Bose-Einstein solitons in highly asymmetric traps

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(Received 29 October 1997)

We obtain analytic solutions to the Gross-Pitaevskii equation with negative scattering length in highly asymmetric traps. We find that in these traps the Bose-Einstein condensates behave like quasiparticles and do not expand when the trapping in one direction is eliminated. The results can be applicable to the control of the motion of Bose-Einstein condensates. [S1050-2947(98)05005-7]

PACS number(s): 03.75.Fi, 03.65.Ge

I. INTRODUCTION

The recent experimental realization of Bose-Einstein condensation (BEC) in ultracold atomic gases [1,2] has triggered the theoretical exploration of the properties of Bose gases. Specifically there has been a great interest in the development of applications that make use of the properties of this state of matter. Perhaps the recent development of the so-called atom laser [3] is the best example of the interest of these applications.

The current model used to describe a system with a fixed mean number N of weakly interacting bosons, trapped in a parabolic potential $V(r)$, is the nonlinear Schrödinger equation (NLSE) [which in this context is called the Gross-Pitaevskii equation (GPE)]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi + V(r)\psi + U_0|\psi|^2\psi, \qquad (1)$$

which is valid when the particle density and temperature of the condensate are small enough. Here $U_0 = 4\pi\hbar^2 a/m$ characterizes the interaction and is defined in terms of the ground-state scattering length a. The normalization for ψ is $N = \int |\psi|^2 d^3\vec{r}$ and the trapping potential is given by

$$V(\vec{r}) = \frac{1}{2} m \nu^2 (\lambda_x^2 x^2 + \lambda_y^2 y^2 + \lambda_z^2 z^2), \qquad (2)$$

the λ_η $(\eta = x,y,z)$ being, as usual, constants describing the anisotropies of the trap [4]. In real experimental systems the geometry of the trap imposes the condition $\lambda_x = \lambda_y = 1$. $\lambda_z = \nu_z/\nu$ is the quotient between the frequency along the z direction ν_z and the radial one $\nu_r \equiv \nu$. Equation (1) is strictly valid in the $T=0$ and low-density limit, but has been validated in different ways for the current experimental systems, e.g., by the comparison of the experimental [5] and theoretical low-energy excitation spectra of the condensates [6,7]. Recent theoretical work extends the applicability of the GPE to the high-density limit [8,9].

When $a>0$ the interaction between the particles in the condensate is repulsive, as in most current BEC experiments [1,5,10,11]. In opposite case $(a<0)$ the interaction is attractive [2,12].

Although the GPE is widely accepted as a valid model for the dynamics of the BEC at $T \simeq 0$ K, the knowledge of the dynamics of the condensates is scarce since the GPE is nonintegrable and explicit solutions are not known. In the positive scattering length case Eq. (1) has been solved numerically for cylindrically symmetric systems and analytically some work has been done in the framework of the Thomas-Fermi approximation [4,7,13]. The negative scattering length case is mostly unexplored, except for some numerical results [4]. Another approach to the dynamics of the condensate is the time-dependent variational technique [14], which assumes a fixed profile and computes the evolution of some parameters such as the width by variational techniques.

An important fact related to negative scattering length condensates is that stable solutions to Eq. (1) exist only under certain conditions for the number of particles and the size of the trap [14–17]. When those conditions are not satisfied the condensate is unstable and destroyed by the collapse phenomenon because the density $|\psi|^2$ increases up to a point where nonlinear losses [not included in Eq. (1)] become dominant. So, to have a *large* stable negative scattering length condensate collapse must be avoided. Having larger condensates is important to get better experimental observations of BEC. The reason is that the critical number of particles that can be put in the condensate without collapse is very small for current experimental parameters and thus it is difficult to perform accurate measurements and to obtain experimental data of the condensation process. Another reason for the interest of large condensates is their future practical applications (atom interferometers, atom clocks, etc.), where coherent atom clouds as large as possible are necessary.

In this paper we concentrate on the analysis of negative scattering length condensates in cigar-shaped traps. We present a class of solitonlike stable solutions that can be of interest in the applications of these coherent atom aggregates.

II. DERIVATION OF THE MODEL EQUATIONS

From now on we will study the solutions of Eq. (1) in cylindrically symmetric parabolic traps (2). More explicitly, we will consider cigar-shaped condensates, i.e., the case in which the trapping potential in s is much weaker than the trapping potential in ρ; mathematically, $\lambda_z \ll 1$.

Let us make the change of variables $\tau = \nu t$, $a_0 \rho = r$, $a_0 s = z$, and $Q = -8\pi a N/a_0$, where $a_0 = \sqrt{\hbar/m\nu}$ is the size of the ground-state solution of the noninteracting GPE with a harmonic potential of frequency ν (except for a $\sqrt{2}$ factor). Let us also define a wave function as $u(\rho,s,\tau) = \psi(r,z,t)\sqrt{a_0^3/N}$; then Eq. (1) reads

$$i\frac{\partial u}{\partial \tau} = \left[-\frac{1}{2}\nabla^2 + \frac{1}{2}(\rho^2 + \lambda_z^2 s^2) - \frac{Q}{2}|u|^2 \right] u, \tag{3}$$

with the normalization condition $\int |u|^2 d\vec{r} = 1$.

The solution of this nonlinear partial differential equation is a challenging problem and no explicit solutions are known. However, due to the different interaction scales involved in our particular problem it is possible to find approximate (but very accurate) analytic forms for the ground-state solutions of Eq. (3). A detailed analysis using multiscale expansions is done in the Appendix. Here we will derive the ground-state solution by simple physical arguments.

We will first assume that it is possible to factor the solution of Eq. (3) as

$$u(\rho,s,\tau) = \phi(\rho)\xi(s,\tau). \tag{4}$$

Then ϕ satisfies

$$-\frac{1}{2}\nabla_\perp^2 \phi + \frac{1}{2}\rho^2 \phi = \nu_\rho \phi. \tag{5}$$

Equation (5) is a well-known eigenvalue problem, the two-dimensional isotropic harmonic oscillator. Its *ground-state* solution is

$$\phi_0(\rho) = e^{-\rho^2/2}. \tag{6}$$

Multiplying Eq. (3) by ϕ^* and integrating to eliminate the ρ dependence, we find

$$i\frac{\partial \xi}{\partial \tau} = -\frac{1}{2}\frac{\partial^2 \xi}{\partial s^2} - \frac{Q}{4}|\xi|^2 \xi + \frac{1}{2}\lambda_z^2 s^2 \xi + \nu_\rho \xi, \tag{7}$$

where the additional factor 1/2 in the nonlinear term comes from the quotient $\int_0^\infty |\phi_0|^4 \rho \, d\rho / \int_0^\infty |\phi_0|^2 \rho \, d\rho = 1/2$.

Finally, let us make the change $\varphi(s,\tau) = \xi(s,\tau) e^{i\nu_\rho \tau}$ to obtain

$$2i\frac{\partial \varphi}{\partial \tau} + \frac{\partial^2 \varphi}{\partial s^2} + \lambda_z^2 s^2 \varphi - \frac{Q}{2}|\varphi|^2 \varphi = 0. \tag{8}$$

This equation is a one-dimensional (1D) NLSE, which in the $\lambda_z = 0$ case can be integrated by the inverse scattering technique [18]. When $\lambda_z = 0$, Eq. (8) has stationary normalized single-soliton solutions of the type

$$\varphi(s) = \frac{\sqrt{Q}}{4\pi}\text{sech}\left(\frac{Qs}{8\pi}\right). \tag{9}$$

From this solution and using the Galilean invariance of the 1D NLSE it is possible to find traveling soliton solutions that propagate without distortion.

The width of the cloud in the s direction is related to the nonlinear coefficient through the relation

$$W_s = \sqrt{\langle s^2 \rangle_\varphi} = 4\pi^2/Q\sqrt{3}. \tag{10}$$

This is remarkable and means that condensates with a small number of particles (Q is proportional to N) would be very long, while condensates with more particles would be shorter. If the number of particles is large enough, the condensate is unstable and collapse occurs.

To justify approximation (4) let us note that in the transverse direction the trapping potential and nonlinear force tend to compress the wave packet competing against the linear dispersion effect provided by the kinetic-energy term. On the other hand, the trapping force in the s direction has been removed so that along that axis there is only a competition between the nonlinear attraction and the dispersion. When the main force on the transverse direction is the one caused by the trapping potential the approximation will be justified. To check this let us compare both potentials $H_{\text{trap}} = \frac{1}{2}\rho^2$ and $H_{\text{self-int}} = \frac{1}{2}Q|u|^2$ for the soliton solution. Their ratio is a function $q(\rho,s)$ given by

$$q(\rho,s) = \frac{16\pi^2 \rho^2 e^{\rho^2}}{Q^2 \text{sech}(Qs/8\pi)}. \tag{11}$$

Since $\text{sech}(x) \leqslant 1/2$, evidently when $32\pi^2/Q^2 \gg 1$, $q(\rho,s) \gg 1$, except for very small values of ρ. When this condition is satisfied, the parabolic potential dominates over the self-interaction and then the only effect of the nonlinear term on the transverse shape is to provide small shape corrections near the center of the trap, which is the place where the parabolic potential is lower and the nonlinear term more relevant.

To see whether the soliton solutions really exist we have computed numerically the ground-state solution of Eq. (3) for different values of λ_z. In the noninteracting limit (small Q) the solution is given by

$$u(\rho,s) = \lambda_z^{1/4}\pi^{-3/4}\exp(-\rho^2/2 - s^2/2). \tag{12}$$

Decreasing λ_z, increasing Q, and preserving the condition $q(\rho,s) \gg 1$, we should obtain the soliton solutions (9). To compute the ground-state solutions we have used the steepest-descent method described in [4] to minimize the Hamiltonian

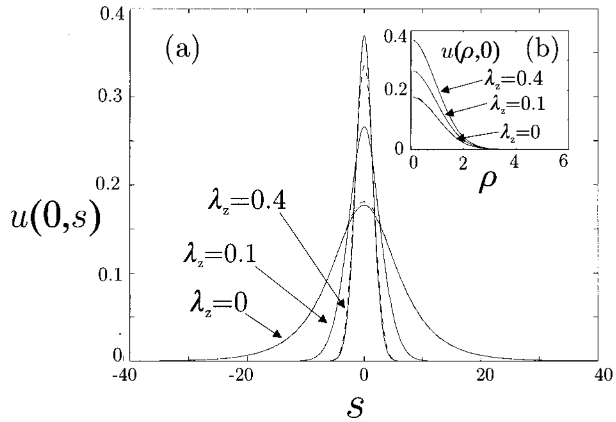

FIG. 1. Sections of the ground-state solution of Eq. (3) with $Q=5$ and different values of λ_z. From the innermost to the outermost curves the λ_z parameter is $\lambda_z=0.4,0.2,0.0$. The dash-dotted line corresponds to the theoretical prediction for $\lambda_z=0$ given by Eq. (9), while the dashed line corresponds to the Gaussian solution given by Eq. (12) for $\lambda_z=0.4$. (a) s section for $\rho=0$ and (b) ρ section for $s=0$.

$$H=\int d\vec{r}\left[|\nabla u|^2+(\rho^2+\lambda_z^2 s^2)|u|^2-\frac{Q}{2}|u|^4\right] \tag{13}$$

over a discrete lattice, where the solution is defined. In Fig. 1 we plot sections of the fundamental state $u(\rho,s)$ for different values of the trapping potential in s (parametrized by the value of λ_z) and $Q=5$. As the λ_z value decreases the solutions get wider, but when $\lambda_z=0$ the solutions do not widen indefinitely. The profile is then very close to the one defined by Eq. (9), while the transverse profile is Gaussian as Eq. (6) predicted. For $Q=10$ it is seen in Fig. 2 that the $\lambda_z=0$ solutions are not so close to the profiles predicted by Eq. (9). This is because when Q is large the approximation involved in the derivation of Eq. (9) is not valid and the nonlinear energy term is comparable to the transverse harmonic trapping energy. However, it is striking that even for this large Q case the differences between the numerically calculated profile and the solitonic profile are less than 10%. In this case it is seen that compact solutions exist when the trapping in z is absent.

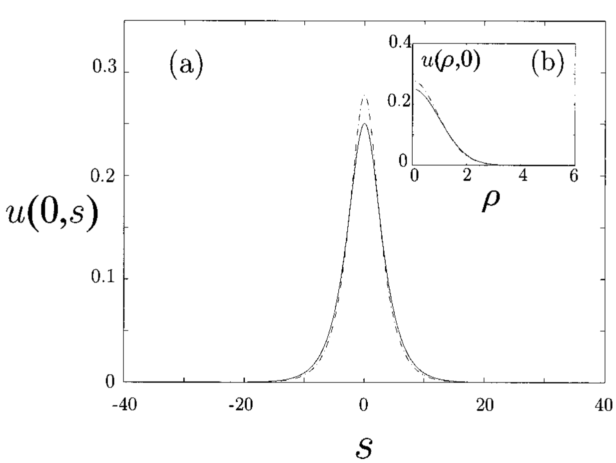

FIG. 2. Sections of the ground-state solution of Eq. (3) with $Q=10$ and $\lambda_z=0$. The dash-dotted line corresponds to the theoretical prediction for $\lambda_z=0$ given by Eq. (9). (a) s section for $\rho=0$ and (b) ρ section for $s=0$.

On the other hand, when a strong trapping potential is applied in the s direction (stronger than the nonlinear self-interaction term) the numerical solution is close to the exact Gaussian ground-state solution (12) and the effect of the nonlinear term is only an enhanced compression of the solution near the center of the trap. This phenomenon is seen in Fig. 1 (a plot with $\lambda_z=0.4$).

The existence of atomic solitons has been put forth in [19] in the context of the motion of an atom beam in the field of a traveling-wave laser and in a similar context in [20]. In those papers, however, the trap effect was not considered and the validity of the transition to 1D equations was not studied.

III. CONTROL OF THE CONDENSATE MOTION

Thus we have numerically established the existence of localized solutions and obtained their analytical form when the trapping in s is eliminated (provided the number of particles $N\propto Q$ is small enough to avoid collapse). Also, there exist traveling-wave solutions of this type that could propagate without distortion. Now it is interesting to study the response of the center of mass of the condensate to an external potential because it could allow one to control the motion of the condensate.

Let us then study the evolution of the center of mass of a condensate governed by Eq. (1) in an *arbitrary* external potential $V(x,y,z)$ (the following results are valid for any potential not only for *parabolic traps*). Defining

$$\vec{X}=\int d^3\vec{r}|\psi|^2\vec{r} \tag{14}$$

and computing its time derivatives using Eq. (1), we find $d\vec{X}/dt=\langle\mathcal{P}\rangle$, where $\mathcal{P}$ is the usual momentum operator $\mathcal{P}=-i\hbar\vec{\nabla}$, and

$$m\frac{d^2\vec{X}}{dt^2}=-\langle\vec{\nabla}V\rangle, \tag{15}$$

which is the Ehrenfest theorem of quantum mechanics. Equation (15) means that this theorem is still valid for the GPE so that the center of mass of the wave packet behaves like a classical particle. It is possible to check the validity of Eq. (15) for more general NLS equations (i.e., more general nonlinear terms), a fact that is not well known [21].

This result implies that one could manipulate a condensate by using an external potential as is known for the 1D NLSE [22]. Joining this result with the previous one, i.e., the existence of localized solutions, we find a way to control the motion of a negative scattering length condensate: Just relax the trap in one direction and apply an external force along that axis; the condensate will respond by preserving its shape and moving like a classical particle. Of course the external force should be smoothly varying since the localized solutions have been derived in the limit where no forces are present [23].

It is not strictly true that a condensate would respond as a whole to the external force. It is well known [18] that any initial data evolving following Eq. (8) decomposes into solitons and ''radiation.'' So it should be simple to find these solitonic objects experimentally by just adiabatically relaxing the trap and applying an external potential. In doing so radiation (which in this context means some free atoms) would be generated and some solitons obtained. Only if the initial data are already solitons would there be no breaking of the initial data into a soliton train plus radiation.

Concerning the motion of a soliton wave packet in a highly asymmetric trap (with λ_z small but different from zero), it must be said that there are no completely stationary solutions as shown by [24]. However, there would be a quasistationary solution with Gaussian tails in the $s\rightarrow\infty$ and near-sech profile in the $s=0$ region as recently proposed in [25]. If the number of particles is large enough so that the soliton size is small compared to the scale of variation of the potential, smooth motion of the ''soliton'' towards the ''boundary'' is expected. However, if the soliton size and trap size are of the same order of magnitude, there will be a competition of both scales that can result in quasiperiodic motion or even chaotic motion as discussed in [26]. Another ''tool'' to control the motion of the condensate could be a laser field, as has been put forth in [19,27] but in those cases the interaction between the transverse laser field and the atoms should be carefully considered.

IV. APPLICATION TO LITHIUM CONDENSATES

Let us analyze the relevance of our results for the lithium Bose-Einstein condensates [2,12]. Following Ref. [12], we will take as parameters $a=-14.5$ Å and the usual trapping potentials for the cigar-shaped trap that are about $\nu=150$ Hz corresponding to $a_0\simeq 3$ μm.

To ensure the validity of Eq. (4) it is necessary that $W_s \gg 1$ and then we find that $N\sim 300$. However, in Fig. 2 it is seen how even in the case $Q=10$ ($N\simeq 900$) the differences between the soliton profile and the real ground state are small.

Another interesting limit corresponds to collapse. In principle, one would expect that the cigar-shaped trap would allow a larger number of particles to be put in the condensate before collapse occurs. The physical reason is that keeping free the condensate in one spatial direction collapse would not occur along that axis, but through compression of the orthogonal (transverse) directions, which are smaller and thus ''feel'' stronger interactions. This means that the system would behave in a two-dimensional-like manner and then the collapse conditions should be less severe [14,15].

To test this hypothesis we have performed simulations of the largest Q value allowed using the same code as for the computation of the ground state. The upper limit found for the cigar-shaped trap is $Q=17$, corresponding to $N\simeq 1500$. This number is somewhat lower than the Gaussian bound given in [14], which is $Q=19.5$, corresponding to $N\simeq 1710$. These numbers compare favorably with the spherically symmetric results. In that case the limit found using the steepest-descent method is $Q=13.7$, corresponding to about $N \simeq 1200$, again lower than the Gaussian bound $Q=16.7$ and then $N=1460$. So the cigar-shaped trap allows one to increase the maximum number of particles by 25%. This is a small but significant increase. We have checked by numerical solution of the Gaussian equations of Ref. [14] and numerical simulations of Eq. (3) that the cigar-shaped trap is the optimal one; i.e., there is no other parabolic trap configuration with better collapse-avoiding properties.

As stated in the Introduction, one would like to increase the number of particles in the condensate as much as possible. This would allow the control of a large coherent pulse of atoms. To do so one could try to use a higher-order soliton [28]. However, for those solutions the shape performs complicated (but periodic) oscillations and develops high spatial and temporal gradients that probably would rule out the approximation and induce collapse if the order of the soliton is large enough. It is also possible to generate a soliton train where the solitons have different global phases so that the interaction between them is repulsive. This idea should work for some situations allowing many particles to be put in the ground state and we plan to elaborate upon it in future work. Another possibility is to use a non-Gaussian fundamental mode for the transverse solution as in [4,29]. However, the question of the stability of those solutions under general three-dimensional perturbations is not trivial and is the subject of current research. Finally, other possibilities proposed in the literature could be of use here such as using two condensates [30] or the control of the value of the scattering length [31].

V. CONCLUSION

We have found compact solutions of Eq. (3) that exist due to nonlinear effects even when the z trapping potential is absent. Joining this result with Eq. (15), we conclude that it is possible to control the motion of the condensate, which could propagate without distortion by using smoothly varying external potentials. Thus the atom cloud could be manipulated very easily, e.g., with an atom guide. It is interesting and curious that this cigar-shaped packet could be transported in that rigid way behaving like a quasiparticle. This behavior is specific of negative scattering length condensates and an advantage over the positive scattering length ones, which tend to fill all the available space due to the repulsive atom-atom interaction. Additionally, we have pointed out that relaxing the trapping potential in one direction in current traps would allow one to increase the number of particles that can be put into a negative scattering length condensate.

We hope that this study will stimulate the experimental efforts in performing BEC with negative scattering length and think that the soliton solutions here studied will be of practical applicability in Bose-Einstein condensate ''engineering.''

ACKNOWLEDGMENTS

V.M.P.-G. and H.M. acknowledge the hospitality of the Institut für Theoretische Physik, Universität Innsbruck, where part of this work was done. We thank F. Dalfovo for his help with the numerical simulations. This work was supported in part by the Spanish Ministry of Education and Culture under Grants Nos. PB95-0389 and PB96-0534.

APPENDIX: DERIVATION OF THE 1D NLSE BY MULTIPLE-SCALE ANALYSIS

Here we will give the details of a more formal derivation of Eqs. (5) and (8) from Eq. (3) using multiple scale analysis [32]. Let us first choose

$$u=\varepsilon^{1/2}u_0(\tau,\tau',x,y,z')+\varepsilon^3 u_1+\varepsilon^{11/2}u_2+\cdots, \quad \text{(A1a)}$$

$$\lambda_z=\varepsilon^4\lambda_z, \quad \text{(A1b)}$$

$$z'=\varepsilon z, \quad \text{(A1c)}$$

$$\tau'=\varepsilon^2\tau, \quad \text{(A1d)}$$

$$Q=\varepsilon Q_0. \quad \text{(A1e)}$$

This scaling satisfies the desirable property that the L^2 norm of u is conserved and that the potential in z is weaker than the nonlinear interaction (and the later weaker than the transverse potential) when $\varepsilon\to 0$. Inserting Eqs. (A1) into Eq. (3), we find

$$i\left(\frac{\partial}{\partial\tau}+\varepsilon^2\frac{\partial}{\partial\tau'}\right)(\varepsilon^{1/2}u_0+\varepsilon^3u_1+\varepsilon^{11/2}u_2+\cdots)$$
$$=\left[-\frac{1}{2}\nabla^2_{xy}-\frac{1}{2}\varepsilon^2\frac{\partial^2}{\partial z'^2}+\frac{1}{2}\left(\rho^2+\varepsilon^4\lambda_0^2\frac{1}{\varepsilon^2}z'^2\right)\right.$$
$$\left.-\frac{\varepsilon Q_0}{2}\varepsilon|u_0+\varepsilon^{5/2}u_1+\cdots|^2\right](\varepsilon^{1/2}u_0+\varepsilon^3u_1+\cdots). \quad \text{(A2)}$$

We now separate Eq. (A2) into the different orders in ε:

$$O(\varepsilon^{1/2}):\quad i\frac{\partial u_0}{\partial\tau}=\left(-\frac{1}{2}\nabla^2_{xy}+\frac{1}{2}\rho^2\right)u_0, \quad \text{(A3a)}$$

$$O(\varepsilon^{5/2}):\quad i\frac{\partial u_0}{\partial\tau'}=\left(-\frac{1}{2}\frac{\partial^2}{\partial z'^2}+\lambda_0^2z'^2\right)u_0-\frac{Q_0}{2}|u_0|^2u_0, \quad \text{(A3b)}$$

$$O(\varepsilon^3):\quad i\frac{\partial u_1}{\partial\tau}=\left(-\frac{1}{2}\nabla^2_{xy}+\frac{1}{2}\rho^2\right)u_1, \quad \text{(A3c)}$$

$$O(\varepsilon^5):\quad i\frac{\partial u_1}{\partial\tau'}=\left(-\frac{1}{2}\frac{\partial}{\partial z'^2}+\lambda_0^2z'^2\right)u_1-\frac{Q_0}{2}2|u_0|^2u_1, \quad \text{(A3d)}$$

$$O(\varepsilon^{11/2}):\quad i\frac{\partial u_2}{\partial\tau}=\left(-\frac{1}{2}\nabla^2_{xy}+\frac{1}{2}\rho^2\right)u_2, \quad \text{(A3e)}$$

$$O(\varepsilon^{15/2}):\quad i\frac{\partial u_2}{\partial\tau'}=\left(-\frac{1}{2}\frac{\partial}{\partial z'^2}+\lambda_0^2z'^2\right)u_2-\frac{Q_0}{2}2|u_0|^2u_2$$
$$-\frac{Q_0}{2}u_1^2u_0^*, \quad \text{(A3f)}$$

where $\nabla^2_{xy}=\partial^2/\partial x^2+\partial^2/\partial y^2$. Equation (A3a) implies that the transverse profile of u_0 is given by the isotropic two-dimensional harmonic-oscillator equation and then u_0 can be chosen as $u_0=\phi(x,y)\xi(z',\tau')e^{\nu\tau}$. Substituting into Eq. (A3a), multiplying by ϕ^*, and integrating over the transverse coordinates x,y we obtain

$$i\frac{\partial\xi}{\partial\tau'}=\left(-\frac{1}{2}\frac{\partial^2}{\partial z'^2}+\lambda_0^2z'^2\right)\xi-\frac{Q_0}{4}|\xi|^2\xi. \quad \text{(A4)}$$

This means that the longitudinal profile obeys the nonlinear Schrödinger equation. In the $\lambda_z=0$ case the solutions can be found analytically as discussed in Sec. II. Joining the longitudinal and transverse solutions and changing back to the nonscaled variables, we find that the ground-state solution has the form

$$u(\rho,s,\tau)=\sqrt{\frac{Q}{4\pi}}\text{sech}\left(\frac{Qs}{8\pi}\right)e^{-\rho^2/2}e^{-i\nu_p\tau}, \quad \text{(A5)}$$

at least to the first order in $\varepsilon\propto Q$. The corrections are given by Eqs. (A3c)–(A3f). It is easy to see that the equations have solutions $u_1=u_2=0$, so the solution is determined at least to order $\epsilon^{11/2}$ by u_0. This is the reason why the ground-state solution is close to the approximate profile given by Eq. (A5) even in the nonperturbative region as discussed in Sec. II.

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