Dissipation and decoherence in a quantum register

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A model for a quantum register $\mathcal{R}$ made of N replicas of a d-dimensional quantum system (cell) coupled with the environment is studied by means of a Born-Markov master equation (ME). Dissipation and decoherence are discussed in various cases in which a subdecoherent encoding can be rigorously found. For the quantum-bit case ($d=2$) we have solved, for small N, the ME by numerical direct integration and studied, as a function of the coherence length ξ_c of the bath, fidelity and decoherence rates of states of the register. For large enough ξ_c the singlet states of the global su(2) pseudo spin algebra of the register (noiseless at $\xi_c=\infty$) are shown to have a much smaller decoherence rates than the rest of the Hilbert space. [S1050-2947(98)05705-9]

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I. INTRODUCTION

Preserving coherence in a quantum system is one of the most demanding features required to be able to take practical advantage of the implementation of the objects of quantum information and quantum computation theory [1]. Indeed all the additional power, with respect to the classical case, arising from the quantum nature of the information-processing device depends on the complex linear structure of the state space of a quantum system and on the invariance of such a structure under (unitary) time evolution. The system is therefore endowed with a massive intrinsic parallelism and the capability of exhibiting interference. Unfortunately all of this holds only for *closed* quantum systems. Real systems are unavoidably coupled with the environment in which they are embedded, hence they have to be considered as *open* systems, no matter how weak is the interaction. The relevant state manifold now has a *convex structure* [2]; the dynamics in general is no longer unitary and the interference patterns may disappear. This set of effects is known as the *decoherence* problem [3]. The protection of quantum-encoded information against environmental noise has been, up to now, mainly addressed in the framework of the so-called *error correcting* codes [4]. These are essentially schemes to encode redundant information in such a way that it can be recovered also when (a few) ''errors'' due to external sources have occurred. Such schemes are often based on suitable measurement protocols that have to be performed frequently enough to keep the error level within the scope of the given encoding. Of course this implies that quantum information-processing systems have to be coupled with a classical measurement apparatus: even leaving aside the obvious practical difficulties, such a necessity naturally leads, at least, to a severe slowdown of the computational speed. More recently in [5] (see also [6,7]) the idea has been put forward that, conceptually, a more efficient quantum state protection can be realized by encoding the information in subspaces that the (nonunitary) dynamics makes *intrinsically* more robust against the perturbation due to the environment. Here the attitude is, in some sense, opposite to that at the basis of error correcting codes: now one aims to encode states that cannot be easily corrupted rather than to look for states that can be easily corrected. In this approach one has to assume explicit models of system-environment interaction and try to design the various ingredients in such a way that the algebraic-dynamical structure of the global system gives rise to the stable subspaces one is looking for. Since the typical environment consists of infinitely many degrees of freedom a direct Hamiltonian approach to the problem is not the most suitable except for some simplified situations [8]. In this paper we address the problem of dynamically stable quantum encoding within a master equation formalism that allows us to deal directly with the marginal dynamics of the computational degrees of freedom. The relevant information about the environment is contained in a few parameters appearing in the master equation itself. The system considered is the model of a *quantum register*: N replicas of a given finite-dimensional quantum system (the cell). If the cell is two dimensional one obtains an N–*quantum bit* (qubit) register. The key feature of the existence of the subdecoherent codes is the possibility of partitioning the register in clusters (possibly coinciding with a single cell or with the whole register) such that the cells within each cluster are collectively perturbed by the environment. It is the dynamical symmetry of the cluster that allows one to single out collective (entangled) states that, at least on a short time scale, are unaffected by the noise and therefore evolve unitarily. This mechanism has a well-known counterpart in quantum optics given by the phenomenon of *subradiance* [9].

The paper is organized as follows: in Sec. II we introduce the model, in Sec. III are discussed the general features of both the master equation and the subdecoherent codes. The cases of purely dephasing and dissipative coupling with the environment are analyzed respectively in Secs. IV and V. Section VI contains some conclusions.

II. THE MODEL

We call a system $\mathcal{R}$ a *quantum register* with N d cells, if $\mathcal{R}$ is composed by N replicas of a d-level system. The Hil-

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bert space is given by $\mathcal{H}_{\mathcal{R}}=\otimes_{i=1}^{N}\mathcal{H}_i$, where $\mathcal{H}_i\cong\mathbb{C}^d$ ($i=1,\ldots,N$) is the single d-cell Hilbert space. In particular, if $d=2$ one has an *N-qubit* register. The set of the *states* (density matrices) of $\mathcal{R}$ is

$$S_{\mathcal{R}}=\{\rho\in \mathrm{End}(\mathcal{H}_{\mathcal{R}}):\rho\geqslant 0,\ \ \rho=\rho^\dagger,\ \ \mathrm{tr}^{\mathcal{R}}\rho=1\}. \tag{1}$$

$S_{\mathcal{R}}$ is *not* a linear subspace of $\mathrm{End}(\mathcal{H}_{\mathcal{R}})$ but a convex submanifold. The register is coupled with an uncontrollable environment $\mathcal{B}$ (from now on the *bath*). The time evolution of the states of the closed system $\mathcal{R}+\mathcal{B}$ is generated by a Hamiltonian of the form $H=H_{\mathcal{R}}+H_{\mathcal{B}}+H_{\mathcal{I}}$. We now discuss the structure of each of these terms. The bath will be considered as a single bosonic field, namely, $H_{\mathcal{B}}=\Sigma_k\omega_k b_k^\dagger b_k$ describes a set of noninteracting linear oscillator (field modes). The self-Hamiltonian $\mathcal{H}_{\mathcal{R}}$, of $\mathcal{R}$, is assumed, for the time being, to be the sum of single-cell Hamiltonians H_i^C (i.e., the register is an array of noninteracting cells). The register-bath interaction is given by the sum of the bath-cell interactions:

$$H_{\mathcal{I}}=\sum_{ki\alpha}\ (g_{ki\alpha}\ b_k^\dagger A_i^\alpha+\mathrm{H.c.}). \tag{2}$$

Here the A_i^α's are single-cell operators whose action is nontrivial only on the ith tensor factor of $\mathcal{H}_{\mathcal{R}}$, representing the various interaction channels through which the ith cell can be coupled with the bath. Although this kind of situation can be suitably handled by resorting to the notion of dynamical algebra [10], in the following we shall assume that there is just one dominant interaction, the corresponding operator being A_i. As is well known, the generic effect of $H_{\mathcal{I}}$ on the marginal dynamics of $\mathcal{R}$ is to induce dissipation and decoherence. The first effect, of course, consists in the irreversible loss of register energy into the bath. Decoherence is a pure quantum effect consisting in the destruction of phase coherence of the register states: due to the entanglement with the bath the initial pure preparations of the register become mixed in a very short time scale. The interplay between these two phenomena is strictly related to the nature of the operators $\{A_i\}$. Now we make another simplifying assumption, supposing that the A_i's are eigenvectors of the adjoint action of $H_{\mathcal{R}}$,

$$[H_{\mathcal{R}},A_i]=-\epsilon A_i(\epsilon\in\mathbb{R}_0^+). \tag{3}$$

This means that if $\epsilon>0$, the necessarily non-Hermitian and traceless A_i's ($A_i^\dagger$'s) are the destruction (creation) operators of *elementary cell excitations* of $H_{\mathcal{R}}$. Notice that the energy ϵ does not depend on the cell index i, in that we are considering replicas of the same physical system. If one considers the zero-temperature case, in which only the bath vacuum is involved, the effect of $H_{\mathcal{I}}$, therefore, will be that of letting the register relax to the A vacuum $|A_0\rangle$, ($A_i|A_0\rangle=0,\forall i$) by exciting the bath modes. On the other hand, if $\epsilon=0$ the possibly Hermitian A_i's belong to a symmetry algebra of $H_{\mathcal{R}}$, and no energy exchange occurs at all: the effect of register-bath interaction is pure decoherence. A quantity that will play an essential role in the following is the *bath coherence length* ξ_c, which, in a Hamiltonian approach, can be defined as the spatial scale over which the coupling constants $g_{ki}\equiv g_k(i)$, have a non-negligible variation; when $\xi_c=\infty$, the g_{ki}'s no longer depend on the qubit index i. This limit will be referred to as the *replica symmetric* point, in that for $\xi_c=\infty$, the dynamics becomes invariant under the action of the symmetric group $\mathcal{S}_N$ of the cell permutations and only the collective operators $A=\Sigma_i A_i$ are effectively coupled with the bath. This situation corresponds to the well-known Dicke limit of quantum optics [11]. To exemplify this situation let us consider the $\xi_c=\infty$, limit with $A_i=\sigma_i^-$. In this case, as far as the coupling with the environment is concerned, the relevant register operators are $S^\alpha=\Sigma_{i=1}^N\sigma_i^\alpha,(\alpha=\pm)$. Let $H_{\mathcal{R}}=\epsilon S^z+H_{\mathcal{R}}^1$, where $H_{\mathcal{R}}^1$ is a qubit-qubit interaction term, and suppose that the latter is su(2) invariant [(i.e., $[H_{\mathcal{R}}^1,S^\alpha]=0$, $(\alpha=\pm,z)$]; one then has the commutation relation $[H_{\mathcal{R}},S^\pm]=\pm\epsilon S^\pm$. It follows that for large time the register relaxes to the lowest S^z eigenstate allowed by the total spin conservation. If $H_{\mathcal{R}}^1=0$, this amounts to a ground-state relaxation, whereas if $\epsilon=0$ and $H_{\mathcal{R}}^1\neq 0$, there is no energy loss. This example will be discussed with greater detail in Sec. V.

III. MASTER EQUATION

The quantum dynamics of the system $\mathcal{R}+\mathcal{B}$ is highly nontrivial, and exact results are difficult to obtain. Nevertheless one is mostly interested in the register marginal dynamics (i.e., forgetting about the bath degrees of freedom) in order to study stability against external noise of the information-coding states of $\mathcal{R}$. This issue can be conveniently addressed in the framework of the Liouville–von Neumann equation for open systems, the so-called master equation (ME). Following the standard Born-Markov scheme where one traces out the bath degrees of freedom, which is assumed to be in the state $\rho_{\mathcal{B}}$, one obtains a closed equation for the marginal density matrix of $\mathcal{R}$, of the form

$$\dot\rho=\mathbf{L}(\rho)\equiv(i\ \ \mathrm{ad}\,H'_{\mathcal{R}}+\tilde{\mathbf{L}})(\rho), \tag{4}$$

where as usual $\mathrm{ad}\,H(\rho)\equiv[\rho,H]$, denotes the adjoint action of H. The *superoperator* $\mathbf{L}$ is called the Liouvillian. The action of the non-Hamiltonian (dissipative) part is

$$\tilde{\mathbf{L}}(\rho)=\sum_{ij=0}^{N-1}\left\{\Gamma_{ij}^{(-)}A_i\rho A_j^\dagger-\frac{\Gamma_{ji}^{(-)}}{2}(A_i^\dagger A_j\rho+\rho A_i^\dagger A_j)+\Gamma_{ij}^{(+)}A_i^\dagger\rho A_j-\frac{\Gamma_{ji}^{(+)}}{2}(A_iA_j^\dagger\rho+\rho A_iA_j^\dagger)\right\}, \tag{5}$$

where the $\Gamma_{ij}^{(\pm)}$'s are temperature-dependent coupling constants containing all relevant information about the bath. They are respectively associated with the process of deexcitation and excitation of the qubit system. At $T=0$, one has $\Gamma_{ij}^{(+)}=0$. The renormalized Hamiltonian $H'_{\mathcal{R}}=H_{\mathcal{R}}+\delta H_{\mathcal{R}}$, where, by introducing the Lamb-shift parameters $\Delta_{ij}^{(\pm)}$,

$$\delta H_{\mathcal{R}}=\sum_{ij}\ (\Delta_{ij}^{(-)}A_i^\dagger A_j+\Delta_{ji}^{(+)}A_iA_j^\dagger). \tag{6}$$

At zero temperature the excitation terms $\Delta_{ij}^{(+)}$ are vanishing. Notice that these terms make the cells interacting even though $H_{\mathcal{R}}$ is a free-cell Hamiltonian. On the other hand, it follows from relation (3) that $[\delta H_{\mathcal{R}}, H_{\mathcal{R}}]=0$; this means that, in this model, the Lamb shift terms are not responsible for additional register energy loss, but they are a source of dephasing. Let $n_k = \mathrm{tr}^{\mathcal{B}}(b^\dagger b_k \rho_{\mathcal{B}})$ be the mean occupation number of the mode k in the initial (thermal) bath state $\rho_{\mathcal{B}}$. The explicit form for the coefficients appearing in the ME (4) is

$$\Gamma_{ij}^{(\pm)} = 2\pi \sum_k g_{ki}\overline{g_{kj}}[n_k + \theta(\mp)]\delta(\omega_k - \epsilon),$$

$$\Delta_{ij}^{(\pm)} = \mathrm{P}\sum_k \frac{g_{ki}\overline{g_{kj}}}{\omega_k - \epsilon}[n_k + \theta(\mp)]. \tag{7}$$

θ is the customary Heaviside function, and P denotes the principal part. From these relations it follows that $\mathbf{\Gamma}^{(\pm)}$ and $\mathbf{\Delta}^{(\pm)}$ are Hermitian. Furthermore $\mathbf{\Gamma}^{(\pm)} \geqslant 0$ and $\mathbf{\Gamma}^{(-)} \geqslant \mathbf{\Gamma}^{(+)}$. It is important to notice that the assumption (3) plays an essential role in the derivation of the ME, in that it allows one to move to the interaction picture (with respect to $H_{\mathcal{R}}$) $A_i \rightarrow A_i e^{-i\epsilon t}$. This is necessary in order to separate the (fast) dynamics generated by the self-Hamiltonian from the (slow) one generated by the coupling with the bath. When only the collective cell-operators A are coupled with the bath, $H_{\mathcal{R}}$ has to satisfy condition (3) only with respect to them. Given such $H_{\mathcal{R}}$ one can obtain a family of new register Hamiltonians, fulfilling the same constraint simply by adding terms commuting with $\{A, A^\dagger\}$. Introducing the notation $A_i^\sigma = \theta(-\sigma)A_i + \theta(\sigma)A_i^\dagger$, Eq. (5) can be cast in the compact form

$$\tilde{\mathbf{L}}(\rho) = \frac{1}{2}\sum_{ij,\sigma=\pm} \Gamma_{ij}^{(\sigma)}(2A_i^\sigma \rho A_j^{-\sigma} - \{A_j^{-\sigma}A_i^\sigma, \rho\}). \tag{8}$$

Diagonalizing the Hermitian matrices $\mathbf{\Gamma}^{(\sigma)} = \|\Gamma_{ij}^{(\sigma)}\|$ $(\sigma = \pm)$ one obtains the following canonical form for the dissipative part of the Liouvillian [12]:

$$\tilde{\mathbf{L}}(\rho) = \frac{1}{2}\sum_{\mu,\sigma=\pm} \lambda_\mu^\sigma([L_\mu^\sigma \rho, L_\mu^{-\sigma}] + [L_\mu^\sigma, \rho L_\mu^{-\sigma}]), \tag{9}$$

where $\{\lambda_\mu^\sigma\}$ are the eigenvalues of $\mathbf{\Gamma}^{(\sigma)}$. Moreover $L_\mu^\sigma = \sum_i u_i^\mu A_i^\sigma$, u_i^μ denoting the components of the eigenvectors of $\mathbf{\Gamma}^{(\sigma)}$. The L_μ^σ's will be referred to as the Lindblad operators. Given an initial pure preparation $|\psi_0\rangle$ of the register, one defines $F(t) \equiv \langle \psi_0 | \rho(t) | \psi_0 \rangle$ *fidelity*. Such a quantity measures the degree of similarity with the initial preparation that a state maintains during its time evolution. Another quantity that one introduces in order to study the quantum coherence loss due to the bath is $\delta(t) = \mathrm{tr}[\rho(t) - \rho(t)^2]$, called *linear entropy* (or idempotency deficit). This quantity shares with the von Neumann entropy $S = -\mathrm{tr}\rho \ln \rho$, the fundamental property $\delta[\rho] = 0 \Leftrightarrow \rho^2 = \rho$ (i.e., they both vanish if ρ is a pure state). On the other hand, since the linear entropy does not involve transcendent operatorial functions, it is much simpler to evaluate than S. To characterize the degree of stability of the states it is useful to consider the short-times expansion

$$\delta(t) = \delta(0) + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{t}{\tau_n}\right)^n. \tag{10}$$

In the following τ_n (τ_n^{-1}) will be referred to as the nth-order decoherence time (rate). One straightforwardly finds

$$1/\tau_n^n = -\mathrm{tr}^{\mathcal{R}}\left\{\sum_{k=0}^{n} \binom{n}{k} \mathbf{L}^{n-k}(\rho)\mathbf{L}^k(\rho)\right\} \quad (n \geqslant 1). \tag{11}$$

Since in the following τ_1 will play a major role, here we report explicitly the first decoherence rate

$$\frac{1}{\tau_1} = -2\ \mathrm{tr}\{\rho \tilde{\mathbf{L}}(\rho)\}. \tag{12}$$

In particular, for a pure initial preparation $\rho = |\psi\rangle\langle\psi|$, one has $\delta(0) = 0$ and

$$\tau_1^{-1}[|\psi\rangle] = 2\sum_{\mu,\sigma=\pm} \lambda_\mu^\sigma(\langle\psi|L_\mu^{-\sigma}L_\mu^\sigma|\psi\rangle - |\langle\psi|L_\mu^\sigma|\psi\rangle|^2), \tag{13}$$

whereby one notices that the Hamiltonian component of the Liouvillian does not contribute to the first-order decoherence time [this comes from $\mathrm{tr}^{\mathcal{R}}\{\rho\ \mathrm{ad}\, H'_{\mathcal{R}}(\rho)\} = 0$]. Of course these expressions obtained within the ME equation formalism (which relies on the Born-Markov assumption) differ from the ones that one could get by the exact temporal evolution induced by the interaction Hamiltonian (2) (see, for example [13,14]). Nevertheless, as far as the issue of code stability classification is concerned, this is not crucial in that the (exact) first order decoherence rate $1/\tau_1^{\mathrm{ex}}$ is vanishing for the pure initial state and τ_2^{ex} essentially corresponds to τ_1.

A. Codes

The ME with initial condition ρ has the formal solution $\rho(t) = e^{t\mathbf{L}}(\rho)$, obtained by exponentiation of the Liouville superoperator $\mathbf{L}$. The *stationary* solutions $\rho(t) = \rho$ are therefore the states belonging to $\ker \mathbf{L}$, where $\ker \mathbf{L} = \{\rho \in \mathrm{End}(\mathcal{H}_{\mathcal{R}}) : \mathbf{L}(\rho) = 0\}$. When $\tilde{\mathbf{L}}(\rho) = 0$ it follows, from Eq. (13), that $\delta(t) = O(t^2)$ [whereas for the fidelity one finds $F(t) = 1 - O(t^2)$]. Such a state will be called *subdecoherent*. In general the adjoint action of $H_{\mathcal{R}}$ maps subdecoherent states onto states such that $\tilde{\mathbf{L}}(\rho) \neq 0$; but when $S_{\mathcal{R}} \cap \ker \tilde{\mathbf{L}}$ is $\mathrm{ad}\, H'_{\mathcal{R}}$ invariant the Liouvillian evolution of each state $\rho \in \ker \tilde{\mathbf{L}}$ becomes unitary: $\rho(t) = \exp(-iH'_{\mathcal{R}}t)\rho \exp(iH'_{\mathcal{R}}t)$. In particular one has $\delta(t) = 0, \forall t > 0$ (i.e., $\tau_n^{-1} = 0, \forall n$). This kind of state will be called *noiseless*. A subspace $\mathcal{C} \subset \mathcal{H}_{\mathcal{R}}$ such that each density matrix over it is a subdecoherent (noiseless) state will be referred to as a subdecoherent (noiseless) *code*.

Let us assume $|\psi\rangle$ is subdecoherent. First of all we notice that due to non-negativity of matrices $\mathbf{\Gamma}^{(\sigma)}$ and from the Schwartz inequality, each term of the sum in Eq. (13) is non-negative. Therefore from $\tau_1^{-1}[|\psi\rangle] = 0$ it follows that $\|L_\mu^\sigma|\psi\rangle\|^2 = |\langle\psi|L_\mu^\sigma|\psi\rangle|^2, (\forall \mu, \sigma)$, which in turn implies $|\psi\rangle$ to be a simultaneous eigenvector of *all* the Lindblad operators. Conversely if $|\psi\rangle$ is a simultaneous eigenvector of the

L_μ^σ's then the subdecoherence constraint $\tau_1^{-1}=0$ is trivially fulfilled. In other words a necessary and sufficient condition for the existence of a subdecoherent code is the existence of a simultaneous eigenspace of all Lindblad operators L_μ^σ:

$$\mathcal{C}_\alpha=\{|\psi\rangle\in\mathcal{H}_\mathcal{R}:L_\mu^\sigma|\psi\rangle=\alpha_\mu^\sigma|\psi\rangle,\forall\mu,\sigma\}. \quad (14)$$

The greater $d[\alpha]\equiv\dim\mathcal{C}_\alpha$ is the more efficient the encoding. It is obvious that one has two quite different situations, depending on whether or not the L_μ^σ's are Hermitian. In fact, if $L_\mu^\sigma=(L_\mu^\sigma)^\dagger$ one has that the Lindblad operators commute, $[L_\mu^\sigma,L_\nu^\tau]=\sum_{ij}u_i^\sigma u_j^\tau[A_i,A_j]=0$; then there exists a nontrivial $\mathcal{C}_\alpha$, furthermore $\mathcal{H}_\mathcal{R}=\oplus_\alpha\mathcal{C}_\alpha$. On the other hand, if $L_\mu^\sigma\neq(L_\mu^\sigma)^\dagger$ the Lindblad operators no longer span an abelian algebra and cannot be simultaneously diagonalized. The only candidate of the subdecoherent code is $\mathcal{C}=\cap_{\mu\sigma}\ker L_\mu^\sigma$. Indeed the Lindblad operators satisfy relation (3), from which one derives that the only allowed eigenvalue is $\lambda_\mu^\sigma=0$. The proof is as follows [15]: let $\{|E_i\rangle\}_{i=1}^D$ be a $H_\mathcal{R}$ eigenstates basis of $\mathcal{H}_\mathcal{R}$ ($H_\mathcal{R}|E_i\rangle=E_i|E_i\rangle$, $E_{i+1}\geq E_i$, $D=d^N$). Since the L_μ^+ are raising operators over the spectrum of $H_\mathcal{R}$ one has $L_\mu^+|E_i\rangle\propto|E_{i'}\rangle$, where $i'>i$, $E_{i'}=E_i+\epsilon$ [in particular the maximum eigenvalue vector $|E_D\rangle$ is annihilated by $L_\mu^+,(\forall\mu)$]. Let $|\psi\rangle=\sum_{i=1}^D c_i|E_i\rangle$ be an eigenvector of L_μ^+ with eigenvalue $\lambda\neq0$; then one must have $L_\mu^+|\psi\rangle=\sum_{i=1}^{D-1}c_iL_\mu^+|E_i\rangle=\lambda\sum_{i=1}^D c_i|E_i\rangle$ hence $c_1=0$. Acting on $|\psi\rangle$ with increasing powers of L_μ^+ one analogously finds $c_2=c_3=\cdots=c_D=0$, therefore if $\lambda\neq0$ one would have $|\psi\rangle=0$.

Let $\mathcal{L}$ be the Lie algebra generated by the L_μ^σ's (i.e., the minimal subspace of operators closed under commutation containing $\{L_\mu^\sigma\}_{\mu\sigma}$) then the code $\mathcal{C}$ is nothing but the *singlet sector* of $\mathcal{L}$; each $|\psi\rangle\in\mathcal{C}$ is a one-dimensional representation space of $\mathcal{L}$. From the general form of the Lindblad operators one has $\mathcal{L}\subset\oplus_{i=1}^N\mathcal{L}_i$ where $\mathcal{L}_i$ is the (local) Lie algebra generated by the A_i^σ's. Generically one has $\mathcal{L}_i\cong\mathrm{sl}(d,\mathbf{C})$, therefore if the above inclusion is not strict it follows that $\mathcal{C}=\{0\}$, which has no use for quantum encoding. In order to obtain meaningful codes one has to impose constraints on the algebraic structure generated by the Lindblad operators. The smaller $\mathcal{L}$ is the easier will be the task of finding (by representation theory) nontrivial $\mathcal{C}$. Notice that, given such a subdecoherent code, if $H'_\mathcal{R}$ belongs to the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ then $\mathcal{C}$ is also necessarily noiseless.

The matrices $\mathbf{\Gamma}_{ij}^{(\sigma)}$ and $\mathbf{\Delta}_{ij}^{(\sigma)}$ encode all the information about the spatial correlations among the register cells induced by coupling with the bath. The actual form of these correlations depends [see Eq. (7)] on the detailed form of the coupling functions g_{ki}, on the bath density of the states and on temperature as well.

Leaving aside strongly model-dependent considerations and in view of keeping the form of the ME here considered as general as possible, in the following the matrices $\mathbf{\Gamma}^{(\pm)},\mathbf{\Delta}^{(\pm)}$ will be considered rather as *a priori* data of the problem defining the basic dynamical Eq. (4). In other words they are treated as parameters that have to be ''engineered'' in order to realize an advantageous situation for quantum encoding. In this context the bath coherence length ξ_c is better defined in relation to the spatial behavior of the $\Gamma_{ij}^{(\pm)}$. One can consider the following particular regimes, corresponding to different ''effective'' topologies of $\mathcal{R}$. It is just from these topologies that constraints on the algebraic structure arise.

(i) $\mathbf{\Gamma}_{ij}=\Gamma\,\delta_{ij}$, $(\forall i,j)$: this is the cell limit; the decoherence process occurs independently in each cell. The Lindblad operators coincide with the A_i^σ's ($\mathcal{L}=\oplus_{i=1}^N\mathcal{L}_i$).

(ii) $\mathbf{\Gamma}_{ij}=\Gamma(\forall i,j)$: this is the replica symmetric point; the decoherence is collective. The matrices $\mathbf{\Gamma}^{(\sigma)}$ have constant entries, the only nonzero eigenvalue is N and the corresponding Lindblad operators are given by $L^\pm=N^{-1/2}\sum_iA_i^\pm$ ($\mathcal{L}\cong\mathcal{L}_i$).

The limit (i) is the one usually considered in error correction literature. The case (ii) corresponds to the so-called Dicke limit of quantum optics

An interesting intermediate case between (i) and (ii) is when the register is partitioned in clusters such that in each cluster the cells are coupled in the same way with the environment and different clusters are far enough to feel correlated environments. In other terms, if l (L) is the typical intracluster (intercluster) distance, we are assuming $l\ll\xi_c$ ($L\gg\xi_c$). More formally we assume that there exists a partition $\{C_\lambda\}_{\lambda=1}^M$ of the cell index set $\mathbf{N}_N$, such that

(iii) $\Gamma_{ij}=\Gamma_0$ if $i,j\in C_\lambda$, 0 otherwise. The Lindblad operators are the cluster ones $L_\lambda^\sigma=N_\lambda^{-1/2}\sum_{i\in C_\lambda}A_i^\sigma$, with N_λ being the number cells in the λth cluster ($\mathcal{L}\cong\oplus_{i=1}^M\mathcal{L}_i$). When $M=N$ and $M=1$ we recover respectively the cases (i) and (ii).

For a clustered register the dynamics is invariant under the action of the group $\mathcal{G}\equiv\mathcal{S}_{m_1}\times\cdots\times\mathcal{S}_{m_M}\subset\mathcal{S}_N$; at the replica symmetric point (cell limit) one has $\mathcal{G}=\mathcal{S}_N$ ($\mathcal{G}=\{\mathbf{1}\}$). Some comments are now in order. When the relation (3) holds we see that both in the Hermitian and in the non-Hermitian case the self-Hamiltonian leaves the code invariant; nevertheless also in this rather special situation, due to the renormalizing terms (6) subdecoherence does not necessarily imply noiselessness. The point is that the $\mathbf{\Gamma}^{(\pm)}$'s and the $\mathbf{\Delta}^{(\pm)}$'s in general cannot be diagonalized simultaneously. This can be understood, for example, by looking at the explicit form (7): in the matrix elements $\Delta_{ij}^{(\pm)}$ appears a sum over *all* the bath modes whereas in the $\Gamma_{ij}^{(\pm)}$'s only the modes degenerate with the single cell eigenvalue ϵ are involved. On the other hand, we see, from Eq. (7) that the leading contribution to $\Delta_{ij}^{(\pm)}$ comes from the same bath modes involved in $\Gamma_{ij}^{(\pm)}$, therefore assuming that $\Delta_{ij}^{(\pm)}$ and $\Gamma_{ij}^{(\pm)}$ have the same structure can be in many cases a good approximation. When this is the case also $\delta H_\mathcal{R}$ can be written in terms of the Lindblad operators, namely, each subdecoherent code $\mathcal{C}_\alpha$ is necessarily noiseless.

IV. DECOHERENT COUPLING

In this section we consider the case in which the single-cell operators A_i in Eq. (5) are Hermitian. Although this case is essentially well known we think that it is worthwhile to analyze it in that its exact solvability allows us to shed some light on the general features of the decoherence process of many replicas of a given system coupled with the same environment. Here the ME is considered the starting point of the analysis, we do not assume any *a priori* relation such as

Eq. (3). For the time being we set $H_{\mathcal{R}}=0$. Let $|\alpha\rangle \equiv|\alpha_1,\ldots,\alpha_N\rangle$ denote a simultaneous eigenvector of the A_i's with $A_i|\alpha\rangle=\alpha_i|\alpha\rangle,(i=1,\ldots,N)$. The operators $|\alpha\rangle\langle\alpha'|$ are eigenvectors of the Liouvillian

$$\mathbf{L}(|\alpha\rangle\langle\alpha'|)=W(\alpha,\alpha')|\alpha\rangle\langle\alpha'|,$$
$$W(\alpha,\alpha')=i(\|\alpha\|^2_{\mathbf{\Delta}}-\|\alpha'\|^2_{\mathbf{\Delta}})-\|\alpha-\alpha'\|^2_{\mathbf{\Gamma}}, \tag{15}$$

where $\|\beta\|^2_{\mathbf{M}}=\langle\beta,\mathbf{M}\beta\rangle$ $[\mathbf{M}=\mathbf{\Delta},\mathbf{\Gamma}\equiv\Sigma_{\sigma=\pm}\mathbf{\Gamma}^{(\sigma)} \in \mathrm{End}(\mathbf{C}^N),\beta\in\mathbf{C}^N]$. Notice that $\|\bigcirc\|_{\mathbf{M}}$ is a seminorm only if $\mathbf{M}\geqslant 0$, and a norm only if $\ker\mathbf{\Gamma}=\{0\}$. Each state over $\mathcal{H}_{\mathcal{R}}$ can be written in the form $\rho=\Sigma_{\alpha,\alpha'}R_{\alpha,\alpha'}|\alpha\rangle\langle\alpha'|$, therefore the general solution of Eq. (5) is

$$\rho(t)=\sum_{\alpha,\alpha'} R_{\alpha,\alpha'}e^{W(\alpha,\alpha')t}|\alpha\rangle\langle\alpha'|, \tag{16}$$

whereby one derives the following expressions for fidelity and linear entropy

$$F(t)=\sum_{\alpha\alpha'} |R_{\alpha\alpha'}|^2 e^{W(\alpha,\alpha')t},$$
$$\delta(t)=1-\sum_{\alpha\alpha'} |R_{\alpha\alpha'}|^2 e^{2\Re\, W(\alpha,\alpha')t}. \tag{17}$$

By Eq. (15) the set of subdecoherent and noiseless solutions of the Liouville Eq. (5) is obviously related to the properties of the matrices $\mathbf{\Delta}$ and $\mathbf{\Gamma}$. First at all notice that from the second of Eqs. (17), the imaginary terms in Eq. (15) play no role in decoherence (in the restricted meaning): indeed they give rise to the unitary transformation

$$U_{\mathbf{\Delta}}(t)=e^{-it\delta H_{\mathcal{R}}}=\sum_{\alpha} e^{i\|\alpha\|^2_{\mathbf{\Delta}}t}|\alpha\rangle\langle\alpha|. \tag{18}$$

It is straightforward to verify that the linear entropy is a monotonic nondecreasing function of time, indeed

$$\dot{\delta}(t)=2\sum_{\alpha\alpha'} |R_{\alpha\alpha'}|^2\|\alpha-\alpha'\|^2_{\mathbf{\Gamma}}\geqslant 0, \tag{19}$$

the inequality following from the non-negativity of $\mathbf{\Gamma}$. Cases (i) and (ii) imply, from $W(\alpha,\alpha)=0$, that the diagonal states $\rho_\alpha\equiv|\alpha\rangle\langle\alpha|$ are fixed points of the Liouvillian evolution. Furthermore if $\alpha-\alpha'\in\ker\mathbf{\Gamma}$, one has that the real part of $W(\alpha,\alpha')$ vanishes. Case (i) corresponds to a solution that one could obtain assuming that each cell is interacting with its own independent environment. From Eq. (15) it follows that the maximum decay rate is $O(N)$. In case (i) $\ker\mathbf{\Gamma}=\{0\}$ and only $\alpha=\alpha'$ survives. If the single-cell eigenvalues α_i are nondegenerate the eigenspace $\mathcal{H}(\{\alpha_i\})$ is one dimensional and therefore useless for quantum encoding. If instead the α_i's are m_i-fold degenerate, then $d[\alpha] \equiv \dim\mathcal{H}(\{\alpha_i\})=\Pi_{i=1}^N m_i$. The density matrix corresponding to $|\psi\rangle\in\mathcal{H}(\{\alpha_i\})$ evolves according the unitary transformation $U_{\mathbf{\Delta}}(t)$: these states are noiseless. The largest dimension for the *noiseless* code $\mathcal{H}(\{\alpha_i\})$ is obtained for $\alpha_i=\alpha_M,(\forall i)$ where α_M is the single-cell eigenvalue with the maximum degeneracy. In the qubit case $A_i=\sigma_i^z$ and $\alpha_i=\pm 1/2$. In case even $\mathbf{\Delta}^{(\pm)}$ is proportional to the unit matrix, then the unitary transformation (18) becomes trivial, being $\Sigma_i\alpha_i^2=N/4,(\forall\alpha)$. For the initial state $|\psi_0\rangle=2^{-N/2}\Sigma_\sigma|\sigma\rangle$, uniform linear superposition of all the basis states, one can obtain explicit analytical expressions for the linear entropy and the fidelity

$$\delta(t)=1-e^{-\Gamma Nt}\cosh^N(\Gamma t),$$
$$F(t)=e^{-\Gamma/2Nt}\cosh^N(\Gamma/2t). \tag{20}$$

For $t\to\infty$ one finds $F\sim 2^{-N}$ and $\delta\sim 1-2^{-N}$, results that can be immediately understood from $\rho(\infty)=2^{-N}\Sigma_\sigma|\sigma\rangle\langle\sigma|$. Let us turn to the case (ii). The operator $A=\Sigma_i A_i$ plays the role of *pointer observable* [16]: the diagonal elements with respect to its eigenstate basis of the density matrix do not decohere, whereas the off-diagonal decays with a rate that is proportional to their distances from the diagonal. Now $\dim\ker\mathbf{\Gamma}=N-1$, and the no-damping condition becomes $\Sigma_i\alpha_i=\Sigma_i\alpha_i'$. This means that in that case the space $\mathcal{H}_\alpha$ spanned by the set $B_\alpha=\{|\alpha\rangle:\Sigma_{i=1}^N\alpha_i=\alpha\}$ is decoherence-free. In passing we note that, since A is an extensive observable, at the replica symmetric point the maximum decay rate is $O(N^2)$.

In case (iii) the matrix $\mathbf{\Gamma}$ is block constant and $\alpha-\alpha' \in \ker\mathbf{\Gamma}$, if $\Sigma_{j\in C_\lambda}\alpha=\Sigma_{j\in C_\lambda}\alpha_j'(\lambda=1,\ldots,M)$. Now the relevant operators are the cluster operators $L_\lambda=\Sigma_{j\in C_\lambda}A_i$, the states built over a simultaneous eigenspace of the L_λ's evolve in a noiseless way. As usual, the situation is best exemplified by the qubit case. Let us assume that $A_i=\sigma_i^z$, and N even. At the $\xi_c=\infty$ point the most efficient noiseless encoding is obtained by building states over the eigenspace $S^z=0$. If $\mathbf{\Gamma}$ is partitioned in blocks of m (even) elements one can encode in the subspace with zero cluster z spin. Such a code has dimension

$$d(M)=\binom{m}{m/2}^M. \tag{21}$$

This encoding, with $m=2$ is essentially that proposed in [14]. Until now we have assumed that the self-Hamiltonian was vanishing. If this is not the case, one has that for an initial noiseless preparation the state evolves infinitesimally in a unitary fashion. For finite time the (possible) noncommutativity between $H_{\mathcal{R}}$ and the relevant Lindblad (cell, cluster, register) operators, destroys the coherence of ρ. When relation (3) holds ($\epsilon=0$) $H_{\mathcal{R}}$ commutes with the Lindbald operators. Working in a basis that simultaneously diagonalizes $H_{\mathcal{R}}$ and the A_i^σ's one sees that $U_{\mathbf{\Delta}}(t)\to\exp(-itH_{\mathcal{R}})$, the evolution will remain unitary for finite times; the initial pure states never get mixed.

V. DISSIPATIVE COUPLING

In this section we consider the case of non-Hermitian A_i; namely, the case when the relation (3) holds with $\epsilon>0$. At zero temperature the eigenvalues λ_μ^+ are vanishing. On the other hand since $\mathbf{\Gamma}^{(-)}\geqslant 0$, $\lambda_\mu^-\geqslant 0,(\forall\mu)$ one can immediately check that the register energy $E_{\mathcal{R}}(t)=\mathrm{tr}^{\mathcal{R}}[\rho(t)H_{\mathcal{R}}]$ is a monotonic nonincreasing function. Indeed

$$\dot{E}_{\mathcal{R}}(t) = \mathrm{tr}^{\mathcal{R}}[\tilde{\mathbf{L}}(\rho)H_{\mathcal{R}}] = -\epsilon\sum_{\mu} \lambda_{\mu}^{-} \ \mathrm{tr}^{\mathcal{R}}(L_{\mu}^{+}L_{\mu}^{-}\rho) \leq 0, \tag{22}$$

where we have used the irrelevance of the Hamiltonian component of $\mathbf{L}$ (that is $\mathrm{tr}^{\mathcal{R}}(H_{\mathcal{R}},[H'_{\mathcal{R}},\rho])=0$) the relation $[L_{\mu}^{\sigma}L_{\mu}^{-\sigma},H_{\mathcal{R}}]=0$ [which follows from Eq. (3), which holds for the Lindblad operators as well], and the non-negativity of operators $L_{\mu}^{+}L_{\mu}^{-}$ and ρ. As observed in Sec. III in the present case a subdecoherent code can be obtained if $\mathcal{C}\equiv\cap_{\mu}\ker L_{\mu}\neq\{0\}$.

Restated in this formalism, the essence of the result of Ref. [5] for the qubit case is that at the $\xi_c=\infty$ point the Lindblad operators (and the renormalized self-Hamiltonian) belong to an N-fold tensor representation of a semisimple (dynamical) Lie algebra, out of which a nontrivial $\mathcal{C}$ can be built when N is large enough. In the cell limit (i) if one can find a subspace $\mathcal{C}_i\subset\mathcal{H}$ annihilated by both $A_i^{(+)}$ and $A_i^{(-)}$ then $\mathcal{C}\equiv\mathcal{C}_1\otimes\cdots\otimes\mathcal{C}_N$. An analog construction can be made in the cluster limit. An important example is given by the qubit case. One can design a register that supports noiseless encodings if one is able to build $\mathcal{R}$ in such a way that (iii) is satisfied with $m=4$ qubits for a cluster. Then, according to Ref. [5], a logical qubit can be encoded in each cluster. It is important to note that the dimension of $\mathcal{C}$ decreases passing from (ii) to (iii), and from (iii) to (i).

In general one has $\Gamma_{ij}^{(\pm)}=\Gamma^{(\pm)}(i,j)$. The first-order time scale τ_1 is a functional of $|\psi\rangle$, depending on ξ. The optimal states, with respect to the storage reliability on short times, are those that minimize this functional for a given bath coherence length. Let us assume that $\Gamma_{ij}=\Gamma_0\gamma_{\xi}(i-j)$, where $\gamma_{\xi}(x)\to 1$, when $\xi_c\to\infty$ and $\gamma_{\xi}(x)\to\delta_{x,0}$, when $\xi_c\to 0^{+}$. The latter situation corresponds to the case in which each cell is coupled with an independent bath, therefore $\xi_c\in(0,\infty)$ interpolates between the independent bath limit (i) and the infinite coherence length bath case (ii).

A. Qubit case

Now we specialize to the $d=2$ case: $A_i^{\pm}=\sigma_i^{\pm}$. Let the self-Hamiltonian be of the form $H_{\mathcal{R}}=\epsilon S^{z}+H_{\mathcal{R}}^{1}$, where the second term is a qubit-qubit interaction. In quantum computation applications such a term might arise, for example, during the gate processing. If we assume that $[H_{\mathcal{R}}^{1},S^{\alpha}]=0$ ($\alpha=z,\pm$) then Eq. (3) holds. Now we briefly recall the result of Ref. [5] at the replica symmetric point. When $\xi_c=\infty$, one finds the following.

(i) The total spin operator $S^2=(S^z)^2+1/2\{S^{+},S^{-}\}$, is a constant of the motion.

(ii) Defining in the obvious way an $\mathcal{S}_N$ action T over the density matrices manifold $S_{\mathcal{R}}$, one has $T_{\sigma}\,\mathbf{L}\,T_{\sigma}^{\dagger}=\mathbf{L}$, $(\forall\sigma\in\mathcal{S}_N)$.

(iii) The Lie algebra $\mathcal{L}$ generated by the Lindblad operators $S^{\pm}$ is nothing but the global su(2).

(iv) Since at $\xi_c=\infty$ the coupling functions g_{ki} are assumed to be *strictly* qubit independent also the Lamb-shift matrices $\Delta_{ij}^{(\pm)}$ have constant entries (i.e., $\Delta_{ij}^{(\pm)}=\Delta_0^{(\pm)}\forall i,j$). The renormalizing term can then be written as $\delta H_{\mathcal{R}}=\Delta_0^{-}S^{+}S^{-}+\Delta_0^{+}S^{-}S^{+}$.

From (i)–(iv) it follows that the Hilbert space $\mathcal{H}_{\mathcal{R}}$ splits dynamically according the Clebsch-Gordan decomposition of the n-fold tensor representation of su(2):

$$\mathcal{H}_{\mathcal{R}}=\oplus_{S=S_{\min}}^{N/2}\oplus_{r=1}^{n_N(S)}\mathcal{H}_r(S), \tag{23}$$

where $S_{\min}=0$ ($S_{\min}=1/2$) if N is even (odd). The subspace $\mathcal{H}_r(S)$ is an irreducible su(2) module corresponding to the total spin eigenvalue $S(S+1)$, the latter occurring with multiplicity

$$n_N(S)=\frac{(2S+1)N!}{(N/2+S+1)!(N/2-S)!}. \tag{24}$$

The general state over $\mathcal{H}_r(S)$ has the form $\rho=\Sigma_{M,M'=-S}\rho_{M,M'}|SM\rangle\langle SM'|$, where $S^2|SM\rangle=S(S+1)|SM\rangle$, $S^z|SM\rangle=M|SM\rangle$ $(M=-S,\ldots,S)$ and analogously for $|SM'\rangle$. For a pure state one has

$$\tau_1(\infty)^{-1}=2\,\Gamma_0^{(-)}(\langle\psi|\,S^{+}\,S^{-}\,|\psi\rangle-|\langle\psi|\,S^{-}\,|\psi\rangle|^2), \tag{25}$$

$$+2\,\Gamma_0^{(+)}(\langle\psi|\,S^{-}\,S^{+}\,|\psi\rangle-|\langle\psi|\,S^{+}\,|\psi\rangle|^2). \tag{26}$$

In particular, if $|\psi\rangle=|SM\rangle$ one obtains $(2\,\tau_1)^{-1}=\Gamma_0\,C_{-}^2(S,\,M)+\Gamma_0^{(+)}\,C_{+}^2(S,\,M)$, where $C_{\pm}^2(S,\,M)=S\,(S+1)-M\,(M\pm 1)$. Let us consider the zero-temperature case $(\Gamma_0^{(+)}=0)$ when only the deexcitation processes with strength proportional to $\Gamma_0^{(-)}$ are active. If $|\psi\rangle$ is a *lowest-weight* spin state (i.e., $S^{-}|\psi\rangle=0$), one has $\tau_1(\infty)=\infty$. This result is true for all decoherence times τ_n. At finite temperature the (excitations) terms weighted by $\Gamma_0^{(+)}$ are present as well. On the su(2) *singlets* $|\psi\rangle\in\mathcal{C}\equiv\oplus_{r=1}^{n_N(0)}\mathcal{H}_r(0)$, one has $S^{+}|\psi\rangle=S^{-}|\psi\rangle=0$, and $\delta H_{\mathcal{R}}\,|\psi\rangle=0$; furthermore from the su(2) invariance of $H_{\mathcal{R}}^{1}$ it follows that the unitary part of $\mathbf{L}$ maps the singlet sector onto itself, namely, $\mathcal{C}$ is noiseless. From Eq. (24) it follows that the minimum cluster size to encode a noiseless logical qubit is $N=4$. Defining (in obvious binary notation) the states $|A\rangle\equiv|0011\rangle+|1100\rangle$, $|B\rangle\equiv|0110\rangle+|1001\rangle$, $|C\rangle\equiv|1010\rangle+|0101\rangle$, an orthonormal basis of $\mathcal{C}$ is given by

$$|\mathbf{0}\rangle\equiv 2^{-1}\,(|B\rangle-|A\rangle),$$
$$|\mathbf{1}\rangle\equiv 3^{-1/2}\,(|C\rangle-2^{-1}\,|A\rangle-2^{-1}\,|B\rangle). \tag{27}$$

If $H_{\mathcal{R}}^{1}=0$ these two states are energy degenerate; for nonvanishing qubit-qubit interaction the degeneracy is lifted. For example, if

$$H_{\mathcal{R}}^{1}=J\sum_{\langle ij\rangle}\{\sigma_i^z\sigma_j^z+1/2\,(\sigma_i^{+}\,\sigma_j^{-}+\sigma_i^{-}\,\sigma_j^{+})\} \tag{28}$$

is a Heisenberg coupling between nearest-neighbor qubits arranged on a ring topology, one finds that $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$ are energy eigenstates with eigenvalues respectively given by $E_0=J$ and $E_1=-J$. Since $H_{\mathcal{R}}^{1}$ is su(2) invariant it is always possible to choose the singlet $|\psi\rangle$ among its eigenvectors. It should be emphasized that the su(2) singlet sector is noiseless for a wider class of ME's, with Lindblad operators (and

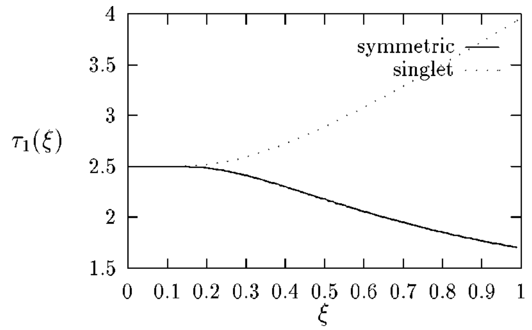

FIG. 1. First-order time scale for the symmetric state $|\psi^{\rm sym}\rangle = (S^+)^2|0\rangle$, and a singlet state, $N=4$ $\Gamma(i,j)=0.1e^{-|i-j|/\xi}$.

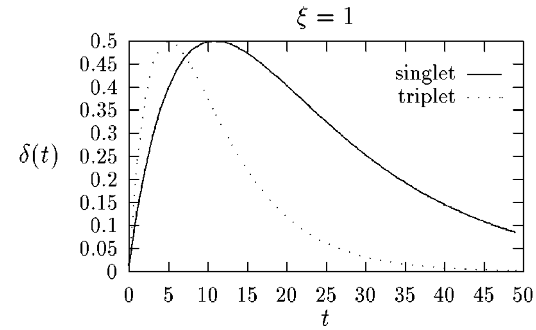

FIG. 3. Linear entropy as a function of the time for the $S^z=0$ singlet and triplet state ($N=2$, $\Gamma_0=0.1e^{-|i-j|/\xi}$).

self-Hamiltonian) given by arbitrary functions of the global operators S^α, ($\alpha=\pm,z$) [17]. Indeed if these operators have the form

$$X=c_1\,\mathbf{I}+F(\{S^\alpha\}), \tag{29}$$

where F is an arbitrary operator-valued analytic function, then—since $F\,|\psi\rangle=0,(\forall\,|\psi\rangle\in\mathcal{C}_N)$—one obtains $X|_{\mathcal{C}_N}=c_1\,\mathbf{I}$. This latter condition is sufficient to preserve the subdecoherence of $\mathcal{C}_N$. Another way to understand this result is that the operators described by Eq. (29) coincide with the $\mathcal{S}_N$-invariant sector (symmetric subspace) of $\mathrm{End}(\mathcal{H}_\mathcal{R})$. Since $\mathcal{C}_N$ is an irreducible $\mathcal{S}_N$ module from the Schur lemma it follows that $X|_{\mathcal{C}_N}\propto\mathbf{I}$. Turning back to the general case $\xi\in(0,\infty)$, if $N=2$, for the initial states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $|\psi_{t,s}\rangle=2^{-1/2}\,(|\uparrow\downarrow\rangle\pm|\downarrow\uparrow\rangle)$, one immediately finds

$$\tau_1^{\uparrow\uparrow}(\xi)=(2\,\Gamma_0^{(-)})^{-1},\quad \tau_1^{\downarrow\downarrow}(\xi)=(2\,\Gamma_0^{(+)})^{-1},$$
$$\tau_1^{t,s}(\xi)=\{2\,(\Gamma_0^{(-)}+\Gamma_0^{(+)})[1\pm\gamma_\xi(1)]\}^{-1}. \tag{30}$$

These equations show that in the generic case ($\Gamma^{(\pm)}\neq 0$) for finite coherence length ξ all the first-order decoherence times are finite as well, whereas for $\xi\to\infty$ the singlet $\tau_1^s(\xi)$ diverges with ξ. Of course for this latter state, since $\mathbf{L}_{\xi=\infty}(|\psi^s\rangle\langle\psi^s|)=0$, all the τ_n's diverge.

In the general case when the matrices $\mathbf{\Gamma}^{(\pm)}$ are not block constant one has to resort to numerical calculations. We have solved Eq. (4) by direct numerical integration in the qubit case with $H_\mathcal{R}=\epsilon\,S^z$. Rather than using the form (7) for the ME parameters, we have chosen a phenomenological parametrization such as $\Gamma_{ij}^{(\pm)}=\Gamma_0^{(\pm)}\,e^{-|i-j|/\xi_c}$ and neglected the self-Hamiltonian renormalization. In Fig. 1 is reported the behavior of $\tau_1(\xi_c)$ for a $N=4$ singlet and the highest-weight su(2) vector belonging the $S=2$ multiplet. We see that for a wide range of ξ_c the decoherence time of the singlet state is much larger than that of the symmetric state.

In Figs. 2 and 3 is compared the behavior of the fidelity and linear entropy of the $N=2$ singlet and ($S^z=0$) triplet states at finite ξ_c. Figures 4 and 5 show, as a function of time, the difference of fidelity and linear entropy, between one of the $N=4$ singlets and the symmetric state $(S^+)^2|0\rangle\in\mathcal{H}_1(2)$, for various bath coherence length. These simple calculations strongly suggest that the *noiseless encoding at infinite coherence length remains, for sufficiently large ξ_c, more robust than all other states.*

B. Gauge transformation

We end this section by showing that for a class of nontrivial qubit couplings, connected to the limits (ii) and (iii) via a local gauge transformation, it is possible to build subspaces annihilated by the dissipative component of the Liouvillian. This transformation is a generalization of the one considered in [11]. Here we give proof for the replica-symmetric case, the cluster case being a straightforward generalization. Let us suppose that $\Gamma_{ij}^{(\pm)}=\Gamma^{(\pm)}e^{i[\phi(i)-\phi(j)]}$, where $\Gamma^{(\pm)},\in\mathbb{R},\phi$: $\mathbb{N}_N\to\mathbb{R}$. This kind of situation is not completely fictitious: for $g_{ki}\sim e^{ik\mathbf{r}_i}$ when there is just one bath mode k degenerate with the qubit energy ϵ, from the first of Eqs. (7) follows that $\Gamma_{ij}^{(\pm)}\sim e^{ik(\mathbf{r}_i-\mathbf{r}_j)}$. Introducing the operators $L_\phi^\sigma=\sum_{j=1}^N e^{i\phi(j)}A_j^\sigma$ the dissipative Liouvillian has the canonical form (9) with $\{\lambda_\mu^\sigma\}=\{\Gamma^{(\sigma)}\}$ and Lindblad op-

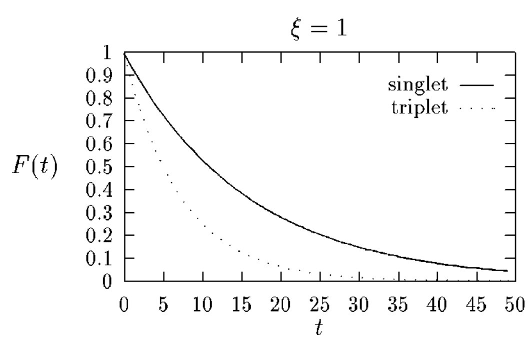

FIG. 2. Fidelity as a function of the time for the $S^z=0$ singlet and triplet state ($N=2$, $\Gamma_0=0.1e^{-|i-j|/\xi}$).

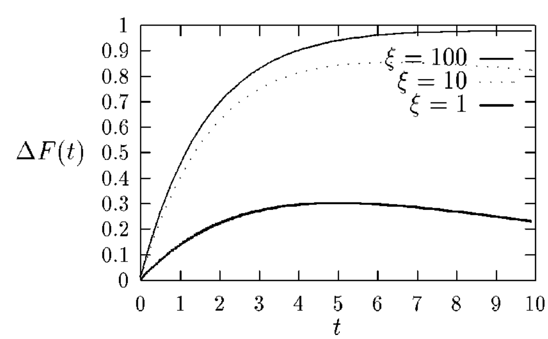

FIG. 4. Fidelity difference, between a $N=4$ singlet and the state $(S^+)^2|0\rangle$, for different bath coherence lengths ξ ($\Gamma_0=0.1e^{-|i-j|/\xi}$).

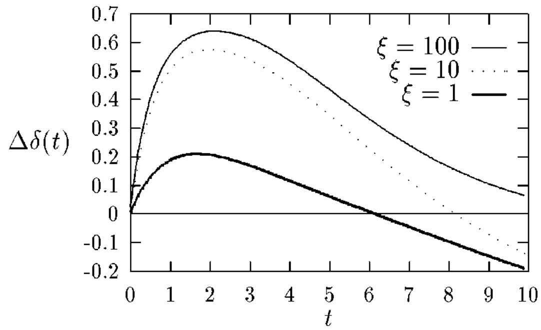

FIG. 5. Linear entropy difference, between the state $(S^+)^2|0\rangle$, and a $N=4$ singlet, for different bath coherence lengths ξ ($\Gamma_0 = 0.1e^{-|i-j|/\xi}$).

erators given by the L^σ_ϕ's. The operators $\{L^\sigma_\phi\}_\sigma$, spanning a Lie algebra $\mathcal{A}_\phi$ isomorphic to $\mathcal{A}$ generated by $\{L^\sigma\}_\sigma$, are obtained from the latter by means of the (local) U(1) gauge transformation:

$$T_\phi\colon \mathrm{End}(\mathcal{H}_\mathcal{R}) \to \mathrm{End}(\mathcal{H}_\mathcal{R}) : X \to U_\phi X U^\dagger_\phi,$$

$$U_\phi = \exp\left\{ i\epsilon^{-1} \sum_{j=i}^{N} \phi(j) H^C_j \right\} \in \otimes^N_{j=1} \mathrm{U}(1)_j, \qquad (31)$$

where we recall that H^C_i is the single-cell Hamiltonian fulfilling relation Eq. (3) with the A_i's. The unitary operator $U \in \mathrm{End}(\mathcal{H}_\mathcal{R})$ maps the singlet sector $\mathcal{C}$ of $\mathcal{A}$ onto the one of $\widetilde{\mathcal{A}}_\phi$. Therefore $\rho \in \mathcal{C} \Rightarrow \widetilde{\mathbf{L}}(T_\phi \rho) = 0$. The new code $U_\phi(\mathcal{C})$ is noiseless depending on the transformation properties of $H'_\mathcal{R}$ under T_ϕ. If $H_\mathcal{R} = T_\phi(H_\mathcal{R})$ (local gauge invariance) it follows that $U_\phi(\mathcal{C})$ is noiseless under $\mathbf{L}_\phi$ if and only if $\mathcal{C}$ is noiseless under $\mathbf{L}_{\phi=0}$ (replica independent case). Let us consider, for example, the qubit case with $N=2$ and $\phi(j) = \phi j$ ($\phi \in \mathbb{R}$) and $H'_\mathcal{R} = \epsilon S^z$. The noiseless state is now the singlet $|\psi_s\rangle = 2^{-1/2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. It is mapped by T_ϕ onto $U_\phi|\psi_s\rangle = 2^{-1/2}(e^{i\phi/2}|\uparrow\downarrow\rangle - e^{-i\phi/2}|\downarrow\uparrow\rangle)$, in particular for $\phi = \pi$, one has $U_\pi|\psi_s\rangle = |\psi_t\rangle$, that is the triplet state becomes the noiseless one. For $\phi \in (0,\pi)$ one has a smooth interpolation from $|\psi_s\rangle$ to $|\psi_t\rangle$. It should be emphasized that even if $T_\phi(H_\mathcal{R}) = H_\mathcal{R}$, generally one has that the many-qubit correction $\delta H_\mathcal{R}$ is not invariant. Nevertheless the Hamiltonian part of $\mathbf{L}$ *does not affect the first-order decoherence rate:* $U_\phi(\mathcal{C})$ *is subdecoherent.*

VI. CONCLUSIONS

In this paper we have studied a model of quantum register $\mathcal{R}$ with N cells made of replicas of a d-dimensional quantum system. The register $\mathcal{R}$ is coupled with the environment, modeled by a thermal bath of harmonic oscillators, through single-cell operators A_i. The latter are step operators over the spectrum of the cell Hamiltonian. The reduced dynamics of $\mathcal{R}$ is studied by a master equation (ME) obtained in the Born-Markov approximation. The ME provides a very natural and powerful tool to discuss, in a unified way, the various aspects of decoherence and dissipation phenomena induced in $\mathcal{R}$ by the bath. The effect of the environment splits into two contributions: a renormalization of the register self-Hamiltonian, that makes the cells effectively interacting, and an irreversible component describing the decay processes. The latter can be cast in canonical Lindblad form by diagonalizing the $N\times N$ matrices $\mathbf{\Gamma}^{(\pm)}$, which contain all information about the effective spatial structure of $\mathcal{R}$ in the given environment state. Three situations that appear to be relevant for quantum encoding have been discussed: (i) all the cells are coupled with the environment in the same way, (ii) different cells feel different environments, (iii) the register can be decomposed in uncorrelated clusters, such that the cells within each cluster satisfy (i). In each of these cases one can show the existence of subspaces $\mathcal{C}$ such that an initial pure preparation $|\psi\rangle \in \mathcal{C}$ has vanishing linear entropy production rate. The states in $\mathcal{C}$ therefore—on a short time scale—maintain quantum coherence: $\mathcal{C}$ can be thought of as a subdecoherent code. The latter is obtained as a simultaneous eigenspace $\mathcal{C}$ of the Lindblad operators L_μ, given by linear functions of the A's associated with the register cells. Depending on the structure of the Lie algebra $\mathcal{L}$ generated by the L_μ's one has to face rather different situations. For A Hermitian $\mathcal{L}$ is Abelian, the Hilbert space splits in a direct sum of the simultaneous eigenspaces $\mathcal{C}$: the ME is exactly solvable. Analytical expressions for decoherence rates can be found in the qubit case. In the non-Hermitian case $\mathcal{L}$ is non-Abelian, the Hilbert space splits according to the $\mathcal{L}$ irreps, $\mathcal{C}$ is the common null space of the L_μ's (singlet sector of $\mathcal{L}$). The latter exists, according to Ref. [5], if the size of the clusters satisfying (i) is large enough. For the qubit case the minimum cluster size required to encode one logical qubit is $N=4$: a register made of M clusters of four qubit each supports a 2^M-dimensional subdecoherent space. If $\mathcal{C}$ is left invariant by the renormalized self-Hamiltonian $H'_\mathcal{R}$ of $\mathcal{R}$ the time evolution of the subdecoherent states is unitary: the code is noiseless. In this case the relevant algebra is $\mathcal{L}'$ generated by the Lindblad operators *plus* $H'_\mathcal{R}$. Furthermore we have shown that there exist cases with nontrivial cell dependence that can be mapped onto (ii) and (iii) by a suitable local gauge transformation. The degree of stability of the resulting codes depends on the covariance properties of the renormalized self-Hamiltonian. When the $\mathbf{\Gamma}^{(\pm)}$'s are not block diagonal one has to resort to numerical calculations. We have integrated the dissipative ME of a qubit register. The results show that for a wide range of bath coherence lengths ξ_c the singlet states (noiseless at $\xi_c = \infty$) are more robust, namely, their entropy increases more slowly on the time scale of decoherence.

The problems related to the practical realizations of the registers satisfying the constraints for the suggested encodings, the preparation as well as the gate manipulations of the code words necessary in the quantum computation applications, are of course open issues that deserve further investigations.

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