Damping in dilute Bose gases: A mean-field approach

S. Giorgini
European Centre for Theoretical Studies in Nuclear Physics and Related Areas, Villa Tambosi, Strada delle Tabarelle 286, I-38050 Villazzano, Italy
and Istituto Nazionale di Fisica della Materia, Unità di Trento, I-38050 Povo, Italy
(Received 24 September 1997)

Damping in a dilute Bose gas is investigated using a mean-field approximation which describes the coupled oscillations of condensate and noncondensate atoms in the collisionless regime. Explicit results for both Landau and Beliaev damping rates are given for nonuniform gases. In the case of uniform systems, we obtain results for the damping of phonons both at zero and finite temperature. The isothermal compressibility of a uniform gas is also discussed. [S1050-2947(98)02804-2]

PACS number(s): 03.75.Fi, 02.70.Lq, 67.40.Db

I. INTRODUCTION

The low-lying collective excitations of magnetically trapped Bose gases have been the object of very accurate experimental measurements [1–3]. The first experiments [1,2] were carried out at low temperature, with approximately all the atoms in the condensate state. These experiments showed almost undamped oscillations of the condensate at frequencies which have been found in excellent agreement with theoretical predictions based on the $T=0$ time-dependent Gross-Pitaevskii equation [4]. In more recent experiments [3] the study of the low-energy collective modes has been extended to higher temperatures, where the condensate oscillates in the presence of a considerably large fraction of above-condensate atoms. In this case, evidence is given of large frequency shifts with unexpected features and of strong damping rates which have not yet been understood theoretically.

On the theoretical side, extensions to finite temperature of the Gross-Pitaevskii equation have been put forward [5–9] and have been very successful in explaining experimental results on the thermodynamic properties of these systems, such as condensate fraction, internal energy, specific heat, and critical temperature [6–8]. These mean-field descriptions, however, seem to be inadequate when applied to the study of collective excitations at finite temperature [6,9]. First of all, the proposed mean-field approximations do not account for damping. Second, the resonance frequencies are predicted to vary with temperature, mainly because the number of atoms in the condensate changes, and the dependence of the frequency shift upon this quantity is expected to show the same behavior as the corresponding $T=0$ dependence on the total number N of atoms in the trap. None of the features exhibited by the experimental data on the frequency shift can be explained using these descriptions.

In the presently available finite-T extensions of the Gross-Pitaevskii equation, it is assumed that the condensate oscillates in a bath of thermally excited atoms at rest and in thermal equilibrium in the effective mean-field potential generated by the average condensate density. This assumption is valid if the time scale on which the thermal cloud oscillates is much larger than the inverse frequency of the condensate oscillations. Since both the condensate and the thermal cloud vary on comparable time scales of the order of the trap frequency, a full dynamic treatment of the coupled oscillations of the two clouds should be developed. The dynamic effects we are aiming to describe are irrelevant in the calculation of the thermodynamic properties of the system, for which the standard finite-T mean-field approaches are well suited. In fact, as pointed out in Refs. [7,10], the fine details of the excitation energies do not affect the thermodynamic behavior, for which what matters is the density of states at a given energy.

Another problem arises as to whether the appropriate regime to describe the experimental situation in [3] is collisional or collisionless. In the collisional regime, which takes place at high temperatures and densities, the damping mechanisms are of dissipative type and the dynamics of the system is described by two-fluid hydrodynamics, recently developed for trapped gases in Ref. [11]. For very dilute systems at low temperatures the mean free path of the elementary excitations becomes comparable with the size of the system and collisions play a minor role. Damping in this regime is not related to thermalization processes but to coupling between excitations, and can be described in the framework of mean-field theories. As suggested in Ref. [12], the collisionless regime may be appropriate for the JILA experiments [1,3], but probably not for the MIT experiments [2].

In the collisionless regime and at finite temperature the damping of the low-lying collective modes is dominated by Landau damping. The idea that this mechanism might be relevant to explain the damping rates in trapped Bose gases was first suggested by Liu and Schieve [13] and was then developed in Refs. [14–16]. In Ref. [14] Pitaevskii and Stringari have derived, using perturbation theory, an expression for Landau damping which is applicable to trapped Bose gases, and have shown that, if applied to the uniform case, it reproduces known results for both the low- and high-temperature asymptotic behavior of the phonon damping.

In this paper we develop a time-dependent mean-field approach based on the Popov approximation, which describes the dynamics of a Bose-condensed gas in the collisionless regime. We obtain a set of coupled equations for the condensate and noncondensate components which allow us to calculate damping coefficients. In nonuniform gases we derive explicit expressions both for Landau damping, which coincides with the finding of Ref. [14], and for Beliaev damping. In the uniform case, we reproduce all known results on the damping rates of phonons both at $T=0$ and at finite T.

However, the calculation of the temperature dependence of the frequency shifts using the Popov approximation is not reliable. For a uniform system the calculation should give, in the long wavelength limit, the correction to the velocity of zeroth sound due to quantum and thermal fluctuations. On the other hand, at $T=0$, the velocity of zeroth sound is directly related to the bulk compressibility of the system, for which we show that the Popov approximation gives an incorrect result. By studying the isothermal compressibility we also find that the Popov approximation is inconsistent at low temperatures, because it neglects fluctuations which are relevant for temperatures smaller than the chemical potential. Since the same fluctuations are also important in the calculation of the velocity of zeroth sound in the low temperature regime, we draw the conclusion that a dynamic mean-field description, which gives a correct account of both damping rates and frequency shifts, can only be developed going beyond the Popov approximation.

The paper is organized as follows. In Sec. II we develop the formalism of the time-dependent mean-field approximation. In Sec. III we study the damping of the oscillations in a nonuniform system obtaining explicit expressions for the Landau and Beliaev damping. In Secs. IV A and IV B, we apply the results of Sec. III to the uniform case. In Sec. IV C we investigate the isothermal compressibility of a uniform gas beyond the Popov approximation.

II. TIME-DEPENDENT MEAN-FIELD APPROXIMATION

In the presence of a nonuniform external field $V_{\rm ext}(\mathbf{r})$, the grand-canonical Hamiltonian of the system has the form

$$K\equiv H-\mu N=\int d\mathbf{r}\psi^{\dagger}(\mathbf{r},t)\left(-\frac{\hbar^2\nabla^2}{2m}+V_{\rm ext}(\mathbf{r})-\mu\right)\psi(\mathbf{r},t)+\frac{g}{2}\int d\mathbf{r}\psi^{\dagger}(\mathbf{r},t)\psi^{\dagger}(\mathbf{r},t)\psi(\mathbf{r},t)\psi(\mathbf{r},t) \qquad (1)$$

in terms of the creation and annihilation particle field operators $\psi^{\dagger}(\mathbf{r},t)$ and $\psi(\mathbf{r},t)$. In the above equation, g is the interaction coupling constant, which to lowest order in the s-wave scattering length a is given by $g=4\pi\hbar^2 a/m$. The equation of motion for the particle field operator then follows immediately and reads

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t)=[\psi(\mathbf{r},t),K]=\left(-\frac{\hbar^2\nabla^2}{2m}+V_{\rm ext}(\mathbf{r})-\mu\right)\psi(\mathbf{r},t)+g\psi^{\dagger}(\mathbf{r},t)\psi(\mathbf{r},t)\psi(\mathbf{r},t). \qquad (2)$$

According to the usual treatment for Bose systems with broken gauge symmetry, we define a time-dependent condensate wave function $\Phi(\mathbf{r},t)$ [17],

$$\Phi(\mathbf{r},t)=\langle\psi(\mathbf{r},t)\rangle, \qquad (3)$$

which allows us to describe situations where the system is displaced from equilibrium and the condensate is oscillating in time. The average $\langle\cdots\rangle$ in Eq. (3) is thus intended to be a nonequilibrium average, while time-independent equilibrium averages will be indicated in this paper with the symbol $\langle\cdots\rangle_0$. The particle field operator can then be decomposed into a condensate and a noncondensate component,

$$\psi(\mathbf{r},t)=\Phi(\mathbf{r},t)+\tilde{\psi}(\mathbf{r},t), \qquad (4)$$

and the noncondensate term satisfies, by definition, the condition $\langle\tilde{\psi}(\mathbf{r},t)\rangle=0$. By applying the decomposition (4) to the Heisenberg equation (2), the term cubic in the field operators becomes (all quantities depend on $\mathbf{r}$ and t)

$$\psi^{\dagger}\psi\psi=|\Phi|^2\Phi+2|\Phi|^2\tilde{\psi}+\Phi^2\tilde{\psi}^{\dagger}+2\Phi\tilde{\psi}^{\dagger}\tilde{\psi}+\Phi^*\tilde{\psi}\tilde{\psi}+\tilde{\psi}^{\dagger}\tilde{\psi}\tilde{\psi}. \qquad (5)$$

We assume that for dilute systems the cubic product of the noncondensate operators [last term in Eq. (5)] has a negligible effect on the dynamics of the condensate and we set its average value equal to zero:

$$\langle\tilde{\psi}^{\dagger}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\rangle=0. \qquad (6)$$

One thus obtains the following equation for the time rate of change of $\langle\psi(\mathbf{r},t)\rangle$:

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r},t)=\left(-\frac{\hbar^2\nabla^2}{2m}+V_{\rm ext}(\mathbf{r})-\mu\right)\Phi(\mathbf{r},t)+g|\Phi(\mathbf{r},t)|^2\Phi(\mathbf{r},t)+2g\Phi(\mathbf{r},t)\tilde{n}(\mathbf{r},t)+g\Phi^*(\mathbf{r},t)\tilde{m}(\mathbf{r},t), \qquad (7)$$

where we have introduced the normal and anomalous time-dependent densities defined, respectively, as

$$\tilde{n}(\mathbf{r},t)=\langle\tilde{\psi}^{\dagger}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\rangle,$$
$$\tilde{m}(\mathbf{r},t)=\langle\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\rangle. \qquad (8)$$

The equation of motion (7) with $\tilde{n}=\tilde{m}=0$ corresponds to the usual $T=0$ Gross-Pitaevskii equation for the condensate wave function, while its extension including the normal and anomalous densities has been already discussed by many authors both in the study of thermodynamic properties and of the collective modes at finite temperature [5–9,11]. The novelty here is that we will treat both terms within a time-dependent mean-field approximation holding in the collisionless regime, which is the object of the present work [18].

We are interested in the small-amplitude regime in which the condensate is only slightly displaced from its stationary value $\Phi_0(\mathbf{r})=\langle\psi(\mathbf{r})\rangle_0$,

$$\Phi(\mathbf{r},t)=\Phi_0(\mathbf{r})+\delta\Phi(\mathbf{r},t), \qquad (9)$$

where $\delta\Phi(\mathbf{r},t)$ is a small fluctuation. In the same way we consider small fluctuations of the normal and anomalous densities,

$$\tilde{n}(\mathbf{r},t)=\tilde{n}^0(\mathbf{r})+\delta\tilde{n}(\mathbf{r},t),$$
$$\tilde{m}(\mathbf{r},t)=\tilde{m}^0(\mathbf{r})+\delta\tilde{m}(\mathbf{r},t), \qquad (10)$$

around their equilibrium values $\tilde{n}^0(\mathbf{r})=\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle_0$ and $\tilde{m}^0(\mathbf{r})=\langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle_0$.

In the so-called Popov approximation the effects arising from the equilibrium value of the anomalous density in Eq. (7) are neglected and the following ansatz is introduced in the mean-field scheme:

$$\tilde{m}^0(\mathbf{r})=0. \tag{11}$$

This approximation was first introduced by Popov in the study of a uniform weakly interacting Bose gas at finite temperature [19] (for a detailed discussion, see Refs. [5] and [20]). More recently, the Popov approximation has been extensively used in the study of properties of magnetically trapped Bose gases at finite temperature [5–7,9,11]. The Popov approximation gives a gapless spectrum of elementary excitations, which at $T=0$ coincides with the well-known Bogoliubov dispersion relation, while at high T it approaches the finite-temperature Hartree-Fock spectrum [21].

By using the Popov prescription (11), the real wave function $\Phi_0(\mathbf{r})$ satisfies the stationary equation

$$\left(-\frac{\hbar^2\nabla^2}{2m}+V_{\mathrm{ext}}(\mathbf{r})-\mu+g[n_0(\mathbf{r})+2\tilde{n}^0(\mathbf{r})]\right)\Phi_0(\mathbf{r})=0, \tag{12}$$

where $n_0(\mathbf{r})=|\Phi_0(\mathbf{r})|^2$ is the condensate density, while the time-dependent equation for $\delta\Phi(\mathbf{r},t)$ is obtained by linearizing the equation of motion (7),

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\delta\Phi(\mathbf{r},t)=&\left(-\frac{\hbar^2\nabla^2}{2m}+V_{\mathrm{ext}}(\mathbf{r})-\mu+2gn(\mathbf{r})\right)\delta\Phi(\mathbf{r},t)\\&+gn_0(\mathbf{r})\delta\Phi^*(\mathbf{r},t)+2g\Phi_0(\mathbf{r})\delta\tilde{n}(\mathbf{r},t)\\&+g\Phi_0(\mathbf{r})\delta\tilde{m}(\mathbf{r},t),\end{aligned} \tag{13}$$

where we have introduced the total equilibrium density $n(\mathbf{r})=n_0(\mathbf{r})+\tilde{n}^0(\mathbf{r})$.

From Eq. (13) one clearly sees that the oscillations of the condensate are coupled to the fluctuations $\delta\tilde{n}(\mathbf{r},t)$ and $\delta\tilde{m}(\mathbf{r},t)$ of the normal and anomalous densities. It is worth reminding at this point that the Popov ansatz (11) imposes a constraint on the equilibrium value of the anomalous density, but not on its fluctuations, which are important to describe properly the coupling between the oscillations of the condensate and noncondensate part. In order to obtain the equations of motion for $\delta\tilde{n}(\mathbf{r},t)$ and $\delta\tilde{m}(\mathbf{r},t)$ it is convenient to express the noncondensate operators $\tilde{\psi},\tilde{\psi}^\dagger$ in terms of quasiparticle operators $\alpha_j,\alpha_j^\dagger$ by means of the Bogoliubov linear transformations

$$\tilde{\psi}(\mathbf{r},t)=\sum_j\,[(u_j\mathbf{r})\alpha_j(t)+v_j^*(\mathbf{r})\alpha_j^\dagger(t)],$$

$$\tilde{\psi}^\dagger(\mathbf{r},t)=\sum_j\,[u_j^*(\mathbf{r})\alpha_j^\dagger(t)+v_j(\mathbf{r})\alpha_j(t)]. \tag{14}$$

The normalization condition for the functions $u_j(\mathbf{r}),v_j(\mathbf{r})$, which ensures that the quasiparticle operators $\alpha_j,\alpha_j^\dagger$ satisfy Bose commutation relations, reads

$$\int d\mathbf{r}[u_i^*(\mathbf{r})u_j(\mathbf{r})-v_i^*(\mathbf{r})v_j(\mathbf{r})]=\delta_{ij}. \tag{15}$$

By using the transformations (14), the quantities $\tilde{n}(\mathbf{r},t)$ and $\tilde{m}(\mathbf{r},t)$ can be expressed in terms of the normal and anomalous quasiparticle distribution functions defined by

$$f_{ij}(t)=\langle\alpha_i^\dagger(t)\alpha_j(t)\rangle,$$

$$g_{ij}(t)=\langle\alpha_i(t)\alpha_j(t)\rangle. \tag{16}$$

The time evolution of these functions is fixed by the following equations of motion:

$$i\hbar\frac{\partial}{\partial t}f_{ij}(t)=\langle[\alpha_i^\dagger(t)\alpha_j(t),K]\rangle,$$

$$i\hbar\frac{\partial}{\partial t}g_{ij}(t)=\langle[\alpha_i(t)\alpha_j(t),K]\rangle. \tag{17}$$

In order to calculate the commutators of Eq. (17), one notices that, after substituting the decomposition (4) of the particle field operator into the Hamiltonian (1), only the terms quadratic and quartic in the noncondensate operators $\tilde{\psi},\tilde{\psi}^\dagger$ give nonvanishing contributions, because, according to Eq. (6), we set to zero all averages of cubic products of the noncondensate operators. The terms in the grand-canonical Hamiltonian relevant for the calculation of the commutators are thus given by

$$\begin{aligned}K_2+K_4=&\int d\mathbf{r}\Bigg[\tilde{\psi}^\dagger(\mathbf{r},t)\left(-\frac{\hbar^2\nabla^2}{2m}+V_{\mathrm{ext}}(\mathbf{r})-\mu\right)\tilde{\psi}(\mathbf{r},t)\\&+2g|\Phi(\mathbf{r},t)|^2\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\\&+\frac{g}{2}\Phi^2(\mathbf{r},t)\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}^\dagger(\mathbf{r},t)\\&+\frac{g}{2}\Phi^{*2}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\Bigg]\\&+\frac{g}{2}\int d\mathbf{r}\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t),\end{aligned} \tag{18}$$

where we have indicated with K_2 the term quadratic in $\tilde{\psi},\tilde{\psi}^\dagger$, while K_4 is the term quartic in these operators. By expanding K_2 up to terms linear in the fluctuations $\delta\Phi(\mathbf{r},t)$, we can rewrite it as the sum $K_2=K_2^{(0)}+K_2^{(1)}$, where $K_2^{(0)}$ does not contain the fluctuations of the condensate

$$K_2^{(0)}=\int d\mathbf{r}\bigg[\tilde{\psi}(\mathbf{r},t)\bigg(-\frac{\hbar^2\nabla^2}{2m}+V_{\rm ext}(\mathbf{r})-\mu+2gn_0(\mathbf{r})\bigg)\tilde{\psi}(\mathbf{r},t)+\frac{g}{2}n_0(\mathbf{r})[\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}^\dagger(\mathbf{r},t)+\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)]\bigg], \tag{19}$$

while $K_2^{(1)}$ is linear in the fluctuations $\delta\Phi$,

$$K_2^{(1)}=\int d\mathbf{r}\{2g\Phi_0(\mathbf{r})[\delta\Phi(\mathbf{r},t)+\delta\Phi^*(\mathbf{r},t)]\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)+g\Phi_0(\mathbf{r})[\delta\Phi(\mathbf{r},t)\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}^\dagger(\mathbf{r},t)+\delta\Phi^*(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)]\}. \tag{20}$$

In the quartic term K_4 we first use the mean-field decomposition (all quantities depend on $\mathbf{r}$ and t)

$$\tilde{\psi}^\dagger\tilde{\psi}^\dagger\tilde{\psi}\tilde{\psi}=4\tilde{n}\tilde{\psi}^\dagger\tilde{\psi}+\tilde{m}\tilde{\psi}^\dagger\tilde{\psi}^\dagger+\tilde{m}^*\tilde{\psi}\tilde{\psi}, \tag{21}$$

and then we expand the resulting expression up to linear terms in the fluctuations $\delta\tilde{n}$ and $\delta\tilde{m}$, thereby obtaining

$$\begin{aligned}K_4&=K_4^{(0)}+K_4^{(1)}\\&=2g\int d\mathbf{r}\tilde{n}^0(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\\&\quad+\frac{g}{2}\int d\mathbf{r}\{4\delta\tilde{n}(\mathbf{r},t)\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\\&\quad+[\delta\tilde{m}(\mathbf{r},t)\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}^\dagger(\mathbf{r},t)+\delta\tilde{m}^*(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)]\},\end{aligned} \tag{22}$$

where we have made use of the Popov ansatz (11). In the above equation, $K_4^{(0)}$ is the zeroth order term, while $K_4^{(1)}$ is linear in the fluctuations $\delta\tilde{n}$ and $\delta\tilde{m}$. In the equations of motion (17) the commutators of $\alpha_i^\dagger\alpha_j$ and $\alpha_i\alpha_j$ with $K_2^{(1)}$ yield, in the small-amplitude regime, the coupling to the fluctuations of the condensate, whereas the commutators with $K_4^{(1)}$ give the coupling to the fluctuations of the normal and anomalous particle densities. If the density of the noncondensate particles is much smaller than the density n_0 of the condensate, the coupling to the condensate is more important than the coupling to $\delta\tilde{n}$ and $\delta\tilde{m}$ and we can consequently neglect the contributions to the commutators arising from the term $K_4^{(1)}$.

One can easily show that the operator $K_2^{(0)}+K_4^{(0)}$ is diagonal in the quasiparticle operators $\alpha_j,\alpha_j^\dagger$ if the functions u_j and v_j satisfy the coupled Bogoliubov equations,

$$\begin{aligned}\mathcal{L}u_j(\mathbf{r})+gn_0(\mathbf{r})v_j(\mathbf{r})&=\epsilon_j u_j(\mathbf{r}),\\\mathcal{L}v_j(\mathbf{r})+gn_0(\mathbf{r})u_j(\mathbf{r})&=-\epsilon_j v_j(\mathbf{r}),\end{aligned} \tag{23}$$

where we have introduced the Hermitian operator

$$\mathcal{L}=-\frac{\hbar^2\nabla^2}{2m}+V_{\rm ext}(\mathbf{r})-\mu+2gn(\mathbf{r}). \tag{24}$$

As a consequence, the relevant terms in the Hamiltonian become

$$K_2+K_4=\sum_j \epsilon_j\alpha_j^\dagger(\mathbf{r})\alpha_j(\mathbf{r})+K_2^{(1)}, \tag{25}$$

where the quasiparticle energies ϵ_j are obtained from the solutions of the Bogoliubov equations (23).

The commutators in the equations of motion (17) can be now calculated straightforwardly. To lowest order in the fluctuations one gets

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}f_{ij}(t)&=(\epsilon_j-\epsilon_i)f_{ij}(t)+2g(f_i^0-f_j^0)\int d\mathbf{r}\Phi_0(\mathbf{r})\\&\quad\times\{[\delta\Phi(\mathbf{r},t)+\delta\Phi^*(\mathbf{r},t)][u_i(\mathbf{r})u_j^*(\mathbf{r})\\&\quad+v_i(\mathbf{r})v_j^*(\mathbf{r})]+\delta\Phi(\mathbf{r},t)v_i(\mathbf{r})u_j^*(\mathbf{r})\\&\quad+\delta\Phi^*(\mathbf{r},t)u_i(\mathbf{r})v_j^*(\mathbf{r})\}\end{aligned} \tag{26}$$

for the time evolution of f_{ij}, while the equation of motion for g_{ij} is given by

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}g_{ij}(t)&=(\epsilon_j+\epsilon_i)g_{ij}(t)+2g(1+f_i^0+f_j^0)\int d\mathbf{r}\Phi_0(\mathbf{r})\\&\quad\times\{[\delta\Phi(\mathbf{r},t)+\delta\Phi^*(\mathbf{r},t)][u_i^*(\mathbf{r})v_j^*(\mathbf{r})\\&\quad+v_i^*(\mathbf{r})u_j^*(\mathbf{r})]+\delta\Phi(\mathbf{r},t)u_i^*(\mathbf{r})u_j^*(\mathbf{r})\\&\quad+\delta\Phi^*(\mathbf{r},t)v_i^*(\mathbf{r})v_j^*(\mathbf{r})\}.\end{aligned} \tag{27}$$

In Eqs. (26) and (27), f_j^0 is the equilibrium density of quasiparticles

$$f_j^0=\langle\alpha_j^\dagger\alpha_j\rangle_0=[\exp(\beta\epsilon_j)-1]^{-1}, \tag{28}$$

in terms of which the noncondensate particle density, at equilibrium, is written as

$$\tilde{n}^0(\mathbf{r})=\sum_j \{[|u_j(\mathbf{r})|^2+|v_j(\mathbf{r})|^2]f_j^0+|v_j(\mathbf{r})|^2\}. \tag{29}$$

The last term in Eq. (29) accounts at $T=0$ for the quantum depletion of the condensate. The fluctuations $\delta\tilde{n}$ and $\delta\tilde{m}$ of the normal and anomalous particle densities can be straightforwardly expressed in terms of $f_{ij}(t)$ and $g_{ij}(t)$, and Eq. (13) for the oscillations of the condensate takes the final form

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\delta\Phi(\mathbf{r},t)&=\bigg(-\frac{\hbar^2\nabla^2}{2m}+V_{\rm ext}(\mathbf{r})-\mu+2gn(\mathbf{r})\bigg)\delta\Phi(\mathbf{r},t)\\&\quad+gn_0(\mathbf{r})\delta\Phi^*(\mathbf{r},t)\\&\quad+g\Phi_0(\mathbf{r})\sum_{ij}\{2[u_i^*(\mathbf{r})u_j(\mathbf{r})+v_i^*(\mathbf{r})v_j(\mathbf{r})\\&\quad+v_i^*(\mathbf{r})u_j(\mathbf{r})]f_{ij}(t)+[2v_i(\mathbf{r})u_j(\mathbf{r})\\&\quad+u_i(\mathbf{r})u_j(\mathbf{r})]g_{ij}(t)+[2u_i^*(\mathbf{r})v_j^*(\mathbf{r})\\&\quad+v_i^*(\mathbf{r})v_j^*(\mathbf{r})]g_{ij}^*(t)\}.\end{aligned} \tag{30}$$

The equations of motion (26), (27), and (30) describe the small-amplitude coupled oscillations of the condensate and noncondensate particles in the collisionless regime. They represent the main result of this section.

III. DAMPING OF THE OSCILLATIONS

Let us suppose that the condensate oscillates with frequency ω

$$\delta\Phi(\mathbf{r},t)=\delta\Phi_1(\mathbf{r})e^{-i\omega t}, \quad \delta\Phi^*(\mathbf{r},t)=\delta\Phi_2(\mathbf{r})e^{-i\omega t}. \tag{31}$$

Through the coupling term in the equations of motion (26) and (27), the fluctuations of the condensate act as a time-dependent external drive inducing oscillations in f_{ij} and g_{ij}. The Fourier component of f_{ij} at the driving frequency ω is given by

$$\begin{aligned} f_{ij}(\omega)=2g\,\frac{f_i^0-f_j^0}{\hbar\omega+(\epsilon_i-\epsilon_j)+i0}\int d\mathbf{r} \\ \times\Phi_0[\delta\Phi_1(u_iu_j^*+v_iv_j^*+v_iu_j^*) \\ +\delta\Phi_2(u_iu_j^*+v_iv_j^*+u_iv_j^*)], \end{aligned} \tag{32}$$

where the frequency ω has been chosen with an infinitesimally small component on the positive imaginary axis. Analogously, for the component of g_{ij} oscillating at the frequency ω, one finds

$$\begin{aligned} g_{ij}(\omega)=2g\,\frac{1+f_i^0+f_j^0}{\hbar\omega-(\epsilon_i+\epsilon_j)+i0}\int d\mathbf{r} \\ \times\Phi_0[\delta\Phi_1(u_i^*v_j^*+v_i^*u_j^*+u_i^*u_j^*) \\ +\delta\Phi_2(u_i^*v_j^*+v_i^*u_j^*+v_i^*v_j^*)]. \end{aligned} \tag{33}$$

If one neglects in Eq. (30) the coupling terms involving the fluctuations of the noncondensate particles, the condensate components $\delta\Phi_1^0,\delta\Phi_2^0$ satisfy the unperturbed RPA equation [6],

$$\begin{pmatrix} \mathcal{L} & gn_0(\mathbf{r}) \\ -gn_0(\mathbf{r}) & -\mathcal{L} \end{pmatrix}\begin{pmatrix} \delta\Phi_1^0(\mathbf{r}) \\ \delta\Phi_2^0(\mathbf{r}) \end{pmatrix}=\hbar\omega_0\begin{pmatrix} \delta\Phi_1^0(\mathbf{r}) \\ \delta\Phi_2^0(\mathbf{r}) \end{pmatrix}, \tag{34}$$

and the normalization condition

$$\int d\mathbf{r}(|\delta\Phi_1^0|^2-|\delta\Phi_2^0|^2)=1, \tag{35}$$

where ω_0 is the unperturbed RPA eigenfrequency of the mode. We treat the coupling terms in Eq. (30) as small perturbations and we write the solution of the normal mode as

$$\begin{pmatrix} \delta\Phi_1 \\ \delta\Phi_2 \end{pmatrix}=\begin{pmatrix} \delta\Phi_1^0 \\ \delta\Phi_2^0 \end{pmatrix}+\begin{pmatrix} \delta\Phi_1' \\ \delta\Phi_2' \end{pmatrix}, \tag{36}$$

and the perturbed eigenfrequency as

$$\hbar\omega=\hbar\omega_0+\delta E-i\gamma. \tag{37}$$

In Eq. (37), δE represents the shift in the real part of the frequency and γ is the damping coefficient, while the eigenvector correction in Eq. (36) is chosen to be orthogonal to the unperturbed eigenvector,

$$\int d\mathbf{r}(\delta\Phi_1'^*\delta\Phi_1^0-\delta\Phi_2'^*\delta\Phi_2^0)=0. \tag{38}$$

Starting from the equations for the perturbed components $\delta\Phi_1$ and $\delta\Phi_2$ [Eq. (30) and its complex conjugate] we multiply the first equation by $\delta\Phi_1^*$ and the latter by $\delta\Phi_2^*$, take the difference of the two equations, and finally integrate over space. By using Eqs. (35) and (38) we get the following relation for the perturbed eigenfrequency:

$$\begin{aligned} \hbar\omega=\hbar\omega_0+4g^2\sum_{ij}(f_i^0-f_j^0)\,\frac{|A_{ij}|^2}{\hbar\omega_0+(\epsilon_i-\epsilon_j)+i0} \\ +2g^2\sum_{ij}(1+f_i^0+f_j^0)\left(\frac{|B_{ij}|^2}{\hbar\omega_0-(\epsilon_i+\epsilon_j)+i0}\right. \\ \left.-\frac{|\widetilde{B}_{ij}|^2}{\hbar\omega_0+(\epsilon_i+\epsilon_j)+i0}\right), \end{aligned} \tag{39}$$

where the matrix elements A_{ij}, B_{ij}, and $\widetilde{B}_{ij}$ are, respectively, given by

$$\begin{aligned} A_{ij}=\int d\mathbf{r}\Phi_0[\delta\Phi_1^0(u_iu_j^*+v_iv_j^*+v_iu_j^*) \\ +\delta\Phi_2^0(u_iu_j^*+v_iv_j^*+u_iv_j^*)], \\ B_{ij}=\int d\mathbf{r}\Phi_0[\delta\Phi_1^0(u_i^*v_j^*+v_i^*u_j^*+u_i^*u_j^*) \\ +\delta\Phi_2^0(u_i^*v_j^*+v_i^*u_j^*+v_i^*v_j^*)], \\ \widetilde{B}_{ij}=\int d\mathbf{r}\Phi_0[\delta\Phi_1^0(u_iv_j+v_iu_j+v_iv_j) \\ +\delta\Phi_2^0(u_iv_j+v_iu_j+u_iu_j)]. \end{aligned} \tag{40}$$

The imaginary part of the right-hand side of Eq. (39) gives the damping coefficient γ. There are two distinct contributions to γ: one arises from the process of one quantum of oscillation $\hbar\omega_0$ being absorbed by a thermal excitation with energy ϵ_i, which is turned into another thermal excitation with energy $\epsilon_j=\epsilon_i+\hbar\omega_0$. This mechanism is known as Landau (L) damping and is given by the imaginary part of the second term on the right-hand side of Eq. (39),

$$\gamma_L=4\pi g^2\sum_{ij}|A_{ij}|^2(f_i^0-f_j^0)\,\delta(\hbar\omega_0+\epsilon_i-\epsilon_j). \tag{41}$$

The above result coincides with the finding of Ref. [14] obtained by direct use of perturbation theory. A different mechanism of damping comes from the process where a quantum of oscillation $\hbar\omega_0$ is absorbed and two excitations with energies $\epsilon_i+\epsilon_j=\hbar\omega_0$ are created. The damping associated with the decay of an elementary excitation into a pair of excitations, which has been studied by Beliaev in the case of uniform Bose superfluids [22], is present also at $T=0$, but is not active for the lowest energy modes in the case of a trapping potential because of the discretization of levels. The Beliaev (B) damping is given by the imaginary part of the first term in brackets on the right-hand side of Eq. (39), and reads

$$\gamma_B = 2\pi g^2 \sum_{ij} |B_{ij}|^2 (1 + f_i^0 + f_j^0)\, \delta(\hbar\omega_0 - \epsilon_i - \epsilon_j). \quad (42)$$

The total damping coefficient is the sum of the two contributions $\gamma = \gamma_L + \gamma_B$, the Beliaev damping becomes dominant at low temperatures ($k_B T \ll \hbar\omega_0$), while the Landau damping becomes dominant in the opposite regime of temperatures ($k_B T \gg \hbar\omega_0$).

In the case of magnetically trapped Bose gases, results (41) and (42) give the damping coefficient γ both at low and high temperatures in the collisionless regime. This quantity has been measured at JILA over a wide range of temperatures for the $m=0$ and $m=2$ modes [3], revealing a very strong temperature dependence. To carry out the calculation at a given temperature T one must start from the unperturbed condensate eigenfrequency with the required symmetry, obtained from Eq. (34), and the elementary excitation energies ϵ_j calculated from Eqs. (23). One has then to calculate the matrix elements A_{ij} and B_{ij} [see Eq. (40)] and then carry out the double summation in Eqs. (41) and (42). The explicit calculation of γ in magnetically trapped gases will be the object of a future work.

By analyzing the real part of Eq. (39), one can calculate the eigenfrequency shift δE in the Popov approximation. However, for temperatures $k_B T \leq \mu$ the effects arising from the equilibrium anomalous density $\tilde{m}^0(\mathbf{r})$ are important for the calculation of the frequency shift and cannot be neglected. In Sec. IV C we will present a calculation of the velocity of sound at $T=0$ for a uniform gas, and it will be shown that the inclusion of $\tilde{m}^0$ is crucial to obtain the correct result. From this finding we conclude that a reliable calculation of the frequency shift δE should be based on a more accurate dynamic theory, beyond the simple Popov ansatz (11).

IV. UNIFORM BOSE-CONDENSED GASES

In this section we apply the results (41) and (42) to uniform Bose gases, for which we reproduce well known results for the damping of phonons, both at finite and zero temperature, in the collisionless regime.

For homogeneous systems the stationary condensate wave function is constant throughout space $\Phi_0 = \sqrt{n_0}$, while the condensate fluctuations and the excitations are described by plane wave functions,

$$\delta\Phi_1(\mathbf{r}) = \frac{u_q}{\sqrt{V}} e^{i\mathbf{q}\cdot\mathbf{r}}, \quad \delta\Phi_2(\mathbf{r}) = \frac{v_q}{\sqrt{V}} e^{i\mathbf{q}\cdot\mathbf{r}},$$

$$u_{\mathbf{p}}(\mathbf{r}) = \frac{u_p}{\sqrt{V}} e^{i\mathbf{p}\cdot\mathbf{r}}, \quad v_{\mathbf{p}}(\mathbf{r}) = \frac{v_p}{\sqrt{V}} e^{i\mathbf{p}\cdot\mathbf{r}}, \quad (43)$$

where u_p and v_p are real functions defined through the usual Bogoliubov relations

$$u_p^2 = 1 + v_p^2 = \frac{(\epsilon_p^2 + g^2 n_0^2)^{1/2} + \epsilon_p}{2\epsilon_p},$$

$$u_p v_p = -\frac{g n_0}{2\epsilon_p}, \quad (44)$$

and ϵ_p is the energy of the elementary excitations as obtained from the Bogoliubov equations (23),

$$\epsilon_p = \left[\left(\frac{p^2}{2m} + g n_0\right)^2 - g^2 n_0^2\right]^{1/2}. \quad (45)$$

In these equations, $n_0 \equiv n_0(T)$ is the condensate density at the given temperature T obtained from the self-consistent calculation in thermodynamic equilibrium (see Ref. [7]).

A. Thermal regime $\hbar\omega_0 \ll k_B T$

We consider the long-wavelength limit for the fluctuations of the condensate

$$\hbar\omega_0 \equiv \epsilon_q \simeq cq, \quad (46)$$

where $c = \sqrt{g n_0 / m}$ is the velocity of sound calculated at temperature T. In this limit the u and v functions associated with the oscillations of the condensate can be expanded as

$$u_q \simeq \left(\frac{mc^2}{2\epsilon_q}\right)^{1/2} + \frac{1}{2}\left(\frac{\epsilon_q}{2mc^2}\right)^{1/2},$$

$$v_q \simeq -\left(\frac{mc^2}{2\epsilon_q}\right)^{1/2} + \frac{1}{2}\left(\frac{\epsilon_q}{2mc^2}\right)^{1/2}. \quad (47)$$

As has been shown in Ref. [14], using the above expansion for u_q and v_q one obtains the following low-q behavior for the matrix elements $A_{\mathbf{pp}'}$ defined in Eqs. (40):

$$A_{\mathbf{pp}'} = \delta_{\mathbf{p}',\mathbf{p}+\mathbf{q}} \frac{\sqrt{n_0}}{\sqrt{V}} \left(\frac{\epsilon_q}{2mc^2}\right)^{1/2} \Bigg(u_p^2 + v_p^2 + u_p v_p - \frac{v_g}{c} \cos\theta \frac{2u_p^2 v_p^2}{u_p^2 + v_p^2} \Bigg), \quad (48)$$

where $\delta_{\mathbf{pp}'}$ is the Kronecker symbol, θ is the angle formed between the directions of $\mathbf{p}$ and $\mathbf{q}$, and $v_g = \partial\epsilon_p/\partial p$ is the group velocity of the excitations. By introducing the above expansion into Eq. (41) and carrying out the integration over θ, one finds the result (see Ref. [14])

$$\frac{\gamma}{\epsilon_q} \simeq \frac{\gamma_L}{\epsilon_q} = (a^3 n_0)^{1/2} F(\tau), \quad (49)$$

where $\tau = k_B T / mc^2$ is a reduced temperature and $F(\tau)$ is the dimensionless function

$$F(\tau) = 8\sqrt{\pi} \int dx (e^x - e^{-x})^{-2} \left(1 - \frac{1}{2u} - \frac{1}{2u^2}\right)^2, \quad (50)$$

where we have introduced the quantity $u = \sqrt{1 + 4\tau^2 x^2}$.

For temperatures $\tau \gg 1$ the function F takes the asymptotic limit $F \to 3\pi^{3/2}\tau/4$, and the damping coefficient is given by

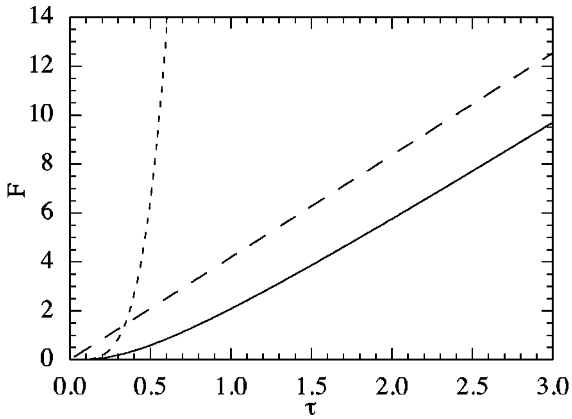

FIG. 1. Dimensionless function F as a function of τ (solid line). The asymptotic behaviors for $\tau \ll 1$ (dashed line) and for $\tau \gg 1$ (long-dashed line) are also reported.

$$\frac{\gamma}{\epsilon_q} = \frac{3\pi}{8} \frac{k_B T a}{\hbar c}. \tag{51}$$

This regime of temperatures was first investigated by Szépfalusy and Kondor [23]. As discussed in Ref. [14], the numerical coefficient in Eq. (51) coincides with the one obtained in [20], while it slightly differs from the one of [23].

In the opposite regime of temperatures, $\tau \ll 1$, one finds the asymptotic limit $F \to 3\pi^{9/2}\tau^4/5$ and one recovers the well known result for the phonon damping [17,24,25,14],

$$\frac{\gamma}{\epsilon_q} = \frac{3\pi^3}{8} \frac{(k_B T)^4}{m n \hbar^3 c^5}. \tag{52}$$

In Fig. 1, the dimensionless function $F(\tau)$ is plotted as a function of τ together with its asymptotic behavior both at small and large τ's. One can see that F departs rather soon from the low-temperature τ^4 behavior, while it approaches the high-temperature linear law very slowly.

B. Quantum regime $\hbar\omega_0 \gg k_B T$

At $T=0$, the damping of the long-wavelength fluctuations of the condensate with energy (46) is obtained through result (42) with $f_j^0 = 0$. For uniform gases the matrix elements $B_{\mathbf{pp}'}$ entering Eq. (42) read

$$B_{\mathbf{pp}'} = \delta_{\mathbf{p}',-\mathbf{p}+\mathbf{q}} \frac{\sqrt{n_0}}{\sqrt{V}} [u_q(u_p v_{p-q} + v_p u_{p-q} + u_p u_{p-q}) + v_q(u_p v_{p-q} + v_p u_{p-q} + v_p v_{p-q})]. \tag{53}$$

In the Beliaev damping mechanism the momenta of the three excitations involved in the process are comparable, $q \simeq p \simeq |\mathbf{p}-\mathbf{q}|$. For $p \ll mc$ one can use the following expansions for the excitation energies ϵ_p and the functions u_p and v_p:

$$\epsilon_p \simeq cp + \frac{p^3}{8m^2c}, \tag{54}$$

$$u_p \simeq \left(\frac{mc}{2p}\right)^{1/2} + \frac{1}{2}\left(\frac{p}{2mc}\right)^{1/2} + \frac{1}{8}\left(\frac{p}{2mc}\right)^{3/2} - \frac{1}{8}\left(\frac{p}{2mc}\right)^{5/2},$$

$$v_p \simeq -\left(\frac{mc}{2p}\right)^{1/2} + \frac{1}{2}\left(\frac{p}{2mc}\right)^{1/2} - \frac{1}{8}\left(\frac{p}{2mc}\right)^{3/2} - \frac{1}{8}\left(\frac{p}{2mc}\right)^{5/2}.$$

If one substitutes the above expressions for all the u's and v's in Eq. (53) and makes use of the condition for energy conservation

$$|\mathbf{p}-\mathbf{q}| + p = q + \frac{1}{8m^2c^2}(q^3 + p^3 + |\mathbf{p}-\mathbf{q}|^3), \tag{55}$$

after long but straightforward algebra one gets the result

$$B_{\mathbf{pp}'} = \delta_{\mathbf{p}',-\mathbf{p}+\mathbf{q}} \frac{\sqrt{n_0}}{\sqrt{V}} \frac{3}{4\sqrt{2}} \frac{(qp|\mathbf{p}-\mathbf{q}|)^{1/2}}{(mc)^{3/2}}. \tag{56}$$

Once the low-energy behavior of the matrix elements of B has been obtained, the calculation of the damping coefficient follows directly:

$$\gamma \simeq \gamma_B = \frac{3q^5}{640\pi\hbar^3 mn}, \tag{57}$$

and coincides with the well known result first obtained by Beliaev using diagrammatic techniques [22,26] (for a review of the second-order Beliaev approximation at $T=0$ and its extension to finite temperature, see Ref. [20]).

C. Bulk compressibility

In this subsection we calculate the isothermal inverse compressibility $\kappa_T^{-1} = n^2(\partial\mu/\partial n)_{V,T}$. In a uniform system at $T=0$ it is fixed by the velocity of sound through the relation

$$\kappa_{T=0}^{-1} = \rho\left(\frac{\partial P}{\partial \rho}\right)_{N,T=0} = \rho c^2(T=0), \tag{58}$$

where $\rho = mn$ is the mass density. We will show that the Popov approximation gives an incorrect result for the inverse compressibility in the low-temperature regime and that one must go beyond this approximation in order to obtain the correct low-T behavior of κ_T^{-1}.

Without making the Popov assumption (11) and including the term proportional to $\tilde{m}^0(\mathbf{r})$, the stationary equation for the real wave function $\Phi_0(\mathbf{r})$ becomes

$$\left(-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu + g[n_0(\mathbf{r}) + 2\tilde{n}^0(\mathbf{r}) + \tilde{m}^0(\mathbf{r})]\right)\Phi_0(\mathbf{r}) = 0, \tag{59}$$

instead of Eq. (12). In terms of the quasiparticle operators defined in Eq. (14), the anomalous density at equilibrium is given by

$$\widetilde{m}^0(\mathbf{r})=\sum_j [2u_j(\mathbf{r})v_j^*(\mathbf{r})f_j^0+u_j(\mathbf{r})v_j^*(\mathbf{r})]. \tag{60}$$

For a uniform system the above equation (59) fixes the value of the chemical potential

$$\mu=gn+g(\widetilde{n}^0+\widetilde{m}^0) \tag{61}$$

which differs from the value $\mu_P=g(n+\widetilde{n}^0)$ obtained in the Popov approximation [27]. At $T=0$ this approximation gives $\mu_P(T=0)=4\pi\hbar^2an[1+8(a^3n)^{1/2}/3\sqrt{\pi}]/m$, where the correction to gn comes from the quantum depletion of the condensate. The $a^{5/2}$ order of the correction is correct *but* the numerical coefficient is wrong. To obtain the $T=0$ chemical potential to order $a^{5/2}$, one must use Eq. (61) and expand the coupling constant g to second order in the scattering length,

$$g=\frac{4\pi\hbar^2a}{m}\left(1+\frac{4\pi\hbar^2a}{m}\frac{1}{V}\sum_{\mathbf{p}}\frac{m}{p^2}\right). \tag{62}$$

The need of the above renormalization of g for the calculation of the ground-state energy of a Bose gas has been first pointed out by Lee, Huang, and Yang [28] and is discussed in many textbooks (see, e.g., Ref. [29]). By substituting the renormalized value of g into the first term of Eq. (61), one gets

$$\begin{aligned}\mu(T=0)&=\frac{4\pi\hbar^2an}{m}\left[1+\frac{8}{3\sqrt{\pi}}(a^3n)^{1/2}+\frac{4\pi\hbar^2a}{m}\frac{1}{V}\sum_{\mathbf{p}}\left(\frac{m}{p^2}-\frac{1}{2\epsilon_p}\right)\right]\\&=\frac{4\pi\hbar^2an}{m}\left(1+\frac{32}{3\sqrt{\pi}}(a^3n)^{1/2}\right),\end{aligned} \tag{63}$$

which coincides with the well-known result for the chemical potential of a dilute Bose gas at $T=0$ [30]. In the above equation we have used Eqs. (43) and (44) for the product $u_{\mathbf{p}}(\mathbf{r})v_{\mathbf{p}}^*(\mathbf{r})$ with the Bogoliubov spectrum (45). Notice that in the integral over $\mathbf{p}$ the ultraviolet divergencies arising from the renormalization of g and from $\widetilde{m}^0$ cancel out. By differentiating Eq. (63) with respect to n one obtains for the velocity of sound

$$c(T=0)=\sqrt{\frac{4\pi\hbar^2an}{m^2}}\left(1+\frac{8}{\sqrt{\pi}}(a^3n)^{1/2}\right), \tag{64}$$

a result which was first derived by Beliaev [22]. In a self-consistent dynamic theory, result (64) should also be obtained from the frequency shift of the long-wavelength excitations. However, it cannot be found within the Popov approximation because the crucial ingredients, renormalization of g and equilibrium value of the anomalous density, are not accounted for in this approximation.

At finite T one gets from Eq. (61)

$$\begin{aligned}\mu(T)=&\frac{4\pi\hbar^2an}{m}\left(1+\frac{32}{3\sqrt{\pi}}\frac{n_0}{n}(a^3n_0)^{1/2}\right)\\&+\frac{4\pi\hbar^2an}{m}\frac{\sqrt{32}}{\sqrt{\pi}}\frac{n_0}{n}(a^3n_0)^{1/2}\tau\\&\times\int_0^\infty dx\,\frac{(\sqrt{1+\tau^2x^2}-1)^{3/2}}{\sqrt{1+\tau^2x^2}}\frac{1}{e^x-1},\end{aligned} \tag{65}$$

where $\tau=k_BT/mc^2$ is the reduced temperature. Result (65) coincides with the finding of Ref. [20] obtained using the second-order Beliaev approximation. In the low-temperature regime, $\tau\ll 1$, one has

$$\mu(T)=\mu(T=0)+\frac{\pi^2}{60}\frac{(k_BT)^4}{n\hbar^3c^3} \tag{66}$$

for the chemical potential, and

$$\kappa_T^{-1}=\rho c^2(T=0)-\frac{\pi^2}{24}\frac{(k_BT)^4}{\hbar^3c^3} \tag{67}$$

for the inverse compressibility. The above results coincide with the ones obtained from the thermodynamic relation $\mu=(\partial F_{\rm ph}/\partial N)_{V,T}$, where $F_{\rm ph}$ is the free energy of a phonon gas. In the same temperature regime, $k_BT\ll mc^2$, the Popov approximation gives $\mu_P(T)=\mu_P(T=0)+m^2c(k_BT)^2/(12n\hbar^3)$ and the inverse compressibility exhibits an unphysical T^2 dependence which is not consistent with the T^4 dependence obtained by differentiating the phonon free energy.

In the low-temperature limit the frequency shift of collisionless phonons is proportional to $T^4\log(T)$ [24,25], whereas the Popov approximation again yields an unphysical T^2 dependence arising from the low-T expansion of $n_0(T)$. The analogy with the result for the chemical potential suggests that also for this calculation the inclusion of $\widetilde{m}^0$ is crucial to obtain the correct result.

From the above results we conclude that a self-consistent dynamic theory, aiming to describe both the damping and the frequency shifts of the oscillations of the condensate, should go beyond the Popov ansatz (11). The ingredients that this theory should contain are the following: (i) the equilibrium anomalous density $\widetilde{m}^0$ has to be taken into account, (ii) the renormalization (62) of the interaction coupling constant is needed in order to reproduce the energetics at $T=0$, and (iii) the elementary excitation energies ϵ_p have to be gapless and must coincide with the Bogoliubov spectrum at low temperatures.

ACKNOWLEDGMENTS

The author wishes to thank L. P. Pitaevskii and S. Stringari for enlightening discussions and useful criticisms about the manuscript. Useful remarks made by A. Griffin and F. Dalfovo are also gratefully acknowledged.

[1] D. S. Jin, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **77**, 420 (1996).
[2] M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, C. G. Townsend, and W. Ketterle, Phys. Rev. Lett. **77**, 988 (1996).
[3] D. S. Jin, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **78**, 764 (1997).
[4] K. G. Singh and D. S. Rokhsar, Phys. Rev. Lett. **77**, 1667 (1996); S. Stringari, *ibid.* **77**, 2360 (1996); M. Edwards, P. A. Ruprecht, K. Burnett, R. J. Dodd, and C. W. Clark, *ibid.* **77**, 1671 (1996); B. D. Esry, Phys. Rev. A **55**, 1147 (1997).
[5] A. Griffin, Phys. Rev. B **53**, 9341 (1996).
[6] D. A. W. Hutchinson, E. Zaremba, and A. Griffin, Phys. Rev. Lett. **78**, 1842 (1997).
[7] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A **54**, 4633 (1996); Phys. Rev. Lett. **78**, 3987 (1997); J. Low Temp. Phys. **109**, 309 (1997).
[8] A. Minguzzi, S. Conti, and M. P. Tosi, J. Phys.: Condens. Matter **9**, L33 (1997).
[9] R. J. Dodd, M. Edwards, C. W. Clark, and K. Burnett, e-print cond-mat/9708139.
[10] F. Dalfovo, S. Giorgini, M. Guilleumas, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A **56**, 3840 (1997).
[11] E. Zaremba, A. Griffin, and T. Nikuni, e-print cond-mat/9705134.
[12] A. Griffin, W.-C. Wu, and S. Stringari, Phys. Rev. Lett. **78**, 1838 (1997).
[13] W. V. Liu and W. C. Shieve, e-print cond-mat/9702122.
[14] L. P. Pitaevskii and S. Stringari, Phys. Lett. A **235**, 398 (1997).
[15] W. V. Liu, e-print cond-mat/9708080.
[16] P. O. Fedichev, G. V. Shlyapnikov, and J. M. T. Walraven, e-print cond-mat/9710128.
[17] P. C. Hohenberg and P. C. Martin, Ann. Phys. (N.Y.) **34**, 291 (1965).
[18] A similar time-dependent mean-field approach has also been developed in A. Minguzzi and M. P. Tosi, e-print cond-mat/9709323; and in the framework of the T-matrix approximation in N. P. Proukakis, K. Burnett, and H. T. C. Stoof, e-print cond-mat/9703199.
[19] V. N. Popov, Zh. Eksp. Teor. Fiz. **47**, 1759 (1964) [Sov. Phys. JETP **20**, 1185 (1965)]; V. N. Popov, *Functional Integrals and Collective Excitations* (Cambridge University Press, Cambridge, 1987).
[20] H. Shi, Ph.D. thesis, University of Toronto (1997).
[21] V. V. Goldman, I. F. Silvera, and A. J. Leggett, Phys. Rev. B **24**, 2870 (1981); D. A. Huse and E. D. Siggia, J. Low Temp. Phys. **46**, 137 (1982).
[22] S. T. Beliaev, Zh. Eksp. Teor. Fiz. **34**, 433 (1958) [Sov. Phys. JETP **7**, 299 (1958)].
[23] P. Szépfalusy and I. Kondor, Ann. Phys. (N.Y.) **82**, 1 (1974).
[24] A. Andreev and I. Khalatnikov, Zh. Eksp. Teor. Fiz. **44**, 2508 (1963) [Sov. Phys. JETP **17**, 1384 (1963)].
[25] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Pergamon, Oxford, 1981), pp. 327–328.
[26] Beliaev damping can also be calculated at finite temperature and it is given by $\gamma_B(T)=\gamma_B(T=0)n/n_0\{1+60\int_0^1 dx\, x^2(x-1)^2/[\exp(xcq/k_BT)-1]\}$. In the quantum regime $(k_BT\ll cq)$ one has $\gamma_B(T)=\gamma_B(T=0)[1+120(k_BT/cq)^3]$, while in the thermal regime $(k_BT\gg cq)$ one finds $\gamma_B(T)=5\gamma_B(T=0)k_BTn/(n_0cq)$ and the Beliaev damping is overwhelmed by Landau damping.
[27] For a discussion on the chemical potential in various approximations for Bose-condensed systems and its connection to the Hugenholtz-Pines theorem, see Ref. [5].
[28] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).
[29] R. K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1972), pp. 350–355.
[30] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics* (Pergamon, Oxford, 1980), Pt. 2, p. 101.