Bell's inequality and detector inefficiency

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In this paper, a method of generalizing the Bell inequality is presented that makes it possible to include detector inefficiency directly in the original Bell inequality. To enable this, the concept of ''change of ensemble'' will be presented, providing both qualitative and quantitative information on the nature of the ''loophole'' in the proof of the original Bell inequality. In a local hidden-variable model lacking change of ensemble, the generalized inequality reduces to an inequality that quantum mechanics violates as strongly as the original Bell inequality, irrespective of the level of efficiency of the detectors. A model that contains change of ensemble lowers the violation, and a bound for the level of change is obtained. The derivation of the bound in this paper is not dependent upon any symmetry assumptions such as constant efficiency, or even the assumption of independent errors. [S1050-2947(98)07405-8]

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I. INTRODUCTION

The Bell inequality [1] and its descendants have been the main argument on the Einstein-Podolsky-Rosen paradox [2,3] for the past 30 years. A new research field of ''experimental metaphysics'' has formed, where the goal is to show that the concept of local realism is inconsistent with quantum mechanics, and ultimately with the real world. The experiments that have been performed to verify this have not been completely conclusive, but they point quite decisively in a certain direction: Nature cannot be described by a realistic local hidden-variable theory (see Refs. [4–6], for instance).

The reason for saying ''not been completely conclusive'' is that there is an implicit assumption in the proof of the Bell inequality that the detectors are 100% effective. There has been considerable discussion in the literature on this (see Refs. [7–10], among others), and the main issue is to obtain a limit for the detector inefficiency, but previously inequalities other than the original Bell inequality had to be used, for example, the Clauser-Horne inequality in [7], which in itself contains the case of inefficient detectors, or the Clauser-Horne-Shimony-Holt (CHSH) inequality first presented in [11], which is generalized to the inefficient case in [9].

Since a hidden-variable model is really a probabilistic model, formalism and terminology from probability theory will be used in this paper (see, e.g., Ref. [12]). The sample space Λ is the mathematical analog to the state space used in physics, and a sample λ is a point in that space corresponding to a certain value of the ''hidden variable.'' The measurement results are described by random variables (RV's) $X(\lambda)$, which take their values in the value space V.

To be a probabilistic model, a probability measure P on the space Λ is needed, by which we can define the expectation value E as

$$E(X) \overset{\text{def}}{=} \int_\Lambda X(\lambda)\,dP(\lambda) = \int_\Lambda X\,dP, \tag{1}$$

suppressing the parentheses in the last expression (this will be done in the following when no confusion can arise). Furthermore, $a,b,\ldots$ are the detector orientations used in the various measurements, and to shorten the presentation a notation will be used where $A,B,\ldots$ are the RV's corresponding to the above orientations. The RV's describe measurement results from one detector if it is unprimed (A), and from the other when primed (A'), so that $E(AB')$ is the correlation between A and B'.

I have chosen the ''deterministic'' case here, and will not discuss the ''stochastic'' case as the generalization is straightforward. In this formalism, the Bell inequality can be stated as follows.

Theorem 1 (the Bell inequality). The following four prerequisites are assumed to hold except at a null set.

(i) Realism. Measurement results can be described by probability theory, using (two families of) RV's,

$$\left.\begin{aligned} A(a,b):&\quad \Lambda\to V\\ &\quad \lambda\mapsto A(a,b,\lambda)\\ B'(a,b):&\quad \Lambda\to V\\ &\quad \lambda\mapsto B'(a,b,\lambda)\end{aligned}\right\} \quad \forall a,b.$$

(ii) Locality. A measurement result should be independent of the detector orientation at the other particle,

$$A(a,\lambda) \overset{\text{def}}{=} A(a,b,\lambda) \quad \text{independently of } b,$$

$$B'(b,\lambda) \overset{\text{def}}{=} B'(a,b,\lambda) \quad \text{independently of } a.$$

(iii) Measurement result restriction. Only the results $+1$ and -1 should be possible:

$$V=\{-1,+1\}.$$

(iv) Perfect anticorrelation. A measurement with equally oriented detectors must yield opposite results at the two detectors,

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$$A=-A', \quad \forall a,\lambda.$$

Then,

$$|E(AB')-E(AC')|\leqslant 1+E(BC').$$

To include detector inefficiency in the above inequality, previously two approaches have been used. The first is to use probabilities instead of correlations and derive an inequality on the probabilities (see Ref. [7]). The second is to *assign* the measurement result 0 (zero) to an undetected particle, which makes the original Bell inequality inappropriate because of prerequisite (iii) in Theorem 1. Thus the CHSH inequality must be used and subsequently generalized to obtain an inequality valid in this case (see Ref. [9]).

A third approach, presented here, uses correlations but makes no assignment of any measurement result to the undetected particles. Thus it is possible to obtain a direct generalization of the original Bell inequality.

II. GENERALIZATION OF THE ORIGINAL BELL INEQUALITY

The measurement results are modeled as RV's, which in the ideal case would be defined as in prerequisite (i) in Theorem 1. In the case with inefficiency the situation is quite different, as there are now points of Λ where the particle would be undetected. To avoid the quite arbitrary assignment used in the second approach above, the RV's will be said to be *undefined* at these points, i.e., they will only be defined at subsets of Λ, which will be denoted $\Lambda_A(a,b)$ and $\Lambda_{B'}(a,b)$, respectively.

In this setting, a new expression for the expectation value is needed. The averaging must be restricted to the set where the RV in question is defined, and the probability measure adjusted accordingly,

$$E_X(X)\stackrel{\text{def}}{=}\int_{\Lambda_X} X\,dP_X, \quad \text{where} \quad P_X(E)=P(E|\Lambda_X). \tag{2}$$

The correlation is in this case $E_{AB'}(AB')$, the expectation of AB' on the set at which both factors in the product are defined, the set $\Lambda_{AB'}=\Lambda_A\cap\Lambda_{B'}$. This is the correlation that would be obtained from an experimental setup where the coincidence counters are told to ignore single particle events. In an experiment the pairs that are detected are the ones with λ's in $\Lambda_{AB'}$, so the ensemble is restricted from Λ to $\Lambda_{AB'}$.

It is now easy to see what makes the proof of the original Bell theorem break down. The start of the proof is

$$|E_{AB'}(AB')-E_{AC'}(AC')| = \left|\int_{\Lambda_{AB'}} AB'\,dP_{AB'} - \int_{\Lambda_{AC'}} AC'\,dP_{AC'}\right|. \tag{3}$$

The integrals on the right-hand side cannot easily be added when $\Lambda_{AB'}\neq\Lambda_{AC'}$, so a generalization of Theorem 1 is needed.

Theorem 2 (the Bell inequality with ensemble change). The following four prerequisites are assumed to hold except at a P-null set.

(i) Realism. Measurement results can be described by probability theory, using (two families of) RV's, which may be undefined on some part of Λ, corresponding to measurement inefficiency,

$$\left.\begin{aligned} A(a,b)&:\Lambda_A(a,b)\to V\\ &\lambda\mapsto A(a,b,\lambda)\\ B'(a,b)&:\Lambda_{B'}(a,b)\to V\\ &\lambda\mapsto B'(a,b,\lambda)\end{aligned}\right\} \quad \forall a,b.$$

(ii) Locality. A measurement result should be independent of the detector orientation at the other particle,

$$A(a,\lambda)\stackrel{\text{def}}{=}A(a,b,\lambda) \ \text{on} \ \Lambda_A(a)\stackrel{\text{def}}{=}\Lambda_A(a,b)$$

$$\text{independently of } b,$$

$$B'(b,\lambda)\stackrel{\text{def}}{=}B'(a,b,\lambda) \ \text{on} \ \Lambda_{B'}(b)\stackrel{\text{def}}{=}\Lambda_{B'}(a,b)$$

$$\text{independently of } a.$$

(iii) Measurement result restriction. Only the results $+1$ and -1 should be possible,

$$V=\{-1,+1\}.$$

(iv) Perfect anticorrelation. A measurement with equally oriented detector must yield opposite results if both particles are detected,

$$A=-A', \quad \text{on} \ \Lambda_{AA'}=\Lambda_A\cap\Lambda_{A'}.$$

Define the bound on the ensemble change when the measurement setup is changed as

$$\delta=\inf_{A,B',C,D'} P_{AB'}(\Lambda_{CD'}) \quad (\Rightarrow 0\leqslant\delta\leqslant 1).$$

Then,

$$|E_{AB'}(AB')-E_{AC'}(AC')|\leqslant 3-2\delta+E_{BC'}(BC').$$

Proof. The proof consists of two steps. The first part is similar to the proof of Theorem 1, using the ensemble $\Lambda_{ABB'C'}$, on which all the RV's A, B, B', and C' are defined. This is to avoid the problem mentioned above. This ensemble may be empty, but only when $\delta=0$ and then the inequality is trivial, so $\delta>0$ can be assumed in the rest of the proof. Now (i)–(iv) yields

$$|E_{ABB'C'}(AB')-E_{ABB'C'}(AC')|\leqslant 1+E_{ABB'C'}(BC'). \tag{4}$$

The second step is to translate this into an expression with $E_{AB'}(AB')$ and so on. Using (i)–(iii) and the triangle inequality ($\Lambda_{BD'}^{\mathsf{C}}$ is the complement of $\Lambda_{BD'}$),

$$
\begin{aligned}
&\left|E_{AC'}(AC')-\delta E_{ABC'D'}(AC')\right| \\
&\quad\leqslant\left|P(\Lambda_{BD'}^{\mathsf{C}})E_{AC'}(AC'|\Lambda_{BD'}^{\mathsf{C}})\right| \\
&\qquad+\left|P(\Lambda_{BD'})E_{AC'}(AC'|\Lambda_{BD'})-\delta E_{ABC'D'}(AC')\right| \\
&\quad=P_{AC'}(\Lambda_{BD'}^{\mathsf{C}})\left|E_{AC'}(AC'|\Lambda_{BD'}^{\mathsf{C}})\right| \\
&\qquad+[P_{AC'}(\Lambda_{BD'})-\delta]\left|E_{ABC'D'}(AC')\right| \\
&\quad\leqslant P_{AC'}(\Lambda_{BD'}^{\mathsf{C}})E_{AC'}(\left|AC'\right||\Lambda_{BD'}^{\mathsf{C}}) \\
&\qquad+[P_{AC'}(\Lambda_{BD'})-\delta]E_{ABC'D'}(\left|AC'\right|) \\
&\quad\leqslant 1-\delta.
\end{aligned} \tag{5}
$$

The inequalities (4) and (5) together with the triangle inequality yield the desired result. ■

III. CHANGE OF ENSEMBLE

An important concept to understand the above inequality is ‘‘change of ensemble.’’ Assume that a sequence of experiments is performed, where the ‘‘hidden variable’’ has the values $\lambda_1,\lambda_2,\ldots,\lambda_n$. If the orientations of the detectors were a and b, the detected pairs would be the ones with λ_i's in $\Lambda_{AB'}$. Now if the orientations instead were c and d, the detected pairs would be the ones with λ_i's in $\Lambda_{CD'}$. Then, if $\Lambda_{AB'}\neq\Lambda_{CD'}$, different pairs would be detected if it was possible to do the same run of λ_i's in different setups, i.e., *the ensemble would change*.

The importance of this is most easily seen in the following example. Assume that the nondetections are distributed independently of the detector orientations, i.e.,

$$
\begin{aligned}
&\Lambda_A \text{ is independent of } a,\\
&\Lambda_{B'} \text{ is independent of } b,
\end{aligned} \tag{6}
$$

which yields $\Lambda_{AB'}=\Lambda_{CD'}$ and

$$\delta=P_{AB'}(\Lambda_{CD'})=1. \tag{7}$$

Now, it is easy to see that when $\delta=1$, the result resembles that of the original Bell inequality:

$$|E_{AB'}(AB')-E_{AC'}(AC')|\leqslant 1+E_{BC'}(BC'). \tag{8}$$

Evidently, for this kind of model, inequality (8) is valid at all levels of inefficiency. But we have discarded all events but coincidences in the correlations, so quantum mechanics with detector inefficiency would violate inequality (8). Thus, to exploit the ‘‘loophole’’ in the Bell inequality, the nondetections must be distributed in such a way that the ensemble changes. They cannot be simple independent errors, but must be included in the model at a deeper level.

IV. MEASUREMENT INEFFICIENCY

In Theorem 2, $\delta>3/4$ is sufficient for a violation to occur from the quantum-mechanical correlation. But experiment does not yield an estimate of δ easily, so an inequality involving measurement inefficiency would be useful.

Corollary 3 (the Bell inequality with measurement inefficiency). Assume that Theorem 2(i)–(iv) hold except on a P-null set. Two ways of defining measurement efficiency are used.

(a) Detector efficiency. The least probability that a particle is detected is

$$\eta_1 \overset{\text{def}}{=} \inf_{A,i} P(\Lambda_{A^{(i)}}),$$

where the infimum is taken over all orientations of both detectors. If $\eta_1\geqslant 3/4$, then

$$|E_{AB'}(AB')-E_{AC'}(AC')|\leqslant\frac{3-2\eta_1}{2\eta_1-1}+E_{BC'}(BC').$$

(b) Coincidence efficiency. The least probability that a particle is detected at one detector given that it is detected at the other one is

$$\eta_{2,1} \overset{\text{def}}{=} \inf_{A,B,i\neq j} P_{A^{(i)}}(\Lambda_{B^{(j)}}),$$

where the infimum is taken over all orientations of A,B' and A',B. If $\eta_{2,1}\geqslant 2/3$, then

$$|E_{AB'}(AB')-E_{AC'}(AC')|\leqslant\frac{4}{\eta_{2,1}}-3+E_{BC'}(BC').$$

Proof. First, to prove (b), use the simple inequality

$$
\begin{aligned}
P_{AC'}(\Lambda_B)&=\frac{P_{C'}(\Lambda_{AB})}{P_{C'}(\Lambda_A)}\\
&=\frac{P_{C'}(\Lambda_A)+P_{C'}(\Lambda_B)-P_{C'}(\Lambda_A\cup\Lambda_B)}{P_{C'}(\Lambda_A)}\\
&\geqslant 1+\frac{\eta_{2,1}-1}{P_{C'}(\Lambda_A)}\geqslant 2-\frac{1}{\eta_{2,1}},
\end{aligned} \tag{9}
$$

which gives

$$
\begin{aligned}
P_{AC'}(\Lambda_{BD'})&=P_{AC'}(\Lambda_B)+P_{AC'}(\Lambda_{D'})-P_{AC'}(\Lambda_B\cup\Lambda_{D'})\\
&\geqslant 2\left(2-\frac{1}{\eta_{2,1}}\right)-1=3-\frac{2}{\eta_{2,1}}.
\end{aligned} \tag{10}
$$

This means that

$$\delta=\inf_{A,B',C,D'} P_{AB'}(\Lambda_{CD'})\geqslant 3-\frac{2}{\eta_{2,1}}, \tag{11}$$

and when the right-hand side is non-negative ($\eta_{2,1}\geqslant 2/3$), the (b) part follows from Theorem 2. Now, to prove (a) the same approach gives

$$\eta_{2,1}=\inf_{A,B,i\neq j} P_{A^{(i)}}(\Lambda_{B^{(j)}})\geqslant 2-\frac{1}{\eta_1}. \tag{12}$$

If $\eta_1 \geqslant 3/4$, the above inequality yields $\eta_{2,1} \geqslant 2/3$, and (a) then follows from (b). ■

The two different definitions of measurement efficiency are motivated by models for which $\eta_1 \neq \eta_{2,1}$ (see, e.g., Ref. [10], Chap. 6), and as these results are aimed to be as general as possible, the definitions do not need auxiliary assumptions. There is also the fact that it is possible to estimate $\eta_{2,1}$ from coincidence data, whereas η_1 is more difficult to obtain. A quantum-mechanical violation of these inequalities is obtained when $\eta_1 > 9/10$ or $\eta_{2,1} > 8/9 \approx 0.889$. Both of these bounds are higher than the previous bound in Ref. [9].

V. PREVIOUS RESULTS

The reason for the bound to be higher than previously is that quantum mechanics violates the CHSH inequality more strongly than Theorem 1. Using this inequality, the following is obtained.

Theorem 4 (the CHSH inequality with ensemble change). The following three prerequisites are assumed to hold except at a P-null set.

(i) Realism. As in Theorem 2(i).

(ii) Locality. As in Theorem 2(ii).

(iii) Measurement result restriction. The results may only range from -1 to $+1$,

$$V = \{x \in \mathbb{R}; -1 \leqslant x \leqslant +1\}.$$

With δ as in Theorem 2, this yields

$$|E_{AC'}(AC') - E_{AD'}(AD')| + |E_{BC'}(BC') + E_{BD'}(BD')| \leqslant 4 - 2\delta.$$

The proof is similar to that of Theorem 2.

A quantum-mechanical violation of this inequality would demand $\delta > 2 - \sqrt{2} \approx 0.586$. This is, as expected, significantly lower than obtained from Theorem 2. The measurement inefficiency result is as follows.

Corollary 5 (the CHSH inequality with measurement inefficiency). Assume that Theorem 4(i)–(iii) hold except on a P-null set, and define η_1 and $\eta_{2,1}$ as in Corollary 3. Then, we get the following.

(a) Detector efficiency.

$$|E_{AC'}(AC') - E_{AD'}(AD')| + |E_{BC'}(BC') + E_{BD'}(BD')| \leqslant \frac{2}{2\eta_1 - 1}.$$

(b) Coincidence efficiency.

$$|E_{AC'}(AC') - E_{AD'}(AD')| + |E_{BC'}(BC') + E_{BD'}(BD')| \leqslant \frac{4}{\eta_{2,1}} - 2.$$

The proof is simply to apply the inequalities (11) and (12) to Theorem 4. The "coincidence efficiency" part in (b) is similar to the generalization presented Ref. [9] and the bound is of the same size, $\eta_{2,1} > 2(\sqrt{2} - 1) \approx 0.828$. In Ref. [9], the assumptions of independent errors and constant detector efficiency are used, and then $\eta_{2,1} = \eta_1 = \eta$, and Corollary 5 yields the same bound as in Ref. [9], $\eta > 2(\sqrt{2} - 1) \approx 0.828$. Note that these assumptions are not needed when using the formal detector efficiency definitions above.

VI. CONCLUSIONS

The original Bell inequality is possible to generalize itself to the inefficient case, although the bound on the inefficiency is not as low as obtained from the CHSH inequality. This is because of the stronger violation of the CHSH inequality by quantum mechanics. The generalization of the CHSH inequality in Ref. [9] is possible to obtain in this approach, and the inefficiency bound is the same, $\eta > 82.8\%$.

The formal definition of the term "measurement inefficiency" in this paper uses no auxiliary assumptions such as constant efficiency or independent errors. It is nevertheless possible to obtain the bounds. An estimate of the coincidence efficiency is possible to extract from the coincidence data, so that an easy check of the bound $\eta_{2,1} > 82.8\%$ is possible.

"Change of ensemble" is an essential property needed in the local hidden-variable model to enable it to approach the quantum-mechanical correlation. If the model does not have this property, the generalized inequality reduces to the inequality (8), which resembles the original Bell inequality. The quantum-mechanical predictions including measurement inefficiency violate that inequality as strongly as quantum mechanics without measurement inefficiency violates the original Bell inequality. A quantitative bound on the change of ensemble for quantum-mechanical violation in Theorem 2 (generalized Bell inequality) is $\delta > 75\%$ and in Theorem 4 (generalized CHSH inequality), $\delta > 58.6\%$.

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