidal terms in the wave functions (Fig. 2). For example, referring to Fig. 1, $|i\alpha,-i\beta\rangle_+$, which is the even state at $t/T_- = 2$, has an oscillatory wave function $\sqrt{2/\pi P_+}\ \exp[-(x^2+y^2)/2]\cos[\sqrt{2}(\alpha x - \beta y)]$ whereas $|\alpha,\beta\rangle_+$ does not. Using these wave functions and Eqs. (6), we find that at $t/T_- = 4$ or 12 and $T_+/T_- = 2/3$, $\psi(x,y,t) = [\exp(-i\pi/4)\rho_-(x,y) + \exp(i\pi/4)\rho_+(x,y)]/\sqrt{2}$. The components of the wave function do not interfere with each other as they are $\pi/2$ out of phase. This represents a Schrödinger cat state [Fig. 2(c), left column], a two-way fractional revival in accordance with $|A(t)|^2$ being equal to 1/2. However, at $t/T_- = 2$, $U(t)|\alpha,\beta\rangle = \sqrt{P_+/2}|i\alpha,-i\beta\rangle_+ - \exp(-i\pi/4)\sqrt{P_-/2}|\alpha,\beta\rangle_-$. The corresponding probability density will have a central modulated peak due to $|i\alpha,-i\beta\rangle_+$ and two smooth peaks at the edges due to $|\alpha,\beta\rangle_-$ [Fig. 2(a), left column]. This is not a four-way fractional revival. Furthermore, for the same value of time, a change in the ratio of T_+/T_- will drastically alter the characteristics of fractional revival (Fig. 2).

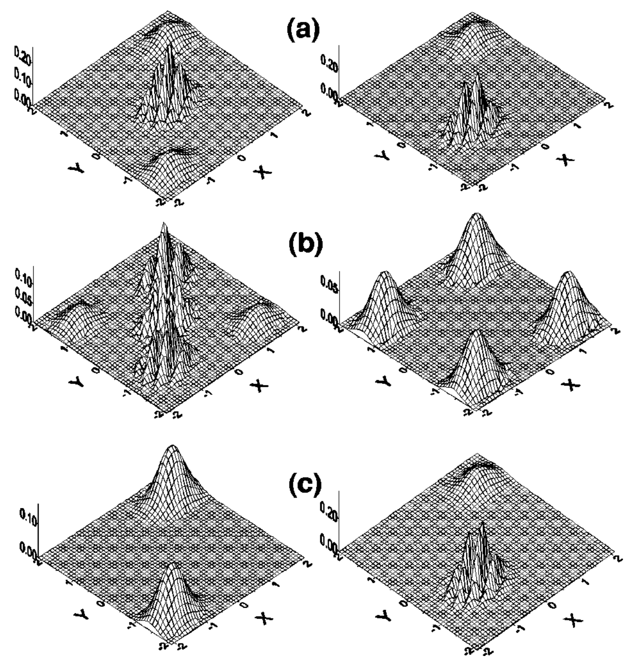

FIG. 2. Two-dimensional probability densities $|\psi(x,y,t)|^2$ as functions of $X = x/\alpha$ and $Y = y/\beta$ for $\alpha = 2$ and $\beta = 3$ when (a) $t/T_- = 2$, (b) $t/T_- = 3$, and (c) $t/T_- = 4$. Left column, $T_+/T_- = 2/3$; right column, $T_+/T_- = 3/5$.

The Gaussian pairs that represent the odd and even states at $t=0$ are well separated for $\alpha,\beta > 1$ since the interference between ρ_- and ρ_+ is negligible. However, as the wave packets spread, there can be significant overlap between the components of the wave packets even at fractional revival times (Fig. 3, left column). For a given set of (m_1, n_1) and (m_2, n_2), this interference can be reduced only for larger values of α and β (Fig. 3, right column). Since we get superpositions of coherent states with different phases, the phase distribution function [14] will exhibit a quite instructive multipeak structure (Fig. 4).

We briefly consider the limit $T_+ \to \infty$, i.e., the limit $n_2 \to \infty$ and $p_2 \to 0$. In this case, the double sums in Eqs. (7), (8), and (11) collapse to single ones. Also, α and β are multiplied by the same phase factor. A similar observation can be made in the earlier context of revival and collapse also. Thus an immediate consequence of two different time scales is that $|\alpha\rangle$ and $|\beta\rangle$ are rotated differently in phase

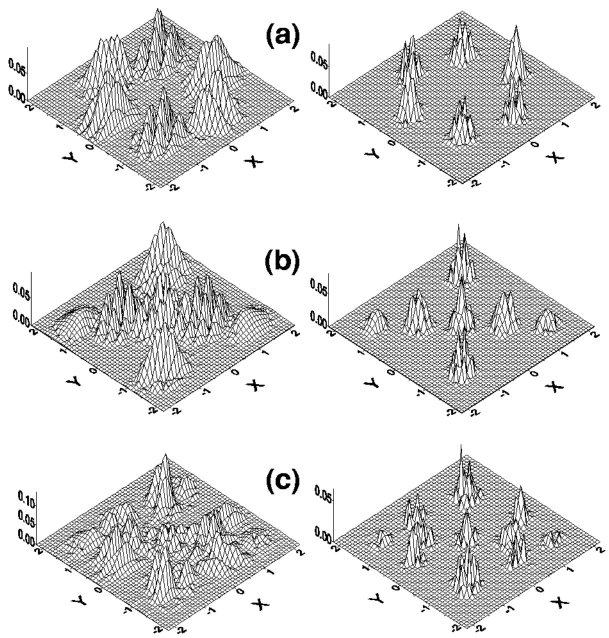

FIG. 3. Probability densities (a) $|\psi_-(x,y,t)|^2$, (b) $|\psi_+(x,y,t)|^2$, and (c) $|\psi(x,y,t)|^2$ as functions of $X=x/\alpha$ and $Y=y/\beta$ when $t=T_+/2=T_-/3$. Left column, $\alpha=2$ and $\beta=3$; right column, $\alpha=4$ and $\beta=5$.

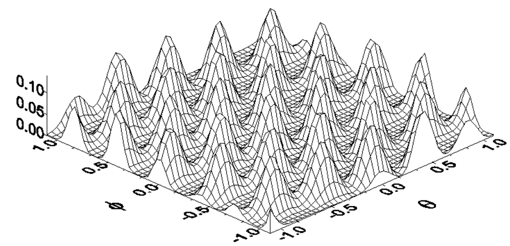

FIG. 4. Two-dimensional continuous phase distribution function $P(\theta,\phi)$ for $U(T_-/3)|2,3\rangle$ when $T_+/T_-=2/3$; angles are in units of π.

space. Since for a coherent state revival corresponds to a 2π rotation in phase space, the situation is analogous to two (or, better still, two packs of) ''runners on a track.''

As mentioned in the introduction, the results of this paper are applicable to a large class of systems. We cite some examples that are experimentally realizable. (a) The anharmonic terms in Eq. (1) are precisely the ones occurring in the stretching of bonds [15] in a linear molecule such as H_2O. Thus suitable wave packets of such molecules will exhibit quantum dynamic behavior found in this paper. (b) Consider the motion of an ion or atom in a trap in the presence of a two-dimensional electromagnetic field with a field profile $\mathcal{E}(\cos kx - \cos ky)$. If the field is far detuned from resonance, then the motion is in a potential determined by the Stark shift $|\cos kx - \cos ky|^2$. If kx and ky are much smaller than unity, then the potential is equivalent to the anharmonic potential of Eq. (1). (c) Wright *et al.* [11] have given a description of Bose condensate in terms of a single mode and shown the presence of collapses and revivals in the dynamics. Clearly, a generalization of their work to binary condensates [16] will involve Hamiltonians such as Eq. (1).

In conclusion, we have shown how systems with an energy spectrum depending essentially on two quantum numbers can lead to a different time scale in the quantum dynamics of the wave packet. The ratio of two time scales determines the nature of revivals and fractional revivals, which also depend on the spatial symmetry of the initial wave packet. Finally, it should be clear that the dynamics of many other wave packets can be studied using the general decomposition (3) and (4) of the evolution operator.

[1] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986); G. J. Milburn and C. A. Holmes, *ibid.* **56**, 2237 (1986); W. Schleich, M. Pernigo, and Fam Le Kien, Phys. Rev. A **44**, 2172 (1991); K. Tara, G. S. Agarwal, and S. Chaturvedi, *ibid.* **47**, 5024 (1993); V. Buzek, H. Moya-Cessa, and P. L. Knight, *ibid.* **45**, 8190 (1992); R. Tanaś, Ts. Gantsog, A. Miranowicz, and S. Kielich, J. Opt. Soc. Am. B **8**, 1576 (1991); G. C. Gerry, Opt. Commun. **63**, 278 (1987).

[2] I. Sh. Averbukh and N. F. Perelman, Phys. Lett. A **139**, 449 (1989); M. Nauenberg, J. Phys. B **23**, L385 (1990); G. Alber, H. Ritsch, and P. Zoller, Phys. Rev. A **34**, 1058 (1986); Z. D. Gaeta and C. R. Stroud, Jr., *ibid.* **42**, 6308 (1990); R. Bluhm, V. Alan Kostelecky, and B. Tudose, *ibid.* **52**, 2234 (1995); **53**, 937 (1996); J. Wals *et al.*, Phys. Rev. Lett. **72**, 3783 (1994).

[3] I. V. Jyotsna and G. S. Agarwal, J. Mod. Opt. **44**, 305 (1997); G. S. Agarwal and J. Banerji, Phys. Rev. A **55**, R4007 (1997); G. Drobny and I. Jex, *ibid.* **45**, 1816 (1992).

[4] J. H. Eberly, N. B. Narozhny, and J. J. Sanchez-Mondragon, Phys. Rev. Lett. **44**, 1323 (1980); P. L. Knight and B. W. Shore, Phys. Rev. A **48**, 642 (1993); G. Rempe, H. Walther, and N. Klein, Phys. Rev. Lett. **58**, 353 (1987).

[5] C. Leichtle, I. Sh. Averbukh, and W. P. Schleich, Phys. Rev. A **54**, 5299 (1996).

[6] D. L. Aronstein and C. Stroud, Phys. Rev. A **55**, 4526 (1997); F. Grobmann, J. M. Rost, and W. P. Schleich, J. Phys. A **30**, L277 (1997).

[7] M. J. J. Vrakking, D. M. Villeneuve, and A. Stolow, Phys. Rev. A **54**, R37 (1996).

[8] It should be borne in mind that the states that form the wave packet can depend on many quantum numbers though the energy may depend only on a single quantum number as, for example, in the case of Rydberg atoms.

[9] R. Bluhm, V. Alan Kostelecky, and B. Tudose, Phys. Lett. A **222**, 220 (1996).

[10] (a) G. S. Agarwal and R. R. Puri, Phys. Rev. A **40**, 5179 (1989); (b) Ts. Gantsog and R. Tanaś, Quantum Opt. **3**, 33 (1991); (c) R. Bluhm, V. Alan Kostelecky, and B. Tudose, Phys. Rev. A **55**, 819 (1997).

[11] E. M. Wright, T. Wong, M. J. Collett, S. M. Tan, and D. F. Walls, Phys. Rev. A **56**, 591 (1997).

[12] This phenomenon was fruitfully utilized to propose different designs for a variety of optical devices. See, for example, R.

M. Jenkins, R. W. J. Devereux, and J. M. Heaton, Opt. Lett. **17**, 991 (1992); J. Banerji, J. Opt. Soc. Am. B **14**, 2378 (1997); J. Banerji, A. R. Davies, and R. M. Jenkins, Appl. Opt. **36**, 1604 (1997).

[13] J. H. Hannay and M. V. Berry, Physica D **1**, 267 (1980).

[14] S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989); see also Ref. [10(b)].

[15] F. Iachello and R. D. Levine, *Algebraic Theory of Molecules* (Oxford University Press, New York, 1995), Eq. 4.28.

[16] E. V. Goldstein and P. Meystre, Phys. Rev. A **55**, 2935 (1997).