Analytical formulas for the third-order diamagnetic energy of a hydrogen atom

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Analytical expressions are presented for the third-order diamagnetic corrections to the energy of nondegenerate hydrogen levels with arbitrary principal quantum number n and the magnetic quantum number $|m|=n-1, n-2, n-3$. The leading term for the third-order energy correction for levels with high n is determined to be $\Delta E^{(3)} \approx \frac{3}{128} n^{16} B^6$. Together with the well-known first- and second-order corrections $\Delta E^{(1)} \approx \frac{1}{8} n^4 B^2$ and $\Delta E^{(2)} \approx -\frac{1}{32} n^{10} B^4$ it determines the upper and lower bounds for the level energy in field and also the range of magnetic fields where the first- and second-order perturbation theory terms are valid for calculating the Zeeman energy in hydrogenlike states of atoms. [S1050-2947(98)00705-7]

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I. INTRODUCTION

A significant amount of precise information on the Zeeman effect in atoms is now available (see, e.g., [1–4], and references therein). Accurate analytical formulas have been obtained for the magnetic field–induced energy corrections in hydrogenlike states up to the second order in diamagnetic interaction [5]. Witwit and Killingbeck [6] have calculated numerically the third-order corrections for the lowest levels with $n \leq 3$ in a field of $B=0.1$ a.u. The difficulties in such calculations are caused by the lack of a symmetry for the diamagnetic Hamiltonian. In contrast to the case of the Stark effect, where the variables are separable in parabolic coordinates, for Zeeman effect the parabolic coordinates may be used only as a basis for the diagonalization of the diamagnetic Hamiltonian [5], which was constructed up to the second-order terms by Delande and Gay [7].

We have demonstrated that the Sturm series presentation for the Coulomb Green function in spherical coordinates also provides a powerful tool in deriving the analytical expressions for the higher-order diamagnetic matrix elements, which may be used for diagonalization in a subspace of degenerate hydrogenic states. The analytical expressions for the second-order susceptibilities together with numerical data have thus been obtained. A similar representation of the valence electron Green function in many-electron atoms has also made it possible to calculate the diamagnetic susceptibilities for the ground and excited states of alkali and noble gas atoms [8,9].

General analytical expressions for higher-order diamagnetic susceptibilities, at least for the nondegenerate substates of an excited hydrogenic manifold, may also be obtained on the basis of this approach. Besides a purely theoretical interest to derive analytical expressions for diamagnetic susceptibilities of a solvable quantum-mechanical system, there are many practical reasons to search for such expressions. The following are the most important of them:

(i) The relations between coefficients for consecutive orders provide useful information both on the analytical properties of the perturbation series for the Zeeman shift and splitting of atomic states (specifically, Rydberg states with high principal quantum numbers) and on the upper limits in the magnetic field strength for the use of the first few terms of perturbation theory in numerical calculations of the bound-state energy.

(ii) Quite a common situation appears for Rydberg atoms in magnetic fields of modern laboratories, when the lowest-order perturbation theory is not sufficient for calculating the Zeeman effect and therefore the higher-order terms are also needed. Besides that, these terms may be used to control the accuracy of calculations with the first few terms.

(iii) The increase of the accuracy of calculations with the higher-order terms is rather important when they are used to check the results of different nonperturbative approaches in the perturbative region of magnetic field strengths.

It is necessary to note that the terms of the perturbation series for the diamagnetic energy of hydrogenlike states are sign alternating. So, two consecutive terms in the region of their validity determine the upper and the lower bounds for the exact atomic energy. That is why to increase the precision of calculations at a given field strength the terms of higher orders of perturbation theory should be calculated, while they are forming a convergent sequence.

In addition, the analytical results for the higher-order diamagnetic susceptibilities for nondegenerate hydrogen states with high angular l and magnetic m quantum numbers (circular states) hold also for many-electron atoms where such states are also hydrogenlike. So, the derivation of analytical formulas for the high-order diamagnetic susceptibilities is quite a general problem in atomic physics and may be referred to as one of the fundamental quantum mechanical problems, similar to the higher-order perturbation theory for the Stark effect, which has been solved analytically quite a long time ago [10], thanks to the symmetry of the Coulomb field and homogeneous electric field superposition. In this paper we present the analytical results for the third-order diamagnetic susceptibilities of nondegenerate (circular and near circular) hydrogen states with the magnetic quantum number $|m| \geq n-3$.

II. THIRD-ORDER DIAMAGNETIC ENERGY OF ATOM

The third-order diamagnetic energy of an atom in a nondegenerate state $|nlm\rangle$ is determined by the diagonal matrix element

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$$\Delta E_{\lambda}^{(3)} = \langle nlm|V(\mathbf{r})\{[G'_{E_{nl}}(\mathbf{r},\mathbf{r}')V(\mathbf{r}')]^2 - \Delta E^{(1)}$$

$$\times[G'_{E_{nl}}(\mathbf{r},\mathbf{r}')]^2 V(\mathbf{r}')\}|nlm\rangle \qquad (1)$$

of the diamagnetic operator (the atomic units are used with the unit magnetic field of $B_0 = 2.35 \times 10^5$ T)

$$V(\mathbf{r}) = \frac{1}{8}[\mathbf{B}\times\mathbf{r}]^2 = \frac{B^2}{12}r^2[1 - \mathbf{C}_{20}(\theta,\varphi)]. \qquad (2)$$

$G'_{E_{nl}}(\mathbf{r},\mathbf{r}')$ here denotes the reduced atomic Green function with the energy E_{nl} of a free atom.

On analogy with the general relations for the first- and second-order energy corrections [8], the third-order diamagnetic susceptibility may be introduced according to the relation

$$\Delta E_{\lambda}^{(3)} = -\frac{B^6}{6!}\chi_{nlm}^{(3)}. \qquad (3)$$

After integration over angular variables, the susceptibility $\chi_{nlm}^{(3)}$ may be presented in terms of the third-order radial matrix elements,

$$p_{l',l''}^{n_1 n_2 n_3} = \langle nl|r^{n_1}g_{l'}^{(n)}r^{n_2}g_{l''}^{(n)}r^{n_3}|nl\rangle, \qquad (4)$$

as follows

$$\chi_{nlm}^{(3)} = -\frac{45}{4}\{Q_{l-2,m}L_{l-1,m}L_{l,m}p_{l-2,l-2}^{222}$$

$$+2Q_{l,m}L_{l-1,m}L_{l,m}p_{l-2,l}^{222} + (Q_{l,m})^3 p_{l,l}^{222}$$

$$+2Q_{l,m}L_{l+1,m}L_{l+2,m}p_{l+2,l}^{222}$$

$$+Q_{l+2,m}L_{l+1,m}L_{l+2,m}p_{l+2,l+2}^{222} - Q_{l,m}\langle nl|r^2|nl\rangle$$

$$\times[(Q_{l,m})^2 p_{l,l}^{202} + L_{l-1,m}L_{l,m}p_{l-2,l-2}^{202}$$

$$+L_{l+1,m}L_{l+2,m}p_{l+2,l+2}^{202}]\}. \qquad (5)$$

Here the notation is introduced for the factors of the angular integration:

$$Q_{l,m} = \frac{l^2 + l - 1 + m^2}{(2l-1)(2l+3)}, \quad L_{l,m} = \frac{l^2 - m^2}{2(2l-1)(2l+1)}. \qquad (6)$$

Thus the dependence on the magnetic quantum number m for $\chi_{nlm}^{(3)}$ is already given in Eq. (5) explicitly. Together with a simple expression for the mean quadratic radius of the state, $\langle nl|r^2|nl\rangle = \frac{1}{2}n^2[5n^2 + 1 - 3l(l+1)]$, the analytical expressions for the third-order radial matrix elements may also be written in the form of polynomials of n and l. The derivation of these expressions may be carried out with the use of the Sturm series presentation for the reduced radial Green function, $g_l^{(n)}(r,r')$, in the third-order matrix elements (4), similar to the second-order matrix elements [8,11]. In the next section we describe a procedure for such calculations based on the Sturm series for the Coulomb Green function and on the orthogonality properties of Laguerre polynomials.

III. ANALYTICAL CALCULATIONS OF THE THIRD-ORDER RADIAL MATRIX ELEMENTS

When the energy of the Green function equals that of the bound state (this case is usual for calculations of the static atomic susceptibilities) the radial function of this state, say $R_{nl}(r)$, equals one of the Sturm functions from the basis set, $F_{kl}(x)$, in which the Green function is resolved [12]:

$$R_{nl}(r) = \frac{2Z^{3/2}}{n^2}F_{n,l}\left(\frac{2Zr}{n}\right), \qquad (7)$$

where Z is the charge of atomic nucleus, $n_r = n - l - 1$ is the radial quantum number of the state,

$$F_{kl}(x) = \sqrt{\frac{k!}{(k+2l+1)!}}e^{-x/2}x^l L_k^{2l+1}(x), \qquad (8)$$

$L_k^{\alpha}(x)$ is the Laguerre polynomial. The Sturm series resolution for the reduced Green function with the energy of the bound state (7) may be written as follows:

$$g_{l'}^{(n)}(r,r') = \frac{4Z}{n}\sum_{k\neq n_r'}^{\infty}\frac{F_{kl'}(2Zr/n)F_{kl'}(2Zr'/n)}{k+l'+1-n} + \frac{n^2}{Z^2}\left[\frac{5}{2}R_{nl'}(r)R_{nl'}(r') + r\frac{dR_{nl'}(r)}{dr}R_{nl'}(r') + R_{nl'}(r)r'\frac{dR_{nl'}(r')}{dr'}\right]. \qquad (9)$$

When substituting the left-hand side of this expression for the Green functions in the radial matrix elements (4), we deal with a number of integrals of the type:

$$\left\langle F_{kl}\left(\frac{2Zr}{n}\right)|r^q|F_{k'l'}\left(\frac{2Zr}{n}\right)\right\rangle, \qquad (10)$$

where $q = 0,1,2,3$, $l' = l, l\pm2$. These integrals consist of few terms, each including a Kronecker symbol $\delta_{kk'}$, according to the orthogonality properties of Laguerre polynomials [13]. For example,

$$\langle F_{kl}|r^2|F_{k'l}\rangle=\left(\frac{n}{2Z}\right)^5\sqrt{\frac{k!k'!}{(k+2l+1)!(k'+2l+1)!}}\int_0^\infty e^{-x}x^{2l+4}L_k^{2l+1}(x)L_{k'}^{2l+1}(x)dx$$

$$=\left(\frac{n}{2Z}\right)^5\{-\sqrt{(k-2)_3(k+2l-1)_3}\,\delta_{k'k-3}+6(k+l)\sqrt{(k-1)_2(k+2l)_2}\,\delta_{k'k-2}$$

$$-3[(k+2l+2)_2+3(k-1)(k+2l+2)+(k-2)_2]\sqrt{k(k+2l+1)}\,\delta_{k'k-1}+[(k+2l+2)_3+9k(k+2l+2)_2$$

$$+9(k-1)_2(k+2l+2)+(k-2)_3]\delta_{k'k}-3[(k+2l+3)_2+3k(k+2l+3)+(k-1)_2]$$

$$\times\sqrt{(k+1)(k+2l+2)}\,\delta_{k'k+1}+6(k+l+2)\sqrt{(k+1)_2(k+2l+2)_2}\,\delta_{k'k+2}-\sqrt{(k+1)_3(k+2l+2)_3}\,\delta_{k'k+3}\}.$$

$$(11)$$

Common notation for the Pochhammer symbol is used here: $(a)_p=a\times(a+1)\times\cdots\times(a+p-1)$; the integral is reduced to a number of the orthogonality condition integrals with the help of the recursion formula for Laguerre polynomials [13]:

$$L_k^\alpha(x)=L_k^{\alpha+1}(x)-L_{k-1}^{\alpha+1}(x).$$

Hence, only the terms with $k=n_r\pm1, n_r\pm2, n_r\pm3$ contribute to the Sturm series and the matrix elements in Eq. (5) are presented as polynomials of the principal n and orbital l quantum numbers. In a general case of arbitrary n and l these expressions are rather long and complicated. However, they simplify significantly for particular values of l. So, for $l=n-1$ they are

$$p_{l,l}^{222}=\frac{n^{10}}{96Z^{10}}(n+1)(2n+1)(96n^5+1110n^4+4519n^3$$

$$+8706n^2+8171n+3030),\qquad(12)$$

$$p_{l,l+2}^{222}=\frac{n^{10}}{96Z^{10}}(n+1)(n+2)(2n+1)(2n+3)(56n^3+336n^2$$

$$+681n+471),\qquad(13)$$

$$p_{l+2,l+2}^{222}=\frac{n^{10}}{288Z^{10}}(n+1)(n+2)(n+3)(2n+1)(2n+3)$$

$$\times(2n+5)(9n^2+56n+88),\qquad(14)$$

$$p_{l,l}^{202}=\frac{n^8}{48Z^8}(n+1)(2n+1)(48n^3+195n^2+269n+126),$$

$$(15)$$

$$p_{l+2,l+2}^{202}=\frac{n^8}{144Z^8}(n+1)(n+2)(2n+1)(2n+3)$$

$$\times(9n^2+44n+54).\qquad(16)$$

For susceptibilities $\chi_{nlm}^{(3)}$ of the circular hydrogenic states (with the highest angular and magnetic quantum numbers) these equations yield

$$\chi_{n\,n-1\,n-1}^{(3)}=-\frac{5}{64Z^{10}}n^{11}(n+1)(216n^4+1089n^3+2048n^2$$

$$+1700n+528),\qquad(17)$$

$$\chi_{n\,n-1\,n-2}^{(3)}=-\frac{15}{16Z^{10}}n^{11}(n+1)(n-1)$$

$$\times(18n^3+63n^2+62n+10).\qquad(18)$$

For another pair of states, with $l=n-2$, $m=\pm l$, $\pm(l-1)$, corresponding radial matrix elements are

TABLE I. Third-order diamagnetic susceptibilities of the hydrogenlike atom in nondegenerate nlm states as calculated from Eqs. (17) and (18) and (24) and (25) and corresponding values of critical fields B_1 and B_2; the number in brackets represents the power of ten $[k]=10^k$.

State $nl|m|$	$-\chi_{nlm}^{(3)}\cdot Z^{10}$	B_1	B_2
$1s$	872.03125	0.9517	0.4774
$2s$	1.6206 [7]	0.1480	8.414 [−2]
$2p0$	3.0528 [6]	0.1890	9.953 [−2]
$2p1$	1.1658 [7]	0.1610	8.464 [−2]
$3p0$	2.4827 [9]	5.050 [−2]	3.200 [−2]
$3p1$	6.3879 [9]	5.000 [−2]	2.848 [−2]
$3d1$	1.6594 [9]	6.048 [−2]	3.267 [−2]
$3d2$	3.9282 [9]	5.403 [−2]	2.911 [−2]
$4d1$	2.4340 [11]	2.297 [−2]	1.427 [−2]
$4d2$	4.6256 [11]	2.244 [−2]	1.292 [−2]
$4f2$	1.4262 [11]	2.665 [−2]	1.460 [−2]
$4f3$	2.7048 [11]	2.444 [−2]	1.335 [−2]
$5f2$	8.0190 [12]	1.226 [−2]	7.481 [−3]
$5f3$	1.3209 [13]	1.193 [−2]	6.919 [−3]
$5g3$	4.5538 [12]	1.403 [−2]	7.760 [−3]
$5g4$	7.5841 [12]	1.308 [−2]	7.213 [−3]
$6g3$	1.3783 [14]	7.295 [−3]	4.409 [−3]
$6g4$	2.0793 [14]	7.092 [−3]	4.128 [−3]
$6h4$	7.7830 [13]	8.282 [−3]	4.609 [−3]
$6h5$	1.1897 [14]	7.805 [−3]	4.333 [−3]

$$p_{l,l}^{222} = \frac{n^{10}}{96Z^{10}}(2n-1)(288n^6 + 2178n^5 + 17557n^4 + 24977n^3$$

$$+ 62417n^2 + 25165n + 21210), \tag{19}$$

$$p_{l,l+2}^{222} = -\frac{n^{10}}{96Z^{10}}(2n-1)(n+1)(2n+1)(8n^4 - 2480n^3$$

$$- 6169n^2 - 11629n - 10230), \tag{20}$$

$$p_{l+2,l+2}^{222} = \frac{n^{10}}{288Z^{10}}(2n-1)(n+1)(n+2)(2n+1)$$

$$\times (2n+3)(9n^3 + 895n^2 + 3424n + 3432), \tag{21}$$

$$p_{l,l}^{202} = \frac{n^8}{48Z^8}(2n-1)(144n^4 + 153n^3$$

$$+ 1988n^2 - 287n + 1830), \tag{22}$$

$$p_{l+2,l+2}^{202} = \frac{n^8}{144Z^8}(2n-1)(n+1)(2n+1)$$

$$\times (9n^3 + 646n^2 + 1722n + 1188), \tag{23}$$

and for the susceptibilities we have

$$\chi_{n\,n-2\,n-2}^{(3)} = -\frac{5}{8Z^{10}}n^{10}(n-1)(27n^5 + 585n^4 + 26n^3$$

$$+ 3649n^2 - 1239n + 2772), \tag{24}$$

$$\chi_{n\,n-2\,n-3}^{(3)} = -\frac{15}{64Z^{10}}n^{10}(n-2)(72n^5 + 1449n^4 - 1835n^3$$

$$+ 12372n^2 - 12712n + 20856). \tag{25}$$

It is interesting to note that the matrix elements $p_{l,l}^{222}$ and $p_{l,l+2}^{222}$ have the asymptotic power dependence on the principal quantum number of the order 17 with the coefficient of the leading term depending on the orbital quantum number l. The senior term in the power dependence of the matrix element $p_{l+2,l+2}^{222}$ is n^{18} with the l-independent coefficient $\frac{1}{4}$. However, the leading terms cancel out of the susceptibility (5) so that $\chi_{nlm}^{(3)} \sim -\frac{135}{8}n^{16}$ with the l-independent coefficient. This property of susceptibility for hydrogenic states does not hold for the states of many-electron atoms with low angular momenta, similarly to the second-order susceptibility $\chi_{nlm}^{(2)}$ [8]: the dependence on the effective principal quantum number ν of the form $\chi_{nlm}^{(3)} \sim \nu^{18}$, characteristic of the radial matrix elements, may be expected here.

IV. NUMERICAL RESULTS AND DISCUSSIONS

The numerical values for the susceptibilities of some lowest states with $n \leq 6$ are given in Table I. Together with the data for the first-order and second-order susceptibilities [8] they complete the information on the diamagnetic energy of these atomic states up to the third-order terms of perturbation theory. The values of two critical magnetic fields B_1 and B_2 determined from equations $\Delta E_{nlm}^{(1)}(B_1) = \Delta E_{nlm}^{(2)}(B_1)$ and $\Delta E_{nlm}^{(2)}(B_2) = \Delta E_{nlm}^{(3)}(B_2)$ are also presented in the table. These values determine the upper bounds for the use of corresponding orders of the perturbation theory in calculating the diamagnetic energy.

As it follows from the data of Table I, the relation $B_2 < B_1$ holds everywhere (the value B_2 is almost half the value of B_1), as is characteristic of the asymptotic series regularities. In particular, the second-order perturbation theory is valid for the diamagnetic energy correction in the states with $n = 6$ up to the magnetic field of $B_2 \approx 4 \times 10^{-3}$ (for the second-order correction) and of $B_1 \approx 7 \times 10^{-3}$ (for the first-order correction). For the states with $n = 3$ corresponding values of field are $B_2 \approx 3 \times 10^{-2}$ and $B_1 \approx 5 \times 10^{-2}$. The upper magnetic field limit B_3 for the use of the third-order correction to energy can also be determined from a similar equation involving the fourth-order correction: $\Delta E_{nlm}^{(3)}(B_3) = \Delta E_{nlm}^{(4)}(B_3)$. It is evident that $B_3 < B_2$ and the exact ratio of B_3 and B_2 depends on a concrete state $|nlm\rangle$.

We have calculated the values of energy taking into account the diamagnetic corrections up to the first, second, and third orders of perturbation theory,

$$E_{nl0}^{(N)}(B) = E_n - \frac{1}{2}B - \sum_{k=1}^{N}\frac{\chi_{nl0}^{(k)}}{2k!}B^{2k} \tag{26}$$

for the spin-down states $1s$, $2s$, and $3p$ with $m = 0$ (the linear in B term corresponds to the magnetic field interaction with spin). The numerical results have been compared with those of the B-splines method [3], which seems to be the most precise available in the literature at present. This comparison is presented in Table II for the binding energy, $-E_{nlm}(B)$. Within the bounds of validity for perturbation theory, $B < B_2$, the third-order energy, $E_{nlm}^{(3)}(B)$, is much closer to the exact values of the B-splines method, $E_{nlm}^{(B-s)}(B)$, than the second-order energy, $E_{nlm}^{(2)}(B)$ [for $B < B_2/2$, $E_{nlm}^{(3)}(B)$ is an order of magnitude closer to $E_{nlm}^{(B-s)}(B)$ than $E_{nlm}^{(2)}(B)$]. Moreover, the values $E_{nlm}^{(2)}(B)$ and $E_{nlm}^{(3)}(B)$ give in the region $B < B_2$ the upper and the lower bounds correspondingly for the exact energy, $E_{nlm}^{\text{exact}}(B)$, of state in field,

$$E_{nlm}^{(2)}(B) < E_{nlm}^{\text{exact}}(B) < E_{nlm}^{(3)}(B).$$

The results (17)–(18) and (24)–(25) for the third-order susceptibilities of highly excited Rydberg states with $n > 10$ may be even more important for the practice, since the higher-order corrections in states with $n \approx 40$ become significant already in fields of about 1 T, available in laboratories. The asymptotic values of the critical fields for the states with high n, as it follows from the analytical expressions for $\chi_{nlm}^{(1)}$, $\chi_{nlm}^{(2)}$ [8], and $\chi_{nlm}^{(3)}$ presented above, are

$$B_2 \approx \frac{B_1}{\sqrt{3}} \approx \frac{2}{\sqrt{3}n^3},$$

which confirms the general relation $B_2 < B_1$, characteristic of asymptotic series.

TABLE II. The binding energies of spin-down states with $m=0$ (in Rydbergs), as calculated with account of the first-order, $|E_{nl0}^{(1)}(B)|$, second-order, $|E_{nl0}^{(2)}(B)|$, and third-order perturbation theory for the diamagnetic interaction, $|E_{nl0}^{(3)}(B)|$, in comparison with corresponding results of the B-spline method, $|E_{nl0}^{(B-s)}|$ [3].

nl	B	$|E_{nl0}^{(1)}|$	$|E_{nl0}^{(2)}|$	$|E_{nl0}^{(3)}|$	$|E_{nl0}^{(B-s)}|$
$1s$	0.002	1.001 998 0	1.001 998 000	1.001 998 000	1.001998 000 0
	0.02	1.019 800 0	1.019 800 088	1.019 800 088	1.019800 087 8
	0.2	1.180 000 0	1.180 883 333	1.180 728 305	1.180763 123 5
$2s$	0.002	0.251 72 00	0.251 972 005 1	0.251 972 005 1	0.251 972 00
	0.01	0.259 300 00	0.259 303 186 7	0.259 303 141 7	0.259 303 12
	0.02	0.267 200 00	0.267 250 986 7	0.267 248 105 6	0.267 248 35
	0.04	0.278 800 00	0.279 615 786 7	0.279 431 397 5	0.279 479 64
	0.06	0.284 800 00	0.288 929 920 0	0.286 829 612 0	0.287 726 92
	0.1	0.280 000 00	0.311 866 666 7	0.266 849 777 8	0.296 178 30
$2p$	0.002	0.251 988 00	0.251 988 001 3	0.251 988 001 3	0.251 988 00
	0.01	0.259 700 00	0.259 700 840 0	0.259 700 831 5	0.259 700 83
	0.02	0.268 800 00	0.268 813 440 0	0.268 812 897 3	0.268 812 93
	0.04	0.285 200 00	0.285 415 040 0	0.285 380 305 9	0.285 387 41
	0.06	0.299 200 00	0.300 288 640 0	0.299 892 997 1	0.300 032 53
	0.1	0.320 000 00	0.328 400 000 0	0.319 920 000 0	0.324 820 15
$3p$	0.001	0.112 093 11	0.112 093 118 2	0.112 093 118 2	0.112 093 12
	0.002	0.113 039 11	0.113 039 224 1	0.113 039 223 7	0.113 069 76
	0.01	0.119 311 11	0.119 381 736 1	0.119 374 839 8	0.119 375 73
	0.02	0.123 911 11	0.125 041 111 1	0.124 599 749 3	0.124 757 12
	0.04	0.122 311 11	0.140 391 111 1	0.112 143 957 3	0.130 812 75

V. CONCLUSIONS

Analytical expressions for the third-order diamagnetic susceptibilities make it possible to increase essentially the accuracy of the perturbation theory results for the Zeeman effect of hydrogen levels. We believe that the formulas (17)–(18) and (24)–(25) are the first in analytical presentations for the third-order diamagnetic energy of hydrogenlike states. Similar expressions but involving square-root radicals may be obtained for the double-degenerate states with $|m|=n-3$, $n-4$, $l=n-1$, $n-3$, and $|m|=n-4$, $n-5$, $l=n-2$, $n-4$. The method presented above for analytical summation of the Sturm-series expansions for the radial matrix elements may also be useful for calculating the fourth and higher orders of perturbation theory for the diamagnetic energy of atoms, providing more exact information on the upper and lower bounds for the energy in the perturbative region of fields.

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