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Soliton generation in the nonlinear interaction of two waves

Alexander A. Zabolotskii*
Institute of Automation and Electrometry, Siberian Branch of the Russian Academy of Sciences, 630090 Novosibirsk, Russia
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We consider interaction of a laser field with surface waves propagating in a thin layer having a width much less than that of the laser field. It is assumed that the laser field interacts concurrently with a two-level transition. We show that generation of the surface waves in this scheme may be described by the completely integrable Thirring model. We use this model for study of the evolution of a steplike pulse. It is found that a leading edge is described asymptotically by a sequence of solitons. The modulation instability leading to the formation of these solitons is treated in a hydrodynamic approximation. For this aim we find a solution to the Whitham equations associated with deformation of the one-phase solution of the Thirring model. We show that the results obtained here can be applied to other physical situations, for instance, the study of the three-wave mixing in a bulk medium. [S1050-2947(98)07401-0]

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I. INTRODUCTION

The interest in the theoretical and experimental study of coherent phenomena in thin layers appears due to the development of coherent spectroscopy and potential application of thin films [1,2]. Special attention has been paid to the multifrequency interaction in thin films of the molecular aggregates, surface films, interfaces, and so on. Multiwave mixing processes in layers may yield a rich variety of nonlinear phenomena. Novel effects may arise due to mixing of resonant interactions with energetic transitions and multiwave interactions in nonlinear media. It has been found in the experimental studies that nonlinear susceptibilities of thin films could reach high values. The property of small switching time makes such layers prospective candidates for application in microelectronics. This stimulates the special attention of researchers to nonlinear optical phenomena in the films.

It is known that interaction of a powerful laser field with surfaces leads to the generation of surface waves and surface patterns [3,4]. Investigation of the patterns is important for understanding of nonlinear phenomena in layers. In the present paper we consider interaction of external pumping fields with a thin layer placed on a solid surface. We consider two internal waves counterpropagating in the layer. It is assumed that both these waves have initial amplitudes and phases that are time and space independent. Being chosen initially small, one field may be used for modeling of noise. Interaction of internal fields with an external field leads to energy exchange between waves propagating in a layer. It is known that creation of the packets of nonlinear pulses in multiwave processes has to be expected for sufficiently long and powerful pulses. Interaction of nonlinear waves propagating in a thin layer may lead to formation of robust surface patterns consisting of dense packets of solitons.

In this paper the formation of dense packets of solitons near the leading edge of a long steplike pulse is investigated analytically. The interaction of an external wave with two waves propagating in a thin layer is considered concurrently with one-photon resonant interaction with a two-level transition. Maxwell equations describing three-wave mixing in this scheme and the material equations are reduced to a pair of evolution equations. It will be shown that these equations are an integrable generalization of the Thirring model (TM) in some region of physical parameters.

The Thirring model had been derived first in the theory of spinor fields [5]. The quantum and classical versions of this model had been studied in elementary particles physics and in the theory of ferromagnetism. In nonlinear optics the completely integrable TM has been applied for study of polarization effects in the Bragg medium [6] and ''gap'' solitons [7].

Dynamics of the soliton solution of the TM now is well understood mainly due to application of the modern analytical tool — the inverse scattering transform (IST) [8]. The integrable models attract the special attention of theorists because their investigation provides the most detailed analytical information about the nonlinear stage of field evolution. Many physically important systems in one space and one dimension have a Lax pair and are integrable using the IST. The Lax representation for the TM was found by Mikhailov [9]. The classical version of the IST had been developed by Mikhailov and Kuznetsov [10]. In the present paper the physical conditions allowing one to extend the application of the integrable TM for study of nonlinear optical processes are found.

Frequently experimental situations involve a high density of solitons or another nonlinear pulse. These packets of solitons in a system, having small losses, may be approximated by modulated quasiperiodic waves. The interaction of waves may involve a formation of nonlinear robust modes, whose study is interesting for both theoretical physics and application. These modes may be generated due to developing of a modulation instability in a layer. For nonlinear optical systems numerical results show that periodic waves arise, for instance, when the characteristic length of the instability is close to the length of the nonlinear medium and boundary conditions are close to the periodic ones. Dense packets of solitons may arise due to the development of the modulation instability during propagation of a long steplike pulse. For instance, it is proved numerically and analytically for the

*Electronic address: zabolotskii@iae.nsk.su

Korteveg–de Vries equation that the leading edge of the steplike pulse transforms to a dense packet of nonlinear pulses [8]. These pulses tend to asymptotic solitons under some initial conditions.

Treating the nonlinear stage of evolution of the dense packets of pulses, one must operate with a large number of degrees of freedom. Such treatment is possible, as a rule, only for completely integrable models, and even for them the study faces tremendous analytical problems. On the other hand some experimental results of the generation of dense packets of pulses may be modeled using modulated periodic waves [8]. These observations motivate one to use the Whitham approach for studying the behavior of a dense packet of pulses. This approach consists of two steps. The first step is a derivation of an exact one- (two-) phase solution of the original equations with the periodic boundary conditions. Then it is assumed that part of the spectral data associated with the periodic wave depends on space and time variables. This dependence is slow in comparison with that of a single oscillation constituting a packet. Averaging over the period of rapid nonlinear oscillations yields the evolution equations for the parameters of the periodic wave. These equations are the hydrodynamic Whitham equations [8,11]. As shown in [12] these equations can be effectively obtained using the IST directly in a diagonal form.

We use the Whitham approach here for analysis of modulation instability arising in propagation of long steplike pulse in a layer. For this aim we construct the one-phase solution of the TM. We use here the IST version developed for the systems of evolution equations by Marchenko *et al.* [13–15]. This method allows one to construct, in common, the exact N-phase quasiperiodic solutions to the model under consideration and was applied to the TM in Refs. [16,17]. The authors of these works had received expressions for the N-phase solution for the TM in a form that is not efficient for the purposes of the present paper. Therefore we use here another form of the one-phase solution. Then we use this solution as the robust nonlinear mode for modeling the nonlinear stage of evolution of modulation instability. We suppose that a change of the parameters of the nonlinear mode during all stages of evolution obeys the Whitham equations, i.e., the quasiclassical (hydrodynamics) approximation is valid. The dynamics of nonlinear waves is described by the evolution of a few wave parameters obeying the Whitham equations. For physical applications, the situation when the highest soliton arises at the leading edge attracts the particular interest of researchers. We find here the physical conditions leading to this regime. It is shown that under these conditions any change of parameters of the wave is described by the Whitham equations solutions. These solutions describe the deformation of weak periodic modulation of the plane wave to a sequence of asymptotic quasiisolated solitons. The isolated solitons of the TM are known [10]. What is new for the TM is that the approach used here allows one to describe the transform of the steplike pulse in a dense packet of solitons due to evolution of the modulation instability. The approach used here allows one to study such a configuration. The tremendous problem of the study of a system having a large number of degrees of freedom is reduced to the analysis of a few nonlinear evolution equations.

The remainder of this paper is organized as follows. In Sec. II Maxwell and Bloch equations describing three-wave mixing and resonant one-frequency interaction in a layer are reduced to the generalization of the Thirring model. Section III is devoted to a derivation of the one-phase solution to the Thirring model in a form convenient for our purposes. In Sec. IV the Whitham equations are presented and the similarity solution is found. Section V contains the conclusions. Another physical situation leading to the same TM model is described in Sec. V as well. In Appendix A conventional IST techniques are used for an analysis of the asymptotics of the steplike pulse. The time evolution of spectral data is found. In Appendix B we prove that under some initial conditions the solitonic part of the spectrum gives rise to the main contribution to asymptotics. It is found also that the leading edge of the generated train of pulses is associated with the soliton, the parameters of which are fixed by the initial pulse parameters.

II. BASIC EQUATIONS

Let us consider a plane thin layer having width l, which is much less than the length of the light wave λ. In the coordinate system used here the plane position is $z=0$. The incident external wave transmits from the medium $(z>0)$, whose characteristics are labeled by the subindex a and interacts with the layer $(z=0)$, whose characteristics are labeled by the subindex b. For instance, the dielectric susceptibility constants are ϵ_a and ϵ_b, respectively. We consider TE waves only. Let the incident field be

$$E_y(t,x,z)=E_0(t,x,z)\exp[i(k_x^a x+k_z^a z-\omega t)],$$

where $E_0(t,x,z)$ is the amplitude and k_x^a, k_z^a are the projections of the wave vector $\vec{k}^a$, $|\vec{k}^a|=\omega\sqrt{\epsilon_a}/c$. For the y components of reflected and transmitted waves we have

$$(E_r)_y(t,x,z)=E_r(t,x,z)\exp[i(k_x^a x-k_z^a z-\omega t)], \quad (2.1)$$

$$(E_{tr})_y(t,x,z)=E_{tr}(t,x,z)\exp[i(k_x^a x+k_z^a z-\omega t)], \quad (2.2)$$

where $E_r(t,x,z)$, $E_{tr}(t,x,z)$ are the slow amplitudes of the reflected and transmitted waves, respectively. $|\vec{k}^b|=\omega\sqrt{\epsilon_b}/c$, $k_x^a=k_x^b$.

The Maxwell equations for a thin layer are reduced to the following system of boundary conditions:

$$E_y(t,x,+0)=E_y(t,x,-0),$$
$$H_z(t,x,+0)=H_z(t,x,-0), \quad (2.3)$$
$$H_x(t,x,+0)-H_x(t,x,-0)=\frac{4\pi}{c}\partial_t p_y(x,t),$$

where $p=p_y(x,t)$ is a surface density of polarization.

From conditions (2.3) we find the following relations between the slow changing field amplitudes and polarization:

$$E_{tr}=\frac{2A}{A+B}E_0+i\frac{4\pi\omega}{c(A+B)}p,$$
$$E_r=\frac{A-B}{A+B}E_0+i\frac{4\pi\omega}{c(A+B)}p; \tag{2.4}$$

here $A=\sqrt{\epsilon_a}\cos\theta_a$, $B=\sqrt{\epsilon_b}\cos\theta_b$. θ_a and θ_b are the angles between the vector orthogonal to the layer surface and the wave vectors of the reflected and transmitted field, respectively. The contribution to the polarization p arises due to the resonant interaction of the field with energetic transitions of a layer, e.g., interaction with molecular transitions in aggregates, with impurity atoms, and so on. Multiwave nonlinear processes also can make significant contributions to polarization if corresponding nonlinear susceptibilities are sufficiently large. Denoting by p_r and p_n the terms describing contributions of the first and the second mechanisms, respectively, we have $p=p_r+p_n$.

The macroscopic field may differ from the local field due to the contribution of induced atomic polarizations. This contribution is described by the Lorentz field. Let us estimate the contribution of the Lorentz field to the polarization p_r. We assume that the main contribution to p_r is induced by coherent excitation of a resonant mode of the medium by the external field. The contribution of the Lorentz field to the surface polarization p_L is described by the following term: $p_L=\zeta_L p_r$ [18]. The real parameter ζ_L can be estimated for many media as $\zeta_L=2\zeta_0/(3lk)$, where ζ_0 is a scalar having order of unity, k is the wave vector, and l is the width of the film. To find the contribution of the Lorentz field for a thin layer one has to calculate the sum

$$\sum_j \frac{3(\vec{p}_j\vec{r}_j)\vec{r}_j-\vec{p}_j r_j^2}{r_j^5}=\sum_j \frac{2p_{jy}y_j^2}{r_j^5}\approx C_0\frac{p_r}{r_0}=\zeta_L p_r, \tag{2.5}$$

where r_0 is a mean distance between atoms, C_0 is a real constant of the order of 1. The relative contribution of nonresonant terms to the Lorentz field is an order of $|d_{12}|^2/(\hbar\omega r_0^3)$ and can be neglected here.

A two-frequency interaction with a two-level transition also may give a contribution to the polarization p_r, if nonlinear susceptibility of the second order is sufficiently large. This contribution to polarization arises at the frequencies $\omega\pm\omega_2$. We do not consider this contribution here, but the obtained results can be easily generalized.

We find the contribution of the resonant interaction to surface polarization p_r using the Bloch equations of one-photon interaction of a light with a two-level transition:

$$\partial_t Q+\gamma_2 Q+i\nu_0 Q=-\frac{i|d_{12}|^2}{\hbar}EN,$$
$$\partial_t N+\gamma_1(N-N_0)=\frac{i}{2\hbar}(Q^*E-QE^*); \tag{2.6}$$

here d_{12} is the dipole momentum of the transition. N is the difference between level populations, Q is then the off-diagonal part of the density matrix, N_0 is the density of resonant atoms, and $\gamma_{1,2}$ are the relaxation constants. The Bloch equations (2.6) may describe the interaction with either molecular transition or impurity atoms implemented into a thin film. In the present paper we assume that the time scale of the change of the amplitudes of fields is much more than $\gamma_{1,2}^{-1}$. Then Eq. (2.6) can be easily solved:

$$Q=i\frac{N_0|d_{12}|^2(1-i\delta_0)E}{\hbar\gamma_2[1+\delta_0^2+(|d_{12}||E|\hbar^{-1})^2(\gamma_1\gamma_2)^{-1}]}, \tag{2.7}$$

where $\delta_0=\nu_0/\gamma_2$. In addition we will consider the contribution of a simultaneous three-wave interaction to the surface polarization p_n. Let us suppose that the field E interacts with two waves propagating within the layer. In practice, the second-order nonlinearity is the main nonlinearity at the surface due to the violation of the reflection symmetry contrary to a bulk crystal [2]. Thus we may treat the three-wave interaction as the main nonlinear process in a thin layer or interface. Let the external field $E(z,t)$ have an amplitude and phase that do not change during nonlinear interaction (see below). Two surface fields (or polariton waves) are assumed to be generated by some experimental scheme.

Let us introduce the two fields describing surface polariton waves propagating along the layer:

$$E_1(t,x,y)=V_1(t,x)\exp[i(p_x^b x-p_y^b y-\omega_1 t)],$$
$$E_2(t,x,z)=V_2(t,x)\exp[i(q_x^b x+q_y^b y-\omega_2 t)]. \tag{2.8}$$

The resonance conditions are the following:

$$\vec{k}=\vec{q}+\vec{p},\quad \omega=\omega_1\pm\omega_2+\nu_0, \tag{2.9}$$

where ν_0 is a frequency detuning. We assume, for simplicity, that $p_y^b=q_y^b=0$, $p_x^b=q_x^b=q$. The z component of the wave vector of the external field $E(z,t)$ is not included in the resonance conditions (2.9). Indeed, conditions (2.9) are derived under approximations of slow changing amplitudes of the fields and after averaging over the fast oscillations. We assume that the width of the layer is described by the delta function. Then averaging along the z axis leads to the absence of corresponding components of the wave vector of the external field $E(z,t)$ in conditions (2.9). Note that in experimental physics special optical devices are used for effective coupling of the external fields with fields generated in a thin layer [2].

The Maxwell equations for slow changing amplitudes describing the three-wave mixing are the following:

$$\left(-\partial_x+\frac{n(\omega_2)}{c}\partial_t\right)V_1=i\frac{2\pi\omega_2^2\chi^{(2)}(\omega_2)}{k_2^2c^2}E_{tr}V_2^*\exp(i\nu_0 t),$$
$$\left(\partial_x+\frac{n(\omega_1)}{c}\partial_t\right)V_2=i\frac{2\pi\omega_1^2\chi^{(2)}(\omega_1)}{k_1^2c^2}E_{tr}V_1^*\exp(i\nu_0 t); \tag{2.10}$$

here $\chi^{(2)}(\omega_{1,2})$ is the nonlinear susceptibility of the second order, $n(\omega_{1,2})$ is the dielectric constant of the medium. The resonance conditions are fulfilled, for instance, if two fields propagating in opposite directions and $\omega_1=\omega_2$, $|k_1|=|k_2|$.

We rewrite the first equation in system (2.4) in the following form:

$$E=E_{tr}=\frac{2A}{A+B}E_0+\left[\frac{C_0}{r_0}+i\frac{4\pi\omega}{c(A+B)}\right]Q+i\frac{4\pi\omega}{c(A+B)}\chi_2 V_1 V_2. \tag{2.11}$$

The system of Eqs. (2.7), (2.10), (2.11) describes the nonlinear mixing processes in a thin film with a resonant two-level transition. Let us rewrite this system in a form that can be considered as a nonintegrable generalization of the Thirring model:

$$\left[\partial_t-\frac{c}{n(\omega_1)}\partial_x\right]V_1=\frac{C_1J_1}{J_1-(\alpha+\delta_0\beta)-i(\delta_0\alpha-\beta)}\times\left(i\chi_2V_2^*E_0\frac{2A}{A+B}-\chi_2^2\frac{4\pi\omega}{A+B}V_1|V_2|^2\right),$$
$$\left[\partial_t+\frac{c}{n(\omega_2)}\partial_x\right]V_2=\frac{C_2J_2}{J_2-(\alpha+\delta_0\beta)-i(\delta_0\alpha-\beta)}\times\left(i\chi_2V_1^*E_0\frac{2A}{A+B}-\chi_2^2\frac{4\pi\omega}{A+B}V_2|V_1|^2\right), \tag{2.12}$$

where

$$J_1=1+\delta_0^2+\frac{|d_{12}|\hbar^{-2}}{\gamma_1\gamma_2}\frac{|D_1V_1|^2}{|\chi_2V_2|^2},$$

$$J_1=1+\delta_0^2+\frac{|d_{12}|\hbar^{-2}}{\gamma_1\gamma_2}\frac{|D_2V_2|^2}{|\chi_2V_1|^2},$$

$$D_i=\left[\partial_t+(-1)^i\frac{c}{n(\omega_i)}\partial_x\right],\quad i=1,2,$$

$$\alpha=\frac{C_0N_0|d_{12}|^2}{r_0\hbar\gamma_2},\quad \beta=\frac{4\pi\omega N_0|d_{12}|^2}{c(A+B)\hbar\gamma_2},\quad C_i=\frac{2\pi\omega_i^2}{k_icn_i}.$$

System (2.12) seems to be rather complicated for analytical study, therefore we use the additional simplifications. For the intensity of field E having values far from that of saturation relation (2.7) becomes linear in E and $J_i\approx 1+\delta_0^2$. We assume that intensities of the fields $V_{1,2}$ are much less than that of the field E. In this case the dependence of the field E on x,t is fixed and does not change due to interaction. This situation arises when the second-order nonlinearity yields a small contribution to polarization in comparison to nonlinear resonance interaction (see the last section). If the intensity of the field E is far from saturated the former system can be reduced to the following generally nonintegrable generalization of the Thirring model

$$\left[\partial_t-\frac{c}{n(\omega_1)}\partial_x\right]V_1=G_0\left(iC_1E_0\exp(i\phi_1-i\phi_0)V_2^*-\frac{C_1|\chi_2|4\pi\omega}{2A}V_1|V_2|^2\exp(i2\phi_1-i\phi_0)\right),$$
$$\left[\partial_t+\frac{c}{n(\omega_2)}\partial_x\right]V_2=G_0\left(iC_2E_0\exp(i\phi_1-i\phi_0)V_1^*-\frac{C_2|\chi_2|4\pi\omega}{2A}V_2|V_1|^2\exp(2i\phi_1-i\phi_0)\right), \tag{2.13}$$

where

$$d_0=[(1+\delta_0^2-\beta-\alpha\delta_0)^2+(\beta-\delta_0\alpha)^2]^{1/2},$$

$$\phi_0=\arg[(1+\delta_0^2-\beta-\alpha\delta_0)-i(\beta-\delta_0\alpha)],$$

$$G_0=\frac{|\chi_2|F}{d_0}(1+\delta_0^2),\quad \chi_2=|\chi_2|\exp(i\phi_1),\quad F=\frac{2A}{A+B}.$$

We rewrite the last system in the following form:

$$\partial_Y V=imU+g_1V|U|^2\exp(2i\phi_1-i\phi_0),$$
$$\partial_X U=im^*V+g_2U|V|^2\exp(-2i\phi_1+i\phi_0), \tag{2.14}$$

where

$$X=-G_0(x/v),\quad Y=G_0(t-2x/v),\quad v=\frac{c}{n(\omega_1)},$$

$$m=\sqrt{C_1C_2}E_0\ \exp[i(\phi_1-\phi_0)],$$

$$U=V_2^*\left(\frac{4\pi\omega C_1|\chi_2|}{2A}\right)^{1/2},\quad V=V_1\left(\frac{4\pi\omega C_2|\chi_2|}{2A}\right)^{1/2},$$

$$g_{1,2}=G_0\frac{C_{1,2}|\chi_2|4\pi\omega}{2A}.$$

The above equations arise for the sign ''+'' in the right-hand side (rhs) of the frequency resonance condition (2.9). For the sign ''−'' the analogous system appears up to the change of one field amplitude to a complex conjugated one. System (2.14) is valid for large parameter $u=|m|^2/|\lambda_0|^2\gg 1$, where $|\lambda_0^2|$ is the maximum of amplitudes of the fields V and U.

The complexity of the coefficients in the rhs of Eqs. (2.14) and the dependence of m on the variables makes this system nonintegrable in the general case. But for some region of the physical parameters system (2.14) can be reduced to a new generalization of the integrable version of the Thirring model. The integrability conditions impose the restrictions on the coefficients and variable dependence of m. The exact integrability provides the following condition

$\mathrm{Re}\,\exp(i2\phi_2-i\phi_0)/[1+\delta_0^2-(\alpha+\delta_0\beta)-i(\delta_0\alpha-\beta)]=0$. However, in a real physical situation this exact relation has to be replaced by the approximate one

$$\left|\mathrm{Re}\frac{\exp(i2\phi_2-i\phi_0)}{1+\delta_0^2-(\alpha+\delta_0\beta)-i(\delta_0\alpha-\beta)}\right| \ll\left|\mathrm{Im}\frac{\exp(i2\phi_2-i\phi_0)}{1+\delta_0^2-(\alpha+\delta_0\beta)-i(\delta_0\alpha-\beta)}\right|. \quad (2.15)$$

The condition (2.15) can be easily fulfilled in experimental optics. For the real χ_2 and m the nonequality (2.15) means that $|1+\delta_0^2-\alpha-\beta\delta_0|\ll|\delta_0\alpha-\beta|$. The later nonequality may be fulfilled for $\beta\gg1$, $\delta_0=0$ and $\alpha\ll\beta$. Let us estimate the required density of surface atoms. For $|d_{12}|^2/\hbar\omega\approx3\times10^{-25}$ cm^3 we find that the density of impurity atoms $N_0\gg10^2\gamma_2(\mathrm{cm}^{-2})$. For the relaxation constant $\gamma_2\sim10^{11}c^{-1}$ the atomic density may be $\sim10^{14}$ cm^{-2}.

We will treat the integrable version of the model (2.14), where the ''mass'' m depends on variables, in common. The Lax representation for system (2.14) has the following form:

$$\partial_X\psi=\begin{pmatrix} -i\zeta^2+i\frac{g_1}{2}|V|^2 & \sqrt{g_1+g_2}\zeta V \\ -\sqrt{g_1+g_2}\zeta V^* & i\zeta^2-i\frac{g_1}{2}|V|^2 \end{pmatrix}\psi, \quad (2.16)$$

$$\partial_Y\psi=\begin{pmatrix} -i\frac{|m|^2}{4\zeta^2}+i\frac{g_2}{2}|U|^2 & \sqrt{g_1+g_2}\frac{m^*}{2\zeta}U \\ -\sqrt{g_1+g_2}\frac{m}{2\zeta}U^* & i\frac{|m|^2}{4\zeta^2}-i\frac{g_2}{2}|U|^2 \end{pmatrix}\psi, \quad (2.17)$$

where ζ is the spectral parameter. Ψ is the vector function depending on X, Y, and ζ. Analysis shows that the system (2.16), (2.17) is integrable by means of the IST if the condition (2.15) is fulfilled and ''mass'' m admits the following decomposition:

$$m(Y,X)=m_1m_2,\quad \partial_Xm_1=0,\quad \partial_Ym_2=0,\quad \mathrm{Im}(m_2)=0. \quad (2.18)$$

The proof of this statement is direct. The dependence of m on X does not alter the Lax pair, contrary to the dependence on Y. But the latter dependence can be avoided if the spectral parameter ζ has the form $\zeta=\zeta_0m_2$, where ζ_0 is a constant. The condition $\mathrm{Im}(m_2)=0$ is verified directly from the above Lax pair. The dependence of the ''mass'' on ''time'' Y yields the dependence of a spectral parameter on ''time.'' If m has the form $m=m_1\exp[2i\nu_1(Y)X+2i\nu(X)Y]$, where m_1 satisfies the above integrability conditions, then the system (2.14) remains integrable by the IST for the arbitrary functions $\nu(X)$ and $\nu_1(Y)$ (see below).

So, we have shown that the integrable Thirring model may be generalized to the case of a complex variable ''mass'' in comparison to the case of the constant and real mass known in the literature. Note finally that for $g_1+g_2=0$ system (2.14) is reduced to a linear one by a simple gauge transform.

III. NONLINEAR EVOLUTION OF INITIAL PLANE WAVE

In this paper we restrict our consideration to a case of $m=m_0\exp(2i\nu_1X+2i\nu Y)$. Here m_0 is a constant. After the transformation

$$U\to\frac{\sqrt{2}U}{\sqrt{|g_1+g_2|}}e^{[i(g_1-g_2)/2\int_0^Y|U|^2dY+i(\nu_1X+\nu Y)]},$$

$$V\to\frac{\sqrt{2}V}{\sqrt{|g_1+g_2|}}e^{i[(g_1-g_2)/2]\int_0^Y|U|^2dY-i(\nu_1X+\nu Y)}$$

system (2.14) changes to the following:

$$\partial_XV=im_0U+i\epsilon V|U|^2-i\nu_1V,\qquad \partial_YU=im_0V+i\epsilon U|V|^2-i\nu U; \quad (3.1)$$

here $\epsilon=\pm1$ is the sign of $g_{1,2}$. We assume that $\mathrm{sgn}g_1=\mathrm{sgn}g_2$. The Lax pair (2.16), (2.17) can be rewritten in the following form:

$$\partial_X\Phi=\begin{pmatrix} \frac{i}{2}(\lambda^2-\epsilon|V|^2+\nu) & \lambda V \\ -\epsilon\lambda V^* & -\frac{i}{2}(\lambda^2-\epsilon|V|^2+\nu) \end{pmatrix}\Phi, \quad (3.2)$$

$$\partial_Y\Phi=\begin{pmatrix} \frac{i}{2}\left(\frac{m_0^2}{\lambda^2}-\epsilon|U|^2+\nu_1\right) & \frac{m_0U}{\lambda} \\ -\frac{\epsilon m_0U^*}{\lambda} & -\frac{i}{2}\left(\frac{m_0^2}{\lambda^2}-\epsilon|U|^2+\nu_1\right) \end{pmatrix}\Phi. \quad (3.3)$$

Here Φ is the two-component function; λ is the spectral parameter. Additionally, ν and ν_1 are the arbitrary functions $\nu=\nu(X)$, $\nu_1=\nu_1(Y)$.

System (3.1) has the trivial plane-wave solution. To investigate the stability of this solution in the linear approximation we transform the TM (3.1) into one equation of the second order. Using the equation $\partial_Y|V|^2=-\partial_X|U|^2$ following from Eq. (3.1), we obtain

$$\partial_X\partial_YW=-m_0^2W-i2\epsilon|W|^2\partial_YW, \quad (3.4)$$

here $W=U(X,Y)\exp[i\epsilon\int|V|^2(X,Y)dX]$. For simplicity we set $\nu_1=\nu=0$.

The linear function

$$W=W_0\exp[i(hX)] \quad (3.5)$$

is the solution to Eq. (3.4), where $2|W_0|^2\epsilon h=m_0^2;h$ and W_0 are real constants.

The plane-wave solution stability of can be easily investigated in the linear approximation. Let A_1 and A_2 be small-amplitude deviations. After substitution of the perturbed solution

$$W=W_0\exp[i(hX)]\{1+A_1\exp[i(\Omega Y+KX)]+A_2\exp[-i(\Omega Y+KX)]\} \tag{3.6}$$

in Eq. (3.4) and linearization with respect to $A_{1,2}$ we obtain the linear homogeneous algebraic system for perturbations. The compatibility condition yields the dispersion relation for modulated waves:

$$\Omega=\pm\sqrt{\frac{m_0^4}{K^2-m_0^2/h^2}}. \tag{3.7}$$

Equation (3.7) shows that for a sufficiently long wavelength of perturbation, the frequency Ω has an imaginary part. This means that the corresponding perturbation exponentially grows, i.e., the modulation instability of solution (3.4) takes place.

Linear analysis is restricted to small perturbations. To study the nonlinear stage of instability one must apply the methods operating with nonlinear modes. We use here the one-phase solution of the TM as a robust nonlinear mode for the analysis of instability. This modulated robust mode may describe a plane-wave transform to a sequence of solitons in the nonlinear stage of instability. Important information can be obtained by studying the spectrum of the spectral problem (3.2) associated with an initial steplike pulse of the field V. For an infinite length of the steplike pulse the spectral problem can be easily solved. We omit the details of the solution of the spectral problem. We will mention only the main steps. First, one can use a pulse having a triangular form. Then, one finds a set of spectral data in the same way as for the Zakharov-Shabat spectral problem [19]. The associated spectrum consists of a set of poles lying on some finite intervals and on the real axis. If the length of the pulse tends to infinity the number of poles tends to infinity and the distance between each pair of neighboring poles vanishes. In this limit one gets a spectrum consisting of the real axis and the continuous intervals.

Let the steplike pulse have a height V_0. The positions of finite intervals are determined by roots of the polynomial K:

$$K^2=\frac{1}{4}[\lambda^4+2\lambda^2(\epsilon|V_0|^2-\nu)+(\epsilon|V_0|^2+\nu)^2].$$

The equation $K=0$ has four roots. There are two different cases of the roots, which depend on the sign of $\epsilon\ \nu$:

(I) $\mathrm{sgn}(\nu\epsilon)=-1$. Then

$$\eta_{1,3}=(\lambda^2)_{1,3}=(\nu-\epsilon|V_0|^2)\pm 2\sqrt{|V_0|^2|\nu|}.$$

(II) $\mathrm{sgn}(\nu\epsilon)=1$. Then

$$\eta_{1,3}=(\lambda^2)_{1,3}=(\nu-\epsilon|V_0|^2)\pm 2i\sqrt{|V_0|^2|\nu|}.$$

It is known that isolated poles lying in quadrants I and III of the complex λ plane are associated with the solitons of the TM. For the present consideration it is more convenient to use the λ^2 plane (Fig. 1). The spectrum associated with the initial steplike pulse consists of the real axis and the finite continuous interval, which is orthogonal to the real axis. The interval lying in a solitonic region (upper half of a plane, Fig. 1) of a complex plane is associated with a solitonic asymptotic. For physical application the most interesting regime arises when powerful solitons are generated near the leading edge of the steplike pulse. In Appendixes A and B it is shown that such a regime may arise for the TM in an infinite medium. In Appendix B it is shown that the leading front of asymptotics is described by the soliton solution associated with the spectral point η_0. It is shown that for large $|m_0^2/\lambda^2|$ a contribution of the continuous spectrum to the asymptotic solution may be neglected. Therefore, for the description of the evolution of the steplike pulse we may restrict ourselves to the consideration of the solitonic part of the spectrum. As shown in Appendix B, there are regions of the initial parameters of the plane wave satisfying this assumption.

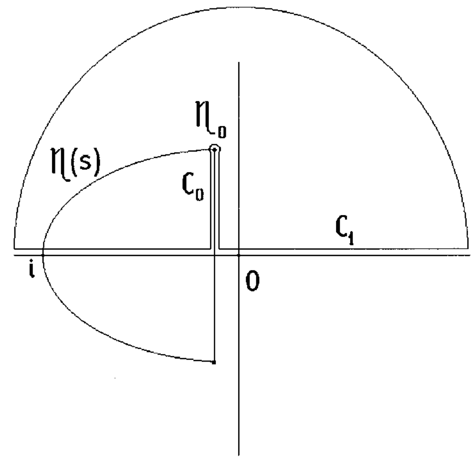

FIG. 1. The complex $\eta=\lambda^2$ plane. The trajectories of $\eta_{2,4}$ as the functions of the similarity variables $s=\chi/\tau$ are shown. The upper (lower) curve corresponds to η_2 (η_4). Transformation from the point i to η_0 corresponds to the transformation from a plane wave to a sequence of solitons. Units are chosen to be arbitrary.

Thus, the long-time modulation is associated with the nonlinear stage of evolution, i.e., generation of asymptotic solitons. The developing of the modulation instability corresponds to the transform of long harmonic waves into a set of asymptotic solitons. Application of the Whitham approach allows one to describe the intermediate region between these asymptotics by using the modulated periodic nonlinear mode.

Let us now find the simplest nonlinear mode — the one-phase solution of the TM. Following the approach developed in Ref. [13], we introduce the following quadratic eigenfunctions:

$$f=(i/2)(\phi_1\psi_2+\phi_2\psi_1),\quad g=\phi_1\psi_1,\quad h=\phi_2\psi_2, \tag{3.8}$$

where $\phi_{1,2}\psi_{1,2}$ are the different solutions of the system (3.2), (3.3).

These functions satisfy the following system:

$$\partial_X f = i(Bh - Cg), \quad \partial_Y f = i(Gh - Hg),$$
$$\partial_X g = 2iBf + 2Ag, \quad \partial_Y g = 2iGf + 2Fg, \tag{3.9}$$
$$\partial_X h = -2iCf - 2Ah, \quad \partial_Y h = -2iHf - 2Fh;$$

here

$$A = \frac{i}{2}(\lambda^2 - \epsilon|V|^2 + \nu), \quad B = \lambda V, \quad C = -\epsilon\lambda V^*,$$

$$F = \frac{i}{2}\left(\frac{m_0^2}{\lambda^2} - \epsilon|U|^2 + \nu_1\right), \quad G = \frac{m_0 U}{\lambda}, \quad H = -\frac{\epsilon m_0 U^*}{\lambda}.$$

It can be easily checked from system (3.9) that the value $P(\lambda) = f^2 - gh$ is independent of both the variables, i.e., $\partial_Y P(\lambda) = 0$, $\partial_X P(\lambda) = 0$. The periodic solution is determined by the dependence of the polynomial P on the spectral parameter λ. For instance, the one-phase solution is fixed by the following polynomial:

$$f^2 - gh = P(\lambda) = \prod_{k=1}^{4} (\lambda^2 - \lambda_k^2) = \sum_{j=0}^{4} P_j \lambda_j^2. \tag{3.10}$$

Here λ_k are the roots of the polynomial. We assume that a pair of the spectral data $\lambda_{1,3}^2$ is fixed by asymptotics. These roots are assumed to be independent of the variables. This choice of P is dictated by the fact that the corresponding solution (see below) must coincide with the plane-wave solution (as $x \to -\infty$) and asymptotically (as $x \to \infty$) coincides with the ‘‘top soliton.’’ On the other hand this form of $P(\lambda)$ includes ‘‘free’’ roots $\eta_2 = \lambda_2^2, \eta_4 = \lambda_4^2$. The variable dependence of these roots describes the deformation of a plane wave to a train of solitons.

It can be shown that the quadratic functions, satisfying the system (3.2), (3.3), have for the one-phase case the form

$$f = \sum_{k=0}^{2} f_k \lambda^{2k}, \quad g = \lambda(g_0 + g_1\lambda^2), \quad h = \lambda(h_0 + h_1\lambda^2). \tag{3.11}$$

From system (3.9), one can find the following relations:

$$\partial_X f_0 = \partial_Y f_2 = 0, \quad g_1 = -2V,$$
$$g_0 = -2Uf_0, \quad h_1 = 2\epsilon V^* f_2, \quad h_0 = 2\epsilon U^* f_0. \tag{3.12}$$

Substituting these relations in Eq. (3.10) and using decomposition in degrees of λ, we find for the zeroth and the fourth degrees of λ:

$$f_0 = \sqrt{P_0}, \quad f_2 = \sqrt{P_4} = 1.$$

We choose P_4 equal to unity without loss of generality. We introduce the ‘‘auxiliary function’’

$$\mu(\lambda, x, y) = -\sqrt{P_0}\frac{U}{m_0 V}. \tag{3.13}$$

To recover Eqs. (3.9), we note the following: if $\{f_j, g_j, h_j\}$ is a solution of Eq. (3.9) then $\{f_j^*, -\epsilon h_j^*, -\epsilon g_j^*\}$ is also a solution of Eq. (3.9). Therefore if the relations $f_j = f_j^*, g_j^* = -\epsilon h_j^*$ hold for the initial conditions, the same relations hold for the solution. The functions g and h have the following representations:

$$g = -2\lambda V(\lambda^2 - \mu), \quad h = 2\epsilon\lambda V^*(\lambda^2 - \mu^*). \tag{3.14}$$

Substituting Eq. (3.14) in Eq. (3.9) and using Eq. (3.14) and $f^2(\lambda^2 = \mu) = P(\mu)$ we have

$$\partial_X g(\lambda^2 = \mu) = 2i\lambda V f(\mu = \lambda^2),$$

$$\partial_Y g(\lambda^2 = \mu) = -2i\frac{m_0^2}{\sqrt{P_0}}\lambda V f(\mu = \lambda^2).$$

The latter pair of the equations yields

$$\partial_\theta \mu = i[P(\mu)]^{1/2}, \quad \theta = X - m_0^2\frac{Y}{\sqrt{P_0}}. \tag{3.15}$$

Using Eqs. (3.10), (3.12) we find that V is a function of μ. To find U it is convenient to use the symmetry property of Eqs. (3.2) and (3.3). Equations (3.2) and (3.3) remain unchanged after the transformation:

$$\lambda \leftrightarrow \pm 1/\lambda, \quad U \leftrightarrow V, \quad X \leftrightarrow Y, \quad \nu \leftrightarrow \nu_1. \tag{3.16}$$

We introduce an inverse auxiliary function $\rho = 1/\mu$. Repeating the above steps and using Eq. (3.16), we obtain

$$\partial_\zeta \rho = i[Q(\rho)]^{1/2}, \quad \zeta = \frac{m_0 X}{\sqrt{P_0}} - Y. \tag{3.17}$$

Here $Q(\rho)$ is the polynomial of the fourth order having the roots $\rho_j = 1/\lambda_j^2$, $j = 1-4$.

Using Eqs. (3.1) and (3.13) one can express solutions to the TM in terms of $\mu(\theta)$. These relations are the following:

$$\partial_Y(\ln|V|^2) = i(\mu - \mu^*)m_0^2/f_0, \tag{3.18}$$

$$\partial_X(\ln|U|^2) = if_0(\rho - \rho^*); \tag{3.19}$$

$$\partial_Y \ln V = i\mu m_0^2/f_0 - i\epsilon|U|^2 + \nu, \tag{3.20}$$

$$\partial_X \ln U = i\rho f_0 - i\epsilon|V|^2 + \nu_1. \tag{3.21}$$

The condition of the reality of f_0 following from Eq. (3.18) requires that the roots λ_k^2 must be in complex conjugated pairs or be either pure imaginary or pure real. Integration of Eqs. (3.15), (3.20), (3.21) yields a one-phase solution. For some physical applications it is convenient to express the solution in terms of the parameters, which can be related with maximal and minimal intensities of the fields. For this aim we solve the following system of algebraic relations:

$$P_3 = 2f_1 - I(\theta), \quad P_1 = 2f_1 f_2 - I(\theta)\mu\mu^*,$$
$$P_2 = f_1^2 + 2f_2 - 2I(\theta)(\mu + \mu^*), \tag{3.22}$$

where $I(\theta) = 4\epsilon|V|^2$. Equations (3.22) are easily derived using the integral of motion $P = f^2 - gh$ as a polynomial in λ

and using the relations (3.11). Let us express μ in terms of $I(\theta)$. Trivial but tedious calculation yields

$$\mu=-\sqrt{S_0}/(8I)+I/8-P_3/4-i\sqrt{-S(I)}/(8I).$$

Here,

$$S(I)=[4P_2\pm 8\sqrt{P_0}-(I-P_3)^2]^2-64I[P_1\pm\sqrt{P_0}(I-P_3)^2],$$

$S_0=I_1I_2I_3I_4$, $I=I(\theta)$. Polynomial $S(I)$ has four roots:

$$I_1=-(\lambda_1^2+\lambda_2^2-\lambda_3^2\mp\lambda_4^2)^2,\quad I_2=-(\lambda_1^2-\lambda_2^2-\lambda_3^2\pm\lambda_4^2)^2, \tag{3.23}$$

$$I_3=-(\lambda_1^2-\lambda_2^2+\lambda_3^2\mp\lambda_4^2)^2,\quad I_4=-(\lambda_1^2+\lambda_2^2+\lambda_3^2\pm\lambda_4^2)^2. \tag{3.24}$$

Here λ_k^2 are the roots of the polynomial $P(\lambda)$ (3.10). From Eqs. (3.15) and (3.18) it can be easily found that the function $I(\theta)$ obeys the equation

$$\frac{\partial I}{\partial\theta}=\frac{\sqrt{-S(I)}}{4}=\frac{\sqrt{-(I-I_1)(I-I_2)(I-I_3)(I-I_4)}}{4}. \tag{3.25}$$

Integrate Eq. (3.25) for the real I_k. Let $I_1>I_2>I_3>I_4$. A common solution to Eq. (3.25) for $I_1\geqslant I(\theta)>I_2$ is the following:

$$I(\theta)=\frac{I_1(I_2-I_4)+I_4(I_1-I_2)\mathrm{sn}^2(\theta_1,\tilde\kappa)}{I_2-I_4+(I_1-I_2)\mathrm{sn}^2(\theta_1,\tilde\kappa)}; \tag{3.26}$$

for $I_3\geqslant I(\theta)>I_4$ we have

$$I(\theta)=\frac{I_4(I_1-I_3)+I_1(I_3-I_4)\mathrm{sn}^2(\theta_1,\tilde\kappa)}{I_1-I_3+(I_3-I_4)\mathrm{sn}^2(\theta_1,\tilde\kappa)}. \tag{3.27}$$

Here the Jacobi function sn has a modulus $\tilde\kappa$: $\tilde\kappa^2=(I_1-I_2)(I_3-I_4)/[(I_1-I_3)(I_3-I_4)]$ and $2\theta_1=\theta[(I_1-I_3)(I_2-I_4)]^{1/2}$. In the limit $I_2\to I_3$ solutions (3.26), (3.27) describe the following isolated solitons, respectively:

$$I(\theta)=I_2+\frac{(I_2-I_4)(I_1-I_2)}{(I_1-I_4)\mathrm{ch}^2(\theta_2)-(I_1-I_2)}, \tag{3.28}$$

$$I(\theta)=I_3+\frac{(I_3-I_4)(I_1-I_3)}{(I_4-I_1)\mathrm{ch}^2(\theta_2)+(I_3-I_4)}. \tag{3.29}$$

Here $2\theta_2=\theta\,[(I_1-I_2)(I_2-I_4)]^{1/2}$. Solution (3.28) is the ''bright'' soliton on the nonzero background, whereas solution (3.29) is the ''gray'' soliton. In the limits $I_1\to I_2$, $I_3\to I_4$ the solutions (3.26), (3.27) are transformed to quasiharmonic periodic waves having a constant intensity I_1 and I_4, respectively. Let us consider, for instance, the case of two pairs of complex conjugated roots λ_k:

$$\lambda_{1,3}=\alpha_0\pm i\beta_0,\quad \lambda_{2,4}=\alpha\pm i\beta. \tag{3.30}$$

Then we have for the roots of the polynomial S for the upper sign in Eq. (3.24):

$$I_1=4(\beta_0+\beta),\quad I_2=4(\beta_0-\beta),\quad I_3=-4(\alpha_0-\alpha),$$
$$I_3=-4(\alpha_0+\alpha).$$

From Eq. (3.20) we have

$$\ln V=\frac{1}{2}\ln I-i\int\left(\frac{\sqrt{S_0}}{8I}-\frac{I}{8}\right)d\theta+i\nu Y+i\epsilon\int I\,d\theta+iC_1X. \tag{3.31}$$

Real constant C_1 can be expressed in terms of P_k. We used the equality $\partial_X|U|^2=-\partial_Y|V|^2$ following from the TM. An analogous relation between U and ρ can be derived using the above formulas. The nonlinear phase factor of the field V can be found from Eq. (3.31) using the table integrals [20].

In addition, initial conditions impose a relation between $\mu(0,0)$ and the initial value of the fields. Following to Kotljarov *et al.* [21] we write down this condition as the following

$$f^2(\varpi)-|V(0,0)|^2[\varpi-\mu(0,0)][\varpi-\mu^*(0,0)]=\prod_{j=1}^{4}(\varpi-\eta_j), \tag{3.32}$$

where η_j are the constant values satisfying to conditions of reality, i.e., coefficients P_k of the polynomial P must be real.

IV. THE WHITHAM EQUATIONS

In this paper we assume that external perturbation is absent. Dispersion phenomena yield a transformation of a dense packet of oscillation. To explain the origin of modulations, we consider the packet of nonlinear oscillations with periodic boundaries. Let the boundaries be extended to infinity. Dispersion produces space and time modulation of the solution, and resulting modulations are required to balance each other and yield the dependence of the spectral parameters on variables.

The exact solutions obtained in Sec. III describe the nonlinear waves repeating themselves after some period T. Smoothed shock waves or a modulated wave train may be described in a quasiclassical approximation. In this approximation it is assumed that the scales of the modulation of the train are much greater than that of each soliton or other nonlinear spikes filling the region of the oscillations. We suggest, in addition, that characteristic parameters of the periodic solution (the roots of a polynomial P: $\lambda_i^2, i=1,\ldots,4$) are smooth functions of the variables X,Y. These parameters obey the equations, which may be derived by averaging some integrals over the period of fast pulsations. By this way one can reduce the cumbersome problem of analysis of the complex system with many degrees of freedom to a solution of a few evolution equations.

The Whitham equations for the one-phase solution to integrable systems can be found directly in a diagonal (Riemann) form. We use here the approach developed by Flaschka *et al.* [12,14]. We present below final results; see Ref. [12], for details,.

We denote $\eta_j=\lambda_j^2$. Averaging over the period of fast oscillations T yields

$$\partial_X\eta_n+\frac{1}{v_n}\partial_Y\eta_n=0. \tag{4.1}$$

Here

$$\frac{1}{v_n}=\frac{1}{v_0}\left[1-\left(\eta_n\left\langle\frac{1}{\eta_n-\mu}\right\rangle\right)^{-1}\right],$$

$$\left\langle\frac{1}{\eta_1-\mu}\right\rangle=\frac{(\eta_2-\eta_3)E(k)+(\eta_1-\eta_2)K(k)}{4(\eta_1-\eta_2)(\eta_1-\eta_3)K(k)},$$

$$\left\langle\frac{1}{\eta_2-\mu}\right\rangle=\frac{(\eta_4-\eta_1)E(k)+(\eta_1-\eta_2)K(k)}{4(\eta_1-\eta_2)(\eta_2-\eta_4)K(k)},\qquad(4.2)$$

$$\left\langle\frac{1}{\eta_3-\mu}\right\rangle=\frac{(\eta_1-\eta_4)E(k)-(\eta_3-\eta_4)K(k)}{4(\eta_1-\eta_3)(\eta_3-\eta_4)K(k)},$$

$$\left\langle\frac{1}{\eta_4-\mu}\right\rangle=\frac{(\eta_3-\eta_2)E(k)-(\eta_3-\eta_4)K(k)}{4(\eta_2-\eta_4)(\eta_3-\eta_4)K(k)}.$$

Here $v_0=\sqrt{P_0}=(\eta_1\eta_2\eta_3\eta_4)^{1/2}$. $K(k),E(k)$ are the complete elliptic integral of the first and the second kind, respectively, with the modulus k: $k^2=[(\eta_1-\eta_2)(\eta_3-\eta_4)]/[(\eta_1-\eta_3)(\eta_2-\eta_4)]$. $\lambda_k^2=\eta_k$ are the roots of the polynomial $P(\lambda^2=\eta)$ such that $\eta_1>\eta_2>\eta_3>\eta_4$.

As considered in the previous section, the roots λ_1^2 and λ_3^2 of the polynomial P are fixed by asymptotics and the two roots λ_2^2 and λ_4^2 may change. We consider the most interesting case of complex roots (3.30), i.e., $\lambda_1^2=\eta_1=\alpha_0+i\beta_0$, $\lambda_3^2=\eta_3=\alpha_0-i\beta_0$. The dynamics of the two remaining ''moving'' roots λ_2 and λ_4 will obey the Whitham equations (4.1). Solving these equations, we find the trajectory of roots in the complex plane associated with the transform of weak quasilinear modulation of a plane wave to a set of isolated solitons.

Let $\lambda_2^2=\eta_2=\alpha+i\beta$, $\lambda_4^2=\eta_4=\alpha-i\beta$, and $\eta_{2,4}$ depend on the similarity variable Y/X. Using an equation for η_2 from system (4.1), we have

$$\frac{Y}{X}=\frac{1}{\sqrt{P_0}}\left\{1-\frac{1}{\alpha+i\beta}\,\frac{4i\beta[\alpha_0-\alpha+i(\beta_0-\beta)]K(\kappa)}{[\alpha_0-\alpha+i(\beta_0-\beta)]K(\kappa)-[\alpha_0-\alpha+i(\beta_0+\beta)]E(\kappa)}\right\}.\qquad(4.3)$$

Let us separate the real and the imaginary parts of Eq. (4.3):

$$\frac{E(\kappa)}{K(\kappa)}=G(k)=\frac{\alpha(\alpha_0^2+\alpha^2+\beta_0^2+\beta^2)-2\alpha(\alpha_0\alpha+\beta_0\beta)}{\alpha(\alpha_0^2+\alpha^2+\beta_0^2+\beta^2)-2\alpha_0(\alpha^2+\beta^2)},\qquad(4.4)$$

$$\left(\frac{Y}{X}\sqrt{P_0}-1\right)(\alpha^2+\beta^2)\{(\alpha_0-\alpha)^2(1-G)+[\beta_0-\beta+(\beta_0+\beta)G]^2\}=4\beta\{(\alpha_0-\alpha)(\alpha_0\beta-\alpha\beta_0)(1-G)+[\beta_0-\beta+(\beta_0+\beta)G](\beta\beta_0-\beta^2+\alpha\alpha_0-\alpha^2)\}.\qquad(4.5)$$

Equations (4.4) and (4.5) can be solved for α and β as functions of $E(\kappa)/K(\kappa)$ and the modulus κ:

$$\kappa^2=\frac{4\beta\beta_0}{(\alpha_0+\alpha)^2+(\beta_0-\beta)^2}.\qquad(4.6)$$

Trajectories of the roots $\eta_2\to\eta_1$, $\eta_4\to\eta_3$ as the functions of Y/X consist of the monotonic curves, which are symmetric with respect to the real axis, Fig. 1. The upper curve starts from the real axis and monotonically tends to the ''top'' value η_0 of the imaginary part of the spectrum. The case of coalescing roots $\eta_2=\eta_4$ corresponds to the plane-wave limit, which we started (point i in Fig. 1). The soliton limit is achieved as $\kappa\to1$. In the vicinity of the point η_0 we obtain

$$\eta_2=\eta_0\left[1+\frac{2\,\mathrm{Im}\,\eta_0(1-\kappa)^{1/2}}{|\eta_0|}\right]+O(1-k).$$

The study of the spectral problem (3.3) (see Appendixes A and B) shows that information can be derived by analysis of the phase factor $\Theta=i[\lambda^2(X+S)+\tilde{\vartheta}(\lambda^2)Y]$. Here $\tilde{\vartheta}(\lambda)$ is determined by the ''time'' (Y) dependence of spectral data. This dependence is nontrivial for $U(x\to-\infty)\neq0$. A group velocity Υ of the soliton solutions to the TM is determined by the conditions $\mathrm{Re}\Theta=0$. Rewrite Υ in the physical variables x, t:

$$\Upsilon=c/n(\omega_1)\ (\beta-1)/(1+\beta),$$

where $\beta=-\mathrm{Im}\tilde{\vartheta}/|\mathrm{Im}(\lambda^2)|>0$ $[\mathrm{Im}\tilde{\vartheta}(\lambda^2)<0]$. The physical conditions used in the above derivation of the TM require $\beta>1$. Let us choose a point η in the finite interval of the soliton part of the continuous spectrum, see Fig. 1. This interval is symmetric and orthogonal to the real axis and its highest values correspond to the point $\eta_0=\lambda_0^2$. It can be easily established that solitons associated with the smaller value Im η move faster. This means that a distance between solitons decreases as x increases. Close behavior is revealed by a numerical study of the above solution; see Fig. 2.

V. DISCUSSION OF THE APPLICATION OF OBTAINED RESULTS

We showed that the TM can be used for a description of the modulation instability in some nonlinear optical phenomena. We investigate the transformation of two plane waves propagating in a thin layer into a set of densely packed nonlinear oscillations due to mutual interaction.

In the above used approach the phenomena related with the modulation instability have to evolve slowly enough. Such a situation can be realized in a ring scheme with small losses. For this scheme periodic boundary conditions may be a good approximation for modeling of optical experiments.

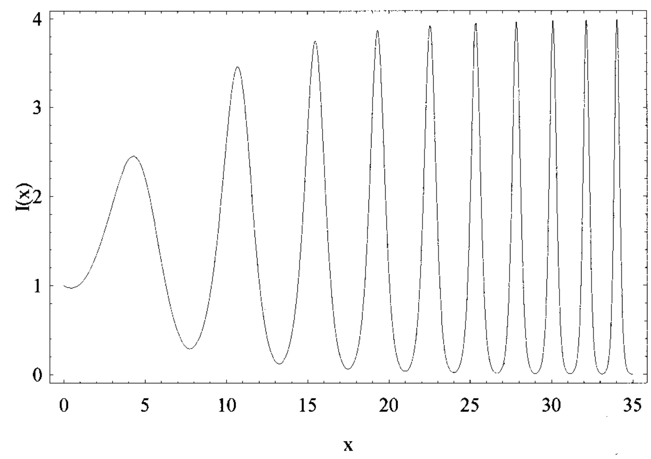

FIG. 2. Transformation of the leading edge of the initially steplike seed pulse as a consequence of solitons due to the modulation instability. The dependence of the intensity $I(x)=|V(x)|^2$ is found by the numerical solution of the Whitham equations for the parameter η_2 and shown in arbitrary units.

The analysis of the modulation instability in a finite medium under the periodic boundary conditions has to include the study of the unstable behavior of a set of discrete modes, which associate with harmonic waves. The spectrum consists of a set of discrete eigenvalues lying in the intervals in the complex plane. As shown above under some initial conditions, such a finite interval arises in the ''solitonic'' region of the complex plane. For a large number of modes and a small interval between the eigenvalues the above results can be applied. For a steplike pulse the experimental scheme may be the following. Consider two plane waves U and V propagating in counterdirections in a medium placed within the interval $[-d<x<d]$. The field U is the plane wave having nonzero amplitude in this interval and the field V has the form of steplike pulse injected in a medium at the point $x_0=-d$. Let a small disturbance initiate development of the modulation instability at the point x_0. The leading edge of pulse will transform in a dense packet of nonlinear oscillations. The shape of these oscillations initially located near $x=0$ tends to solitons as x increases. For sufficiently large d and a long pulse of the field V the dynamics of the leading front may be described by the above solution.

The present study of developing modulation instability is based on an analysis of the spectral problem (3.2). Analogous results can be derived using the linear system (3.3) as the spectral problem instead of Eq. (3.2). One can investigate the initial conditions leading to solitonic asymptotics for the steplike pulse of the second field.

Application of thin films as a nonlinear-optical medium is dictated by the needs of microelectronics. The coefficient $\chi^{(2)}$ for many media is about 10^{-5-6} SGSE, therefore the intensity of the fields about 10^4 SGSE is enough to observe the effects predicted above. If combined frequencies of fields are close to the frequency of a resonant transition of media the coefficient $\chi^{(2)}$ may be increased up to 10^3 times [2]. The required intensities of the field may be decreased correspondingly up to $\sqrt{10^3}$ times.

The results obtained in this paper for the TM may be used not only in nonlinear optics but in the theories of elementary particles and ferromagnetism. The solutions found in this paper can be used to study nonlinear mixing in a bulk medium in optics as well and in others branches of physics.

Application of the TM in the nonlinear optics relates to a specific form of the third-order nonlinearity in resonant media. This nonlinearity associates with nonlinear energy exchange. It may dominate for some schemes of the resonant interaction of weak light fields. This weak limit is important for the application in optical devices having small sizes. It is known that nonlinear structures (solitons and so on) appear when high intensities of the interacting fields are used. At another side such intensities may damage the optical medium. Therefore it is important to find and study the nonlinear processes, which may occur for the lowest possible intensities. The TM describes nonlinear processes, which may be treated as the weak limit of resonant multifrequency interaction. For instance, it may be shown that the TM may be derived as a reduction of integrable four-wave-mixing models [22].

Let us present the physical scheme of nonlinear resonant wave mixing leading to the Thirring model in a bulk medium. We consider the two-frequency interaction of a medium polarization R with laser waves. The material equation for low excitation is

$$\partial_t R+(\Gamma+i\nu_0)R=i\hbar^{-1}[\kappa_{12}E_1E_2\,\exp\{[-i(k_1+k_2)z]\}-\kappa_0E_0^2\exp(-i2k_0z)]; \tag{5.1}$$

here $\kappa_{12,0}$ are the resonant two-photon nonlinear susceptibilities, Γ is the relaxation constant, $E_{0,}\ E_{1,2}$ are the amplitudes of fields having the carrying frequencies ω_0, $\omega_{1,2}$ and vectors k_0, $k_{1,2}$, respectively. The resonance conditions are

$$\omega_1+\omega_2=2\omega_0=\omega+\nu_0,\quad k_1+k_2=2k_0;$$

here ν_0 is a detuning. Let the intensity of the pumping field E_0 be much more than that of fields $E_{1,2}$. Then the dependence of the field E_0 on variables is fixed and does not change due to interaction.

Let R adiabatically follow $E_{0,1,2}$. For large ν_0 one can find R from Eq. (5.1) integrating by parts. Neglecting the terms of order $O(\partial_t/\nu_0)R$, one gets

$$R=\frac{i}{\hbar(\Gamma+i\nu_0)}\{\kappa_{12}E_1E_2\exp[-i(k_1+k_2)z]-\kappa_0E_0^2\exp(-i2k_0z)\}. \tag{5.2}$$

The Maxwell equations for the slow amplitudes for the two-frequency resonance are the following:

$$\left(\partial_z+\frac{1}{v_{1,2}}\partial_t\right)E_{1,2}=-\frac{2\pi\omega_{1,2}\kappa_{1,2}N_0}{cn_{1,2}}E_{2,1}^*R, \tag{5.3}$$

where $v_{1,2}$ are the phase velocities, $n_{1,2}$ are the reflection coefficients, and N_0 is the atomic density.

Substituting the expression (5.2) in Eq. (5.3), one finds that the fields $E_{1,2}$ obey to the following system:

$$\left(\partial_z+\frac{1}{v_{1,2}}\partial_t\right)E_{1,2}=\frac{2\pi\omega_{1,2}N_0}{n_{1,2}c}\left(\frac{\kappa_{12}^2}{\nu_0}E_{1,2}|E_{2,1}|^2-\frac{\kappa_0E_0^2\kappa_{12}}{\nu_0-i\Gamma}E_{2,1}^*\right), \tag{5.4}$$

where $v_1\neq v_2$. The terms having order $O(\kappa_{12}E_{1,2}^2\Gamma/\nu_0\kappa_0E_0^2)$ are neglected.

In the following new notations:

$$\partial_Y=-\frac{n_1c\nu_0}{2\pi\omega_1N_0\kappa_{12}^2}\left[\partial_z+\frac{1}{v_1}\partial_t\right],$$

$$\partial_X=\frac{n_2c\nu_0}{2\pi\omega_2N_0\kappa_{12}^2}\left[\partial_z+\frac{1}{v_2}\partial_t\right],$$

$$m=\frac{\kappa_0E_0^2\nu_0(\Gamma+i\nu_0)}{\kappa_{12}(\Gamma^2+\nu_0{}^2)},$$

$$V=E_1,\quad U=E_2^*,$$

system (5.4) transforms to system (3.1), where $\nu=\nu_1=0$. The physical conditions of complete integrability are the same as above. The last example shows that the results obtained in this paper can be applied to the analysis of multi-wave-mixing in bulk media.

The Thirring model may be used in the study of the two-component field propagation in waveguides in the Bragg medium [6] and of the ''gap'' solitons [7]. It is worth mentioning that the regimes of the optical wave mixing described by the TM may crucially differ from regimes described by the two-component nonlinear Schrödinger equation, which has wide application in nonlinear optics. For instance, the integrable version of the latter model does not describe the counterpropagation of two fields in one-dimensional case [2].

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APPENDIX A: THE INVERSE SCATTERING TECHNIQUES AND THE TIME DEPENDENCE OF SCATTERING DATA

To establish asymptotics as $x\to\infty$ of an initially steplike pulse we apply the conventional ''solitonic'' version of IST. Here we shall follow to the paper of Kaup and Newell [23]. These authors developed the IST for the derivative nonlinear Schrödinger equation. After some modifications their results can be applied to those considered here. This modification relates, for instance, to the ''time'' dependence of the scattering data.

Let us make the following changes:

(1) Change the variables $X\to\chi=G_0x/v$, $Y\to\tau=G_0(t-2x/v)$.

(2) Change the fields $\{{V \atop U}\}\to\{{V \atop U}\}\exp i(\epsilon\int_0^\chi|V|^2\,d\chi-i\nu\chi)$.

(3) Change a spectral parameter $\lambda\to\sqrt{2}\zeta$. Then the spectral problem (3.3) is transformed to the spectral problem studied in Ref. [23]:

$$\partial_\chi\psi_1=-i\zeta^2\psi_1+\zeta q\psi_2, \qquad \partial_\chi\psi_2=i\zeta^2\psi_2+\zeta r\psi_1, \tag{A1}$$

where $q=-\sqrt{2}V\exp i(\epsilon\int_0^\chi|V|^2\,d\chi-i\nu\chi)$, $r=-\epsilon q^*$.

We formulate an initial problem. We consider an infinite medium spread from $-\infty$ to $+\infty$. A long pulse of the ''potential'' $q(\chi)$ having length d propagates from $-\infty$ to $+\infty$. Then, we shall put the length of this pulse as $d\to\infty$.

We consider the vanishing as $\chi\to\pm\infty$ field q. For these asymptotics, the IST techniques had been developed in Ref. [23], therefore we present here only the results required for our purposes. Define

$$\mu^-=\frac{-\epsilon}{2}\int_{-\infty}^{\chi}|V|^2dx,\quad \mu^+=\frac{-\epsilon}{2}\int_{\chi}^{+\infty}|V|^2dx. \tag{A2}$$

The potential q is determined by a diagonal of a kernel $K_1(\chi,\sigma)$ [23]

$$q(\chi)=-2K_1(\chi,\chi)\exp(-2i\mu^+), \tag{A3}$$

where the kernels K_1 and K_2 satisfy the following Marchenko equations:

$$K_2^*(\chi,\gamma)-\int_\chi^\infty K_1(\chi,\sigma)F'(\sigma+\gamma)d\sigma=0,$$
$$-\epsilon K_1(\chi,\gamma)+F(\chi+\gamma)+i\int_\chi^\infty K_2^*(\chi,\sigma)F^*(\sigma+\gamma)d\sigma=0, \tag{A4}$$

where

$$F(\chi)=\frac{1}{2\pi}\int\frac{b(\zeta)}{a(\zeta)}\exp(i\zeta^2\chi)d\zeta,$$
$$F'(\chi)=\frac{dF(\chi)}{d\chi}. \tag{A5}$$

The time dependence of the scattering coefficient $\rho(\tau)=(b/a)(\tau)$ is defined by the second linear system (3.3). We consider the following asymptotic conditions: as $\chi\to-\infty$, field U is a plane wave having a constant amplitude U_0 and $U\to0$ as $\chi\to+\infty$. The time dependence of the scattering data can be obtained using the linear system (3.3). For the matrix $\hat{D}$,

$$\hat{D}=\begin{bmatrix} a & b \\ -\epsilon b^* & -a \end{bmatrix}, \quad \text{(A6)}$$

the following evolution can be easily found:

$$\partial_\tau \hat{D}(\tau)=-\hat{D}\hat{A}(\chi\to-\infty)+\hat{A}(\chi\to\infty)\hat{D}. \quad \text{(A7)}$$

For the zero asymptotics value (as $\chi\to+\infty$) of the off-diagonal part of the matrix $\hat{A}$ and for a constant meaning of $\hat{A}$ at $\chi\to-\infty$ we have the following solution for the components of the matrix $\hat{D}$:

$$a=[(i\vartheta+A_{11})\exp(-i\vartheta\tau)+(i\vartheta-A_{11})\exp(i\vartheta\tau)]a_0 + b_0 A_{21}[\exp(-i\vartheta\tau)-\exp(i\vartheta\tau)], \quad \text{(A8)}$$

$$b=a_0 A_{12}[\exp(-i\vartheta\tau)-\exp(i\vartheta\tau)] + b_o[(i\vartheta-A_{11})\exp(-i\vartheta\tau)+(i\vartheta+A_{11})\exp(i\vartheta\tau)]; \quad \text{(A9)}$$

here A_{ij} are the elements of the matrix $\hat{A}$ such that $A_{11}=-A_{22}$. $\vartheta^2=-A_{11}^2-A_{12}A_{21}$.

Finally we have for the scattering coefficient ρ

$$\rho(\tau)=\frac{b}{a}=\frac{A_{21}[\exp(-i\vartheta\tau)-\exp(i\vartheta\tau)]+\dfrac{b_0}{a_0}[(i\vartheta-A_{11})\exp(-i\vartheta\tau)+(i\vartheta+A_{11})\exp(i\vartheta\tau)]}{(i\vartheta+A_{11})\exp(-i\vartheta\tau)+(i\vartheta-A_{11})\exp(i\vartheta\tau)+(b_0/a_0)A_{21}[\exp(-i\vartheta\tau)-\exp(i\vartheta\tau)]}; \quad \text{(A10)}$$

here $\rho_0=\rho(\tau=0)=b_0/a_0$. For large $u=m_0^2/\lambda^2$ ($|u|\gg1$), we obtain, using decomposition in the degrees of u in Eq. (3.3),

$$\vartheta=\sqrt{(\alpha+u)^2+l_0^2}\approx u\left[1+\frac{\alpha}{u}+\frac{1}{2}\frac{l_0^2-\alpha^2}{u^2}+o(u^{-3})\right],$$

$$A_{21}=ip_1\sqrt{u}, \quad A_{12}=is_1\sqrt{u}, \quad A_{11}=i(l_1-l_2u). \quad \text{(A11)}$$

Neglecting the terms having order $O(u^{-3/2})$ we have

$$\rho(t)=\rho_0\left(\frac{e_+-l_2e_-}{e_++l_2e_-}\right)\left\{1+\frac{e_-}{\sqrt{u}}\left[\frac{p_1}{\rho_0(e_+-e_-l_2)}-\frac{\rho_0 s_1}{(e_++e_-l_2)}\right]\right\}\left[1+O\left(\frac{1}{u}\right)\right]; \quad \text{(A12)}$$

here $e_\pm=\exp(-i\vartheta\tau)\pm\exp(i\vartheta\tau)$. For the Lax pair used above we have $l_2=-1$. As a consequence Eq. (A12) can be simplified to

$$\rho(t)=\rho_0\exp(2i\tilde{\vartheta}\tau)\left(1+\frac{e_-}{\sqrt{u}}Q_1\right)\left[1+O\left(\frac{1}{u}\right)\right], \quad \text{(A13)}$$

where Q_1 is a function of $e_\pm$. We choose the ''minus'' sign of $\vartheta=-(-A_{11}^2-A_{12}A_{21})^{1/2}=-\tilde{\vartheta}$ to link this $\tilde{\vartheta}$ to the solution obtaining for the asymptotics $U\to0$ as $x\to\pm\infty$.

APPENDIX B: EVALUATION OF ASYMPTOTICS

Let us present the integral (A5) as a sum of integrals along the contours C_0 and C_1 (Fig. 1). We estimate this integral, which is calculated along the contour C_0 over the upper part of the ''soliton'' branch

$$J_0(\chi+\gamma)=\frac{1}{2\pi}\int_{C_0}h(\eta)\exp[i\eta(\chi+\gamma)+i2\tilde{\vartheta}(\eta)\tau]d\eta; \quad \text{(B1)}$$

here $h(\eta)$ is some function, which does not coincide with $b(\eta)/a(\eta)$. But this function yields only a shift of the phase of solution and its exact meaning is not required for the present estimation. To estimate the integral we find the maximum value of the factor $\tilde{\Theta}(\eta,\tilde{\tau})=[i\eta+2i\tilde{\vartheta}\tilde{\tau}]$, where $\eta=\lambda^2$, $\tilde{\tau}=\tau/(\chi+\gamma)$. Consider the case of large $|u|=|m_0^2/\eta|$, i.e., $\tilde{\vartheta}\approx u$. The derivative of $\tilde{\Theta}(\xi)$ on C_0 with respect to $\xi=\mathrm{Im}\,\eta$ is equal to zero for $\xi_1^2=-\tilde{\tau}/2\pm\sqrt{\tilde{\tau}^2/4+2\zeta_0^2\tilde{\tau}}$; here $\zeta_0=\mathrm{Re}\,\eta_0$. For large $\tilde{\tau}$ we have $\xi_1^2\approx-\tilde{\tau}/2+\tilde{\tau}/2(1+4\zeta_0^2/\tilde{\tau})-\zeta_0^2\approx\zeta_0^2$. In Sec. II it is found that for the initial steplike pulse having high V_0 $\xi\in[-\xi_0,\xi_0]$, $|\xi_0|=2|V_0|\sqrt{|\nu|}$, $\zeta_0=(\epsilon|V_0|^2-\nu)$. We find the condition when the maximum of $\Theta(\xi)$ lies exterior to the interval $[\zeta_0-i\xi_0,\bar{\zeta_0}+i\xi_0]$. For $\nu>0,\epsilon>0$ this condition is

$$\nu\notin[|V_0|^2(3-\sqrt{8}),|V_0|^2(3+\sqrt{8})]. \quad \text{(B2)}$$

Assume that the condition (B2) is fulfilled. Let us now estimate the integral $J_0(\chi+\gamma)$ over the path C_0 for large $(\chi+\gamma)$. The function $\tilde{\Theta}(\eta,\tilde{\tau})$ has the nonzero derivative with respect to η on C_0 and the maximum of $\mathrm{Re}[\tilde{\Theta}(\eta,\tilde{\tau})]$ is attained at the point $\eta_0=\zeta_0+i\xi_0$, i.e., at the ''top point'' of the solitonic branch of the spectrum. Therefore, by integrating by parts, we obtain the following:

$$J_0(\chi+\gamma)=\frac{h(\eta_0)\exp[i(\chi+\gamma)\tilde{\Theta}(\eta_0,\tilde{\tau})]}{i(\chi+\gamma)\tilde{\Theta}'(\eta_0,\tilde{\tau})}\left[1+O\left(\frac{1}{\chi+\gamma}\right)\right]. \quad \text{(B3)}$$

Now let us estimate the contribution of the real continuous spectrum. For this aim we estimate the integral

$$J_1(\chi+\gamma)=\frac{1}{2\pi}\int_{C_1}\frac{b(\eta)}{a(\eta)}\ \exp[i\eta(\chi+\gamma)+i2\tilde{\vartheta}(\eta)\tau]\times\left[1+O\left(\frac{e_-Q_1}{\sqrt{u}}\right)\right]d\eta. \quad \text{(B4)}$$

We use the τ dependence of spectral data found in Appendix A. Study of the phase $\tilde{\Theta}(\eta,\tilde{\tau})$ for the real η shows that the first term in Eq. (B4) does not yield a significant contribution to asymptotics. The function $\mathrm{Re}\tilde{\Theta}(\eta,\tilde{\tau})$ for a real η attained the maximum at $\eta=0$. The contribution of the second term may be essential. However, according to assumptions used in Sec. II, u is supposed to be large $(|u|\gg 1)$. Thus we are able to neglect the second term in the rhs of Eq. (B4). Consequently integral (B3) gives rise to the main contribution to the leading front of the asymptotic. Substituting this integral and its derivative in the Marchenko equations (A4) one may find the soliton solution to the TM. This asymptotic soliton is characterized by the spectral parameter η_0 and describes the leading edge of the packet of pulses.

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