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Lifshitz theory of Casimir forces at finite temperature

B. W. Ninham[1,2,3] and J. Daicic[1,*]

[1]Department of Applied Mathematics, Research School of Physical Sciences and Engineering, Australian National University, Canberra, Australian Capital Territory 0200, Australia

[2]Physical Chemistry 1, Center for Chemistry and Chemical Engineering, University of Lund, Box 124, SE-22100 Lund, Sweden

[3]Institute for Surface Chemistry, Box 5607, SE-11486 Stockholm, Sweden

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We reexamine the Casimir and Lifshitz theories of intermolecular forces at nonzero temperature. For dilute media and atomic interactions, the limits of validity of the London–van der Waals potential between atoms at finite temperature are established by a detailed asymptotic analysis. In the retarded limit, the Casimir-Polder interaction potential is shown to be rigorously correct only in the limit of zero temperature. At any nonzero temperature a different analytic form obtains and is derived. We then consider Casimir forces between perfectly conducting plates. Existing results for the case of intervening vacuum are recovered by a different method. Moreover, we show that the Mellin transform technique and theory of generalized ζ functions allows a detailed asymptotic treatment of a system of perfectly conducting plates with an intervening electron plasma, useful in the modeling of forces between metal plates, where the finite metallic skin depth is an important consideration. [S1050-2947(98)04203-6]

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I. INTRODUCTION

The remarkable conjecture of Casimir [1] in 1948 and its experimental confirmation in the period since, most recently by Lamoreaux [2], that two uncharged, conducting plates in vacuum attract each other have wide-ranging ramifications for systems where quantum fields mediate interactions and generally for constrained quantum fields [3]. A closely related phenomenon, the attraction between atoms and surfaces due to polarization-induced dipole creation and its modification by the effects of retardation (i.e., the finite velocity of light), was also illuminated by Casimir and Polder [4] and reconciled with the earlier molecular theories of these van der Waals–type forces, such as that of London [5] for the (nonretarded) interaction between neutral atoms.

Lifshitz [6] recast these problems in terms of interactions between continuous media of well-defined (or at least independently measurable) dielectric susceptibilities, mediated by the quantum electromagnetic field. The Casimir and Casimir-Polder results were recovered as limiting cases of this more general theory [7].

Experimental confirmation of the Casimir-Polder interaction has been demonstrated recently by Sukenik *et al.* [8], who measured the deflection of ground-state sodium atoms passing through a parallel-plate cavity, and has now been followed by the work of Lamoreaux, who accurately measured the leading term in an expansion for the Casimir force between conducting gold surfaces at separations of 0.6–6 μm. The latter follows a history of attempts to measure Casimir forces, which began with Abrikosova and Derjaguin [9] and include the important work of Sparnaay [10] and the observation of the crossover from the nonretarded to retarded regimes by Tabor and Winterton [11] and Israelachvili and Tabor [12] using crossed cylinders of muscovite mica.

Brown and Maclay [13], through their formulation of the problem in terms of the energy density of the vacuum, laid the groundwork for the modern theoretical interpretation of these forces in terms of a shift in the zero-point energy of the vacuum due to the presence of constraints or boundaries. As such, the Casimir interaction has found wide application in theoretical physics, from quantum-chromodynamical bag models of nucleons [14] to its role in theories of a universe that is topologically closed or has undergone a dimensional compactification [15]. However, it is the domain of the discovery of Casimir phenomena, colloid and interface science, that continues to be the most readily accessible to experiment. Even here, questions regarding the true nature of (in particular, long-range) intermolecular and surface forces remain, at least in part, unanswered. Experiments are difficult and require extraordinary efforts in system idealization and noise control. In addition, as we shall show herein, the development of theory is not, as it stands, complete. Rather, a return to the Lifshitz theory, with the application of the appropriate mathematical tools, reveals insights into the nature of these intermolecular forces, with significant consequences for real experimental systems, where issues such as nonzero temperature and nonideal surfaces need to be considered.

We are by no means the first to tread this path, but we do hope to shed some light on some pressing issues for these important efforts to experimentally demonstrate these quite fundamental phenomena. These include the effects of nonzero temperature and finite skin depth in the case of metals. It is impossible to describe in full detail the whole gamut of works, in particular theoretical studies, that have dealt with the Casimir effect and its manifestation in colloidal systems. Since we focus on the effect of temperature and later intervening media, it is however possible to draw attention to that part of the literature relevant to these issues. The more general surveys are provided in three recent reviews by Plunien *et al.* [3], Mostepanenko and Trunov [16], and Elizalde and

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[*] Present address: Institute for Surface Chemistry, Box 5607, SE-11486 Stockholm, Sweden.

Romeo [17]. More specifically regarding the Lifshitz theory, the reader is referred to the seminal review by Dzyaloshinskii, Lifshitz, and Pitaevskii [7] and also to the book by Mahanty and Ninham [18].

Temperature corrections to the result of Casimir [1] for two conducting plates in vacuum were calculated by Lifshitz and co-workers [6,7] and later notably by Sauer [19], Mehra [20], Brown and Maclay [13], and Schwinger *et al.* [21]. Importantly, Schwinger *et al.* were able to reconcile what appeared to be discrepancies between the predictions of the Lifshitz theory and the results of others, notably those of Mehra, whose expressions were later confirmed by Brown and Maclay [13]. Some controversy continues [22], but Schwinger *et al.* [21] clearly showed that when the conducting limit was correctly taken in the Lifshitz theory, it yielded results consistent with those of Mehra, Sauer, and Brown and Maclay. We recover these results by a route somewhat different from that of Schwinger *et al.*, as a precursor for a calculation for a classical plasma bounded by two conducting plates using the Lifshitz theory. The related issue of how to treat imperfect conductors in vacuum has been tackled by Lifshitz and co-workers [6,7], Hargreaves [24], and Schwinger *et al.* [21]. A somewhat more sophisticated treatment, allowing for spatial dispersion effects, has been given by Heinrichs [23]. There is also a considerable body of work that deals with the temperature behavior of constrained bosonic and fermionic fields [3], which is complementary to the developments outlined herein.

In his experiment, Lamoreaux [2] was only able to measure the leading, temperature-independent term for the Casimir force between a spherical and flat surface with good accuracy. Corrections for temperature and the imperfect conductivity of the metal could not be deduced from the residuals. It is hoped that future measurements will improve on this and also be able to access separations less than 0.6 μm. On the other hand, Sukenik *et al.* [8] concluded that temperature effects (due, for example, to the blackbody field in the cavity) were not significant in their experiment. We shall show that the van der Waals force between atoms shows a remarkable temperature dependence, whereby the standard Casimir-Polder result for the retarded interaction free energy is only valid strictly at *zero* temperature, and we obtain a result for finite temperatures. We then proceed to consider Casimir forces between conducting plates, in vacuum, and with an intervening electron plasma.

II. LIFSHITZ THEORY

We begin by writing down the essential formulae of the general theory of Lifshitz and co-workers [6,7]. For two dielectric media (denoted 1) separated by a distance l and interacting across a medium (2), the free energy (per unit area) of the interaction is

$$F(l,T)=\frac{kT}{8\pi l^2}\sum_{n=0}^{\infty}{}' I(\xi_n,l), \quad \xi_n=\frac{2\pi nkT}{\hbar}, \tag{1}$$

$$I(\xi_n,l)\equiv\left(\frac{2\xi_n l\sqrt{\varepsilon_2}}{c}\right)^2\int_1^\infty dpp\left(\left\{\ln\left[1-\Delta^2\times\exp\left(\frac{-2p\xi_n l\sqrt{\varepsilon_2}}{c}\right)\right]\right\}+[\Delta\to\overline{\Delta}]\right), \tag{2}$$

$$\Delta=\left(\frac{q\varepsilon_2-p\varepsilon_1}{q\varepsilon_2+p\varepsilon_1}\right), \quad \overline{\Delta}=\left(\frac{q-p}{q+p}\right), \quad q=\sqrt{p^2-1+(\varepsilon_1/\varepsilon_2)}. \tag{3}$$

The prime on the summation symbol of Eq. (1) gives weight 1/2 to the term in $n=0$. The susceptibilities $\varepsilon=\varepsilon(i\xi_n)$ are evaluated on the imaginary frequency axis. For ideal conducting media of conductivities σ_j, magnetic susceptibilities [25] μ_j, Eqs. (3) are modified as

$$\overline{\Delta}\to\frac{\mu_2 q-\mu_1 p}{\mu_2 q+\mu_1 p}, \quad \varepsilon_2(\omega)\to\kappa\to\left(\varepsilon_2(\omega)+\frac{i\sigma_2}{\omega}\right),$$
$$\frac{2l\sqrt{\varepsilon_2}}{c}\to\frac{2l\sqrt{\kappa\mu}}{c}. \tag{4}$$

Formulas (1)–(4) contain a wealth of information, but are hardly transparent. They can be derived simply [26,18] by solving Maxwell's equations with appropriate boundary conditions to determine allowed surface modes and assigning a harmonic-oscillator free energy to each mode.

III. TEMPERATURE EFFECTS ON THE VAN DER WAALS INTERACTION

A. Interaction free energy in the nonretarded regime

As noted above, our first investigation will be that of the effects of temperature on the interaction between two atoms. This can be derived from Lifshitz's general theory by the methods of Ref. [18], p. 59. That is, we consider two nonconducting dilute media of dielectric susceptibility $\varepsilon(\omega)$ separated by a distance l.

Take the Lifshitz formula and expand to leading order in density, i.e., $\varepsilon-1$. Then from Eqs. (3)

$$\Delta^2\sim\frac{(\varepsilon-1)^2}{4}\left(1-\frac{1}{p^2}+\frac{1}{4p^4}\right), \quad \overline{\Delta}^2\sim(\varepsilon-1)^2\frac{1}{16p^4}. \tag{5}$$

The term $n=0$ in the sum over frequencies of Eq. (1) requires a separate treatment to avoid an indeterminacy [18] and yields

$$F_{n=0}(l,T)=-\frac{kT}{64\pi l^2}[\varepsilon(0)-1]^2, \tag{6}$$

This is curious since for nonpolar media, sources of fluctuations that would give rise to an explicitly classical statistical mechanical term ought not to exist. We will show how it in fact disappears from a complete analysis. Substituting Eqs. (5) into Eq. (1) we require

$$F_{n\neq 0}(l,T)=-\frac{kT}{8\pi l^2}(2\pi x)^2\sum_{n=1}^{\infty} n^2\int_1^{\infty}dp\,p \times\exp(-2\pi xpn)\,\frac{(\varepsilon-1)^2}{4}\left(1-\frac{1}{p^2}+\frac{1}{2p^4}\right), \tag{7}$$

where $x\equiv 2kTl/\hbar c$. For a nonpolar medium at low density we can take

$$\varepsilon(\omega)-1=\frac{4\pi\rho\alpha(0)}{1-\omega^2/\omega_0^2},$$

$$\varepsilon(i\xi_n)-1=\frac{4\pi\rho\alpha(0)}{1+\xi_n^2/\omega_0^2}\equiv\frac{B}{1+An^2}. \tag{8}$$

Here ω_0 is the principal absorption frequency of the atoms, ρ is the density, $\alpha(0)$ is the static polarizability, and

$$A=\left(\frac{2\pi kT}{\hbar\omega_0}\right)^2. \tag{9}$$

For, typically, hydrogen atoms, we can take $\alpha(0)=e^2/m\omega_0^2$, e and m are the electronic charge and mass, respectively, $\hbar\omega_0$ is the ground-state energy, and the value of A is around 10^{-1} at room temperature. Then we have

$$F_{n\neq 0}(l,T)=-\frac{kT}{32\pi l^2}(2\pi xB)^2\sum_{n=1}^{\infty}\frac{n^2}{(1+An^2)^2} \times\int_1^{\infty}dp\,p\left(1-\frac{1}{p^2}+\frac{1}{2p^4}\right)\exp(-2\pi xnp). \tag{10}$$

Using the representation

$$e^{-y}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\,y^{-s}\Gamma(s),\quad \mathrm{Re}\ s=c>0, \tag{11}$$

we can carry out the integral over p to obtain

$$F_{n\neq 0}(l,T)=\frac{kT}{32\pi l^2}(2\pi xB)^2\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(s)}{(2\pi x)^s}\left[\frac{1}{2-s}+\frac{1}{s}-\frac{1}{2(2+s)}\right]\sum_{n=1}^{\infty}\frac{1}{n^{s-2}(1+An^2)^2}$$
$$\equiv\frac{kT}{32\pi l^2}(2\pi xB)^2\mathcal{I}(A,x). \tag{12}$$

The sum over n converges for Re $s>-1$ and the contour satisfies Re $s>2$. The integral $\mathcal{I}$ is quite complicated because it involves two parameters $x=2kTl/\hbar c$ and $A=(2\pi kT/\hbar\omega_0)^2$. We seek an asymptotic expansion for small distance l, which can be done by closing the contour c to the left and evaluating the residues of the enclosed poles. The first pole is at $s=2$ due to the denominator $2-s$. The second is a double pole at $s=0$. [The third pole at $s=-1$ is also double, due to $\Gamma(s)$ and the pole of the sum at $s=-1$]. Evaluating the residues at these poles, we have

$$\mathcal{I}=-\frac{1}{(2\pi x)^2}\sum_{n=1}^{\infty}\frac{1}{(1+An^2)^2}+\left[\sum_{n=1}^{\infty}\frac{n^2}{(1+An^2)^2}\right]\left[\ln(2\pi x)+\gamma-\frac{1}{4}\right]+\sum_{n=1}^{\infty}\frac{n^2\ln n}{(1+An^2)^2}+\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(s)}{(2\pi x)^s}\left[\frac{1}{2-s}+\frac{1}{s}-\frac{1}{2(2+s)}\right]\left[\frac{\zeta(s+2)}{A^2}-\frac{1}{A^2}\sum_{n=1}^{\infty}\frac{1+2An^2}{n^{s+2}(1+An^2)^2}\right], \tag{13}$$

where $\zeta(z)$ is the ζ function of Riemann, γ is Euler's constant, and the contour now satisfies $-1<c<0$. We have exhibited the double pole at $s=-1$ explicitly by rearranging the last sum.

To make this expansion a little more explicit, we choose $A=(2\pi kT/\hbar\omega_0)^2\ll 1$ and define $z=\pi/\sqrt{A}\to\infty$. Thus we seek the expansion for the free-energy density in the nonretarded limit, i.e., the temperature corrections to the energy that would be obtained by summing pairwise London forces. Then, after some algebra, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{(1+An^2)^2}=\left(\frac{z}{4}\coth z+\frac{z^2}{4}\mathrm{csch}^2 z-\frac{1}{2}\right)\sim\left(\frac{z}{4}-\frac{1}{2}\right)+\left(z^2+\frac{z}{2}\right)e^{-2z}+O(e^{-4z}), \tag{14}$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(1+An^2)^2}=\frac{z^3}{4\pi^2}\coth z-\frac{z^4}{4\pi^2}\mathrm{csch}^2 z \sim\frac{z^3}{4\pi^2}(1+2e^{-2z})-\frac{z^4}{\pi^2}e^{-2z}+O(e^{-4z}), \tag{15}$$

$$\sum_{n=1}^{\infty}\frac{n^2\ln n}{(1+An^2)^2}=-\frac{\pi}{8A^{3/2}}\ln A+\frac{\pi}{4A^{3/2}}+\frac{\zeta(3)}{4\pi^2}+O(A), \tag{16}$$

$$\sum_{n=1}^{\infty}\frac{1+2An^2}{n(1+An^2)^2}=-\frac{\ln A}{2}+(\gamma+1)-\frac{A}{6}+O(A^2). \tag{17}$$

When these substitutions are made, the net result is that the (classical) $n=0$ term [Eq. (6)] cancels identically with a contribution from the (quantum) $n\neq 0$ terms and

$$F(l,T)=-\frac{\pi\rho^2}{l^2}\alpha^2(0)\frac{\hbar\omega_0}{16}\left\{1+(2l\omega_0/c)^2\left[\ln(2l\omega_0/c)+\gamma-\frac{3}{4}-\frac{\zeta(3)}{\pi^3}A^{3/2}+\cdots\right]+\left(\frac{14}{3\pi}\right)\times(2l\omega_0/c)^3\left[\ln(2l\omega_0/c)+\frac{5}{3}+\gamma-\frac{1}{6}A+\cdots\right]+O(\exp[-\hbar\omega_0/\pi kT])\right\}. \tag{18}$$

The leading term is that which would be obtained by pairwise summation of London forces. Expansions (18) and (13) set the limits of validity of the London expression. Not only is the restriction to the nonretarded regime $2l\omega_0/c\ll 1$, i.e., $l\ll O(100\ \text{Å})$ required, but so is the condition $A=(2\pi kT/\hbar\omega_0)^2\ll 1$. The fact that temperature and separation are linked in this fundamental way should have been evident already from the intermediate form for the free-energy density given in Eq. (7), where the natural scaled quantity $x=2kTl/\hbar c$ first appears. It follows then that expansions for small l will be inextricably linked to those for small T and thus that the low (but *finite*) temperature and nonretarded regimes coincide. Additionally, it is important to note that the temperature-dependent classical term arising from the $n=0$ mode is canceled by a contribution from the higher-frequency, essentially quantum-mechanical, modes in a proper analysis.

B. Interaction potential at finite temperature in the nonretarded regime

The atom-atom pairwise interaction potential can readily be obtained by noting that

$$V(l,T)=-\frac{1}{2\pi\rho^2 l}\frac{d^3}{dl^3}F(l,T). \tag{19}$$

However, evaluation of this directly from an expression such as that given in Eq. (18) would be laborious if more than the first few terms in the retardation parameter $2l\omega_0/c$, and indeed the leading temperature correction, are desired. We can do better by returning to Eq. (12) and applying Eq. (19), so that

$$V_{n\neq 0}(l,T)=\frac{kT}{64\pi^2\rho^2 l^6}(2\pi xB)^2\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(s+3)}{(2\pi x)^s}\left[\frac{1}{2-s}+\frac{1}{s}-\frac{1}{2(2+s)}\right]\sum_{n=1}^{\infty}\frac{1}{n^{s-2}(1+An^2)^2}. \tag{20}$$

This can then be dealt with in a fashion similar to that described for $F(l,T)$ above, i.e., evaluation of residues of the poles enclosed by the contour and appropriate expansion of the sums that arise. The $n=0$ term contributes

$$V_{n=0}(l,T)=-\frac{3kT}{l^6}\alpha^2(0) \tag{21}$$

and will again be canceled by an $n\neq 0$ contribution. Without giving the details, the result obtained is

$$V(l,T)=-\frac{3}{4l^6}\alpha^2(0)\hbar\omega_0\left\{1-\frac{\pi}{12}(2l\omega_0/c)^2+\frac{7}{36}(2l\omega_0/c)^3-\frac{\pi^2}{16}(2l\omega_0/c)^4-\left(\frac{11\pi}{90}\right)\times(2l\omega_0/c)^5\left[\frac{607}{660}-\ln(2l\omega_0/c)-\gamma+\frac{A^3}{3024}+\cdots\right]+O(\exp[-2\pi/\sqrt{A}])\right\}. \tag{22}$$

The leading term is of course the London potential. Note the high (fifth) order in the retardation parameter at which temperature corrections (carried by A) first manifest themselves and only at order A^3. This indicates that these temperature corrections are negligible at very short separations, but the asymptotic nature of the expansion suggests that as the retarded limit is approached by increasing l, even at *fixed* temperature, temperature effects become increasingly significant and ultimately the London description must break down. Again, the interplay between temperature and retardation is evident.

C. The Casimir-Polder (retarded) interaction at finite temperature

So, what happens in the retarded limit? We have emphasized the interplay of retardation and temperature effects. The intrinsic scaling of temperature and separation in the free energy manifested by the parameter x suggests that a useful low-temperature, large-separation expansion may be elusive. At zero temperature, the retarded-limit result for the potential was obtained by Casimir and Polder [4]. In fact, it is important to note that their result can only be recovered from the full Lifshitz free energy, given by the addition of Eqs. (6) and (12), on the assumption that $A\equiv 0$ from the outset. The usual rationalization of this procedure is that the exponential factors appearing in the Lifshitz free energy, Eq. (1) with Eq. (2), should, in the large-distance case $x\gg 1$, significantly contribute to the integrals only when $2pxn/c\simeq 1/\sqrt{\varepsilon}\lesssim 1$, and with $p\gg 1$ that would imply that the static dielectric constants can be used from the outset [18], i.e., $A=0$. We can now show this to be an incorrect limiting procedure.

On setting $A\equiv 0$ in Eq. (12), the sum therein reduces to $\zeta(s-2)$. Then evaluating the residues at the poles of the integrand, we have

$$F(l,T)=-\rho^2\alpha^2(0)\left\{\frac{23\hbar c}{120l^3}+\zeta(3)\frac{\hbar c\pi}{4l^3}\left(\frac{2kTl}{\hbar c}\right)^3-\frac{7\hbar c\pi^4}{360l^3}\left(\frac{2kTl}{\hbar c^4}\right)+\cdots\right\}, \tag{23}$$

where once again the nonretarded temperature-dependent term of Eq. (6) cancels out. The leading term is what would have been obtained by summation of individual retarded Casimir-Polder interaction potentials of the form $V(l)=$

$-23\hbar c\alpha^2(0)/4\pi l^7$. The higher-order terms of the expansion (23) involve the photons and blackbody radiation of the surrounding field. It should be clear that this asymptotic expansion is valid only when $kTl/\hbar c \ll 1$. Hence the original assumption of a large-distance limiting law approximation about a zero-temperature result is manifestly violated by the resulting expansion, rendering it invalid. This is best explained by stating that such an expansion should, were it valid, be useful at low temperatures for any l such that $2l\omega_0/c \gg 1$. This is clearly not so here. The only condition under which Eq. (23) could possibly be valid is actually upon setting $T=0$. This then recovers the Casimir-Polder zero-temperature result for the retarded free energy of interaction, which is indeed the true zero-temperature result. Mathematically, we can see this as follows: The pole at $s=3$ of the integrand of Eq. (12) is an artifact due to the impermissible setting of $A=0$. At any finite temperature the sum that occurs in the integrand has its first pole at $s=-1$. The $T\to0$ limit is therefore nonanalytic.

The important conclusion here is that the Casimir-Polder result is *valid strictly only at zero temperature* and *does not* provide a leading-order approximation for the free-energy density for large separations at finite temperatures. Conversely, the results presented as expansions about the London result, (18) and (22) are only useful at low but finite temperatures and small separations. In general, it is clear that for large distances the correct form of the interaction in Eq. (10) is exponential in form, and rather complicated [27,28]. Furthermore, it should be pointed out that the full Lifshitz theory already has been reconciled with experiments of Tabor and Winterton [11] and Israelachvili and Tabor [12], who observed the shift from the nonretarded to retarded van der Waals forces between mica cylinders [29].

IV. CASIMIR EFFECT BETWEEN CONDUCTING PLATES

A. Plates in vacuum

We now consider Casimir's [1] system of two perfectly conducting plates in vacuum. Since this has received considerable attention in the past, we shall be brief.

Referring again to the general Lifshitz theory of Sec. II, medium 2 is now a vacuum, $\varepsilon_2=1$, and media 1 perfectly reflecting metallic walls $(\sigma=\infty)$. Then $\Delta=\bar{\Delta}\equiv1$ and the formulas simplify. The term in $n=0$ is indeterminate as written because the prefactor ξ_n^2 vanishes while the integral in p diverges. To deal with this replace p by a new variable $y=2p\xi_n l\sqrt{\varepsilon_2(0)}/c$. Then the indeterminacy is removed and

$$F_{n=0}(l,T)=\frac{kT}{8\pi l^2}\int_0^\infty dyy\,\ln(1-e^{-y})=-\frac{kT}{8\pi l^2}\zeta(3). \tag{24}$$

The remaining terms contribute

$$F_{n\neq0}(l,T)=-\frac{kT}{4\pi l^2}(2\pi x)^2\sum_{n=1}^{\infty}n^2\int_1^\infty dpp\sum_{m=1}^{\infty}\frac{1}{m}\times\exp(-2\pi xpmn), \tag{25}$$

for which we require an expansion for small values of x. Using again Eq. (11), we find for the (scaled) free energy $\mathcal{F}$

$$\begin{aligned}\mathcal{F}&\equiv\frac{\hbar cl}{(kT)^2}F(l,T)\\&=-\frac{\zeta(3)}{4\pi x}-\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(s)\zeta(s-2)\zeta(s+1)}{(2\pi x)^{s-1}(s-2)} &(26a)\\&=-\frac{\zeta(3)}{4\pi x}+\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(2-s)\zeta(3-s)\zeta(-s)}{(x/2\pi)^{s-1}s}, &(26b)\end{aligned}$$

where for convergence $c=\text{Re }s>3$. The second form follows from the Riemann relation and the reflection formula for the Γ function, viz.

$$2^{1-s}\Gamma(s)\zeta(s)\cos(\pi s/2)=\pi^s\zeta(1-s),$$

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}. \tag{27}$$

Considering now the form of Eq. (26b), we can translate the contour to the left, evaluate residues at the poles at $s=-1,0,3$, and make the change of variable $2-s=s'$. This then leaves us with the two expression for $\mathcal{F}$

$$\mathcal{F}=-\frac{\zeta(3)}{4\pi x}-\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(s)\zeta(s+1)\zeta(s-2)}{(s-2)(2\pi x)^{s-1}},\quad c>3, \tag{28a}$$

$$\mathcal{F}=-\frac{\zeta(3)}{4\pi x}-\frac{\pi^2}{180}\left(\frac{1}{x^2}-x^2\right)+\frac{\zeta(3)}{4\pi}\left(\frac{1}{x}-x\right)-\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(s)\zeta(s+1)\zeta(s-2)}{(s-2)(2\pi/x)^{s-1}},\quad c>3. \tag{28b}$$

Consequently, on addition and translation of the contour to the left past the pole at $s=2$, we have

$$\mathcal{F}=-\frac{\pi^2}{180x^2}-\frac{1}{2(2\pi i)}\int_c ds\,\frac{\Gamma(s)\zeta(s+1)\zeta(s-2)}{(s-2)(2\pi)^{s-1}}\times\left(\frac{1}{x^{s-1}}+x^{s-1}\right),\quad 0<c<2. \tag{29}$$

The first term in Eq. (29) is the Casimir result (due to zero-point energy) and the remaining, temperature-dependent, terms represented by the integral are symmetric under the inversion $x\to1/x$. This inversion symmetry has been identified previously [13,30,31]. However, it is clear from our treatment that the symmetry of the expressions with respect to inversion in x is to be expected, as the partition function is closely related to Euler's product and its transformation properties [32], or equivalently those of products of ζ functions or the Jacobi θ function transformation.

To obtain an expansion corresponding to large distance or high temperature, $x=2kTl/\hbar c\gg1$, write Eq. (29) as

$$\mathcal{F}=-\frac{\pi^2}{180x^2}+\mathcal{J} \tag{30}$$

and choose the contour Re $s=c=1$. Then with $s=1+it$, we have

$$\mathcal{J}=-\frac{2}{(2\pi)^2}\int_0^\infty dt\,t\cos(t\ \ln x)\,\frac{e^{-\pi t/2}}{1-e^{-\pi t}}\,\zeta(2+it)\zeta(2-it). \tag{31}$$

After writing the ζ function as sums, it then follows after some algebra that

$$\mathcal{F}=-\frac{\zeta(3)}{4\pi x}-\frac{1}{2\pi x}\sum_{n=1}^{\infty}\frac{1}{n^3}\,\frac{e^{-2\pi n x}}{1-e^{-2\pi n x}}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}\,\mathrm{csch}^2(n\pi x),\quad x\gg 1. \tag{32}$$

Writing out the first few terms gives

$$F(l,T)=-\frac{kT}{8\pi l^2}\left[\zeta(3)+\left(\frac{8\pi kTl}{\hbar c}+2\right)e^{-4\pi kTl/\hbar c}+O(e^{-8\pi kTl/\hbar c})\right]. \tag{33}$$

The leading term here arises from current-current correlations [33].

An alternative form for $x\ll 1$ is

$$\mathcal{F}=-\frac{\pi^3}{90}\left(\frac{1}{x^2}-x^2\right)-\frac{x}{2\pi}\left[\zeta(3)+2\sum_{n=1}^{\infty}\frac{1}{n^3}\,\frac{e^{-2\pi n/x}}{1-e^{-2\pi n/x}}\right]-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\,\mathrm{csch}^2\left(\frac{n\pi}{x}\right), \tag{34}$$

so that, explicitly,

$$F(l,T)=-\frac{\pi^2\hbar c}{720 l^3}-\frac{\zeta(3)(kT)^3}{2\pi(\hbar c)^2}+\frac{(kT)^4}{(\hbar c)^3}\,\frac{\pi^2 l}{45}-\frac{(kT)^2}{\hbar c l}\times\left(1+\frac{kTl}{\pi\hbar c}\right)e^{-\pi\hbar c/kTl}+O(e^{-2\pi\hbar c/kTl}). \tag{35}$$

The leading term of this asymptotic expansion is the Casimir result. The second term can be written as $\hbar c/4\pi\rho$, where ρ is the density of photons [34] in blackbody radiation per unit volume. It plays the role of a chemical potential term. The third term is the free energy per unit area L^2 of blackbody radiation in a volume lL^2. The validity of this expansion is restricted to

$$x=\frac{2kTl}{\hbar c}\ll 1\quad\text{or}\quad l\ll\frac{10^{-1}}{T}\ \text{cm}. \tag{36}$$

In practice this means that, even accepting the artificial nature of the boundary conditions with ideal reflecting plates, the Casimir form of the interaction energy breaks down at room temperature at distances beyond several micrometers.

It should be noted that the results quoted above in Eqs. (33) and (35) agree with those of Mehra [20] and thereby confirm the demonstration of Schwinger *et al.* [21] that results obtained from the Lifshitz theory at finite temperature can be reconciled with those of other methods if the conducting boundary condition is properly dealt with.

B. Intervening plasma

Dzyaloshinskii, Lifshitz, and Pitaevskii [7] first considered the effect of the plates being imperfect conductors by assuming that the metals have an effective susceptibility $\varepsilon(\omega)=1-(\omega_p/\omega)^2$, where ω_p is the plasma frequency for the metal. Their resulting calculations for the free energy based on this approximation were later corrected by Hargreaves [24]. As Hargreaves also noted in his paper, an alternative route is to consider the plates as perfect conductors, but with a skin depth of ''free'' electrons. He then presented approximate formulas for the correction to ideal conductivity based on the assumption that to a first approximation it is sufficient to simply account for the skin depth through its thickness alone, i.e., by assuming that it produced an effective separation between ideal conductors. Schwinger *et al.* [21] tackled the imperfect conductivity problem more in keeping with the approach of Dzyaloshinskii *et al.* [7]. In light of all this and the recent experiments outlined in the Introduction, it seems useful to consider the scenario of two conducting plates separated by a plasma of free electrons in the framework of the Lifshitz theory.

Consider then two perfect conductors separated by a free-electron plasma of dielectric susceptibility

$$\varepsilon_2(\omega)=1-\frac{4\pi\rho e^2}{m\omega^2}, \tag{37}$$

where we can identify the plasma frequency of electrons as $\omega_p^2=4\pi\rho e^2/m$, where ρ is the number density of the plasma, m is the mass of the electron, and e is the unit charge. Referring to the Lifshitz free-energy density of Sec. II, and following closely the developments of the interceding sections, we can write down a contour integral representation for $F(l,T)$:

$$F(l,T)=-\frac{kT}{4\pi l^2}\,\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(s)\zeta(s+1)}{(s-2)(2\pi x)^{s-2}}\times\sum_{n=0}^{\infty}{}'\,(n^2+\overline{\rho})^{1-s/2}, \tag{38}$$

where the prime on the sum indicates that the $n=0$ term carries a weight of 1/2, we restrict Re $s=c>3$ for convergence of the integral, and

$$\overline{\rho}\equiv\frac{\rho}{\pi m}\left(\frac{e\hbar}{kT}\right)^2=\frac{1}{4\pi^2}\left(\frac{\hbar\omega_p}{kT}\right)^2$$

is a scaled density.

1. Classical term

Before proceeding any further, it is interesting to consider on its own the $n=0$ term, as it sheds some light on the connection to the related theory of electrolytes [18,35]. Consider

$$F_{n=0}(l,T) = -\frac{kT}{8\pi l^2}\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(s)\zeta(s+1)}{(s-2)(2\kappa l)^{s-2}}$$

$$= \frac{kT}{2\pi}\int_\kappa^\infty dt t\,\ln\,(1-e^{-2lt}), \tag{39}$$

where we have newly defined $\kappa^2 \equiv \omega_p^2/c^2$. This has precisely the form of the $n=0$ term for the classical theory of electrolytes [18,35], with the only difference being that the inverse Debye length of electrolytes $\kappa_D \equiv (8\pi\rho e^2/kT\varepsilon_w)^{1/2}$, where ε_w is the static susceptibility of water, is replaced by κ for the plasma.

Therefore, we can with this connection use results presented in Ref. [18] to obtain directly the large-distance expansion for the classical term

$$F_{n=0}(l,T) = -\frac{kT\kappa^2}{2\pi}e^{-2l\kappa}\left[\frac{1}{2l\kappa}+\frac{1}{(2l\kappa)^2}\right]+O(e^{-4l\kappa}). \tag{40}$$

A small-distance expansion may be found by closing the contour and evaluating residues. This yields

$$F_{n=0}(l,T) = -\frac{kT}{8\pi l^2}\zeta(3)+\frac{kT\kappa^2}{4\pi}\bigg[-\ln(2l\kappa)+\frac{1}{2}+\frac{1}{3}2l\kappa$$

$$-\frac{1}{48}(2l\kappa)^2+\cdots\bigg]. \tag{41}$$

The leading term is precisely that obtained in the vacuum case (and will cancel with a contribution from the $n\neq 0$ terms in the full free energy). The second term represents exactly twice the change in surface free-energy density due to the presence of electrolyte in the analogous case and in this one due to the plasma.

2. Full free energy

We now return to the full free-energy density represented by Eq. (38), with all modes present. We can rewrite the sum in terms of Epstein-Hurwitz ζ functions, which have become widely used in quantum field theory and indeed calculations of the Casimir effect for massive quantum fields [15]. Recently, Elizalde [36] presented their complete analytic structure. Noting that our notation differs from that of Elizalde, the Epstein-Hurwitz ζ function is defined for Re $z > 1/2$ and $a > 0$ by

$$\zeta_{EH}(z,a) \equiv \sum_{n=1}^{\infty}\frac{1}{(n^2+a)^z} \tag{42}$$

and the related generalized Epstein function is

$$\zeta_G(z,a) \equiv \sum_{n=-\infty}^{\infty}\frac{1}{(n^2+a)^z} = 2\zeta_{EH}(z,a)+a^{-z}. \tag{43}$$

It is possible to analytically continue these functions into the region Re $z < 1/2$ using an integral representation for the Γ function and the Jacobi inversion formula [15] and thereby develop asymptotic expansions valid for small and large values of the parameter a. These details are relegated to the Appendix. The important point to note here is that the generalized Epstein-Hurwitz ζ function $\zeta_G(z,a)$ is meromorphic and has simple poles in the complex plane at $z=1/2-k$, where $k=0,1,2,\ldots$. The Laurent series expansion at these poles is given in the Appendix as Eq. (A4).

We can now rewrite Eq. (38) as

$$F(l,T) = -\frac{kT}{8\pi l^2}\frac{1}{2\pi i}\int_c ds\,\frac{\Gamma(s)\zeta(s+1)}{(s-2)(2\pi x)^{s-2}}$$

$$\times\zeta_G(s/2-1,\overline{\rho}), \quad c>3. \tag{44}$$

We again seek an expansion for low temperature or distance $x \ll 1$. This is relatively straightforward, given that we know the analytic structure of the integrand. Closing the contour to the left and evaluating residues yields the result

$$F(l,T) = -\frac{\pi^2\hbar c}{720 l^3}\bigg\{1-15\overline{\rho x}^2-\frac{45}{\pi}\frac{\zeta_G'(-1,\overline{\rho})}{\overline{\rho}^{3/2}}(\overline{\rho x}^2)^{3/2}$$

$$-\frac{45}{2}[1-2\gamma+\mathcal{S}(\overline{\rho},2)-\ln(\overline{\rho x}^2/4)](\overline{\rho x}^2)^2$$

$$-90\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\Gamma(n+7/2)\zeta(2n+3)}{\pi^{1/2}(2n+3)\Gamma(n+4)}(\overline{\rho x}^2)^{n+3}\bigg\}, \tag{45}$$

where $\mathcal{S}(a,k)$ is defined in Eq. (A6).

This expression warrants discussion. Again, the leading term is the vacuum Casimir result. Clearly, there also have been cancellations between the $n=0$ term and the higher mode contributions. Interestingly, $\overline{\rho x}^2$ emerges naturally as the expansion parameter and the asymptotic expansion is of course valid for $\overline{\rho x}^2 \ll 1$. We can rewrite this variable as

$$\overline{\rho x}^2 = \left(\frac{\omega_p l}{\pi c}\right)^2. \tag{46}$$

Note thus that the separation and density are intrinsically coupled in this expansion. The restriction $\overline{\rho x}^2 \ll 1$ means that $l \ll c/\omega_p$ [noting that the factor of $1/\pi^2$ in Eq. (46) somewhat softens this restriction]. This is the opposite limit to that of Dzyaloshinskii *et al.* [7] and Hargreaves [24]. As an example, the number density of free electrons for Au or Ag in a Drude-type model is approximately 5.9×10^{-11} cm^{-3} [37]; in that case this means that our asymptotic expansion is valid for $l \lesssim 0.01$ μm. Thus the low-temperature expansion developed above effectively restricts us to the small-separation limit for a given density. While the expressions in Refs. [6,24,21] are naturally limited by the converse condition $l \gg 0.01$ μm, it should also be noted that in that case there is the added restriction of $l \lesssim 1$ μm, for Eq. (37) to be a useful approximation to the dielectric susceptibility [21].

Given the restriction on $\overline{\rho x}^2$ and that the force per unit area of surface f is related to F via $f = -\partial F/\partial l$, we can calculate the leading correction due to the presence of the electron plasma to the ideal, zero-temperature, Casimir result $f_0 = -\hbar c\pi^2/240 l^4$ as a function of separation and density:

$$\frac{f}{f_0}\approx 1-\frac{5}{\pi^2}\left(\frac{\omega_p l}{c}\right)^2 \tag{47}$$

for $l\ll c/\omega_p$. Note that the result is qualitatively different from that of Dzyaloshinskii *et al.* [7], Schwinger *et al.* [21], and Hargreaves [24] for the $l\gg c/\omega_p$ regime. Hargreave's result is the one most readily comparable to ours, as he considers the electrons in the skin layer to be effectively free, so that his model consists of two perfectly conducting slabs at a separation $l=d+2\delta$, where $\delta=c/\omega_p$ is a (frequency-independent) skin depth and d the width of vacuum between these plasma layers. His result is

$$\frac{f}{f_0}\approx 1-8\left(\frac{c}{\omega_p l}\right). \tag{48}$$

Thus it seems that in crossing from the small- to large-separation limits, the correction for the plasma sees a reversal in the roles of temperature and density: At small separations, increasing separation and plasma density increase the size of the correction; at large separations, this decreases it. This suggests a maximum correction at some intermediate distance, but unfortunately this appears inaccessible to an analytic treatment. Of course, as a caveat we should also consider the imaginary component of the dielectric function at such short distances in a more realistic model [2]. However, we suggest that our model provides insight into the remarkable crossover behavior of the free energy (and force). We should note that the correction in the near-field limit for the presence of the plasma is small (about 0.1% at 10 Å, but increases rapidly to about 10% at 100 Å; however, our asymptotic expansions are becoming poor approximations at the upper extreme).

We can flesh out the more complete result presented as Eq. (45) containing $\zeta_G'(-1,\overline{\rho})$ and $\mathcal{S}(\overline{\rho},2)$ by inserting the expansions for these functions at small and large $\overline{\rho}$. The former serves to confirm that the correct zero-density limit is recovered; the latter is useful for real metals (for Au or Ag, $\overline{\rho}\approx 3\times 10^3$). These expansions are given in the Appendix. Thus, for small densities $\overline{\rho}\ll 1$, we use Eqs. (A13) and (A14) to obtain

$$\begin{aligned}F(l,T)=&-\frac{\pi^2\hbar c}{720l^3}+\frac{\pi^2(kT)^2}{12\hbar cl}\overline{\rho}-\frac{\zeta(3)(kT)^3}{2\pi(\hbar c)^2}\Bigg[1-\frac{\pi^2}{\zeta(3)}(1\\&-2\ln 2\pi)\overline{\rho}+\frac{\pi^2}{\zeta(3)}\overline{\rho}\ln\overline{\rho}+O(\overline{\rho}^2)\Bigg]\\&+\frac{\pi^2 l(kT)^4}{45(\hbar c)^3}[1-15\overline{\rho}-45\overline{\rho}^2(\ln x+1/4)+O(\overline{\rho}^3)]\\&+O(\overline{\rho x}^2)^3.\end{aligned} \tag{49}$$

Note that, aside from exponentially decaying terms in x, which are difficult to obtain from this treatment [as a full analysis of the contribution to the integral from the great arc of the contour in Eq. (44) would be needed], this result exactly recovers the vacuum expression in the limit $\overline{\rho}\rightarrow 0$.

In the limit of high density $\overline{\rho}\gg 1$ (but such that $\overline{\rho x}^2\ll 1$), useful for metals, we use Eqs. (A17) and (A18) and obtain, up to exponentially decaying terms in x and $\overline{\rho}$,

$$\begin{aligned}F(l,T)=&-\frac{\pi^2\hbar c}{720l^3}+\frac{\pi^2(kT)^2}{12\hbar cl}\overline{\rho}-\frac{2\pi^2(kT)^3}{3(\hbar c)^2}\overline{\rho}^{3/2}\\&+\frac{\pi^2 l(kT)^4}{45(\hbar c)^3}\overline{\rho}^2[1-2\gamma-\ln(\overline{\rho x}^2/4)]+O(\overline{\rho x}^2)^3.\end{aligned} \tag{50}$$

V. CONCLUDING REMARKS

Our investigation has shed light on several features of the Lifshitz theory of Casimir forces at finite temperature. The Casimir-Polder result for the atom-atom interaction energy is valid strictly only at zero temperature. This indicates, importantly, that measurement of the Casimir-Polder interaction energy at room temperature needs careful reconsideration. For the two-plate Casimir problem the corrections for density appear to display a remarkable crossover behavior in going from the small- to large-separation limits.

We have not yet fully addressed a quite fundamental issue. We noted in our treatment of the two-plate problem with intervening plasma that the classical term in the free energy is directly analogous to that for an electrolyte in a colloidal system (i.e., with interfaces present), a system investigated at some length in Ref. [35]. There it was shown that the free energy of interaction between the surfaces, explicitly the change in surface energy due to electrolyte, has a term (namely that at zero frequency) where the electrostatic and van der Waals contributions cancel exactly (essentially because of a gauge condition on the electromagnetic field). This indicates that electrostatic and dispersion forces are inextricably entwined in colloidal systems. Moreover, we know because of this that a linear theory of dispersion forces, such as the Lifshitz theory, is inadequate and that a full nonlinear theory incorporating both electrostatic and fluctuation (dispersion) forces is required [35]. The same must be true for an electron plasma between metal plates, where a description of the change in surface energy due to the plasma would, in a more complete theory, need to take into account the intrinsic coupling of electrostatic and dispersion forces. Thus the decomposition of electrostatic and Casimir forces [2] in the interpretation of experiments seems problematic. These issues provide a significant challenge and warrant close attention, both theoretically and experimentally.

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APPENDIX: EPSTEIN-HURWITZ ζ FUNCTION

1. Analytic continuation and Laurent series

An analytic continuation for the Epstein-Hurwitz ζ function $\zeta_{EH}(z,a)=\Sigma_{n=1}^{\infty}(n^2+a)^{-z}$ valid for $z<1/2$ is readily obtainable by use of an integral representation for the Γ function and then the Jacobi θ-function inversion formula. As this has been done repeatedly in the literature [15,38], we simply give the result, which is

$$\zeta_{EH}(z,a)=-\frac{a^{-z}}{2}+\frac{\pi^{1/2}\Gamma(z-1/2)}{2\Gamma(z)}a^{-z+1/2}+\frac{\pi^{1/2}}{\Gamma(z)}\sum_{n=1}^{\infty}\int_0^{\infty}dt\,t^{z-3/2}\exp\left(-at-\frac{\pi^2n^2}{t}\right). \tag{A1}$$

The integral is (up to a proportionality constant) a representation for the modified Bessel functions $K_\nu(z)$ [39], so that

$$\zeta_{EH}(z,a)=-\frac{a^{-z}}{2}+\frac{\pi^{1/2}\Gamma(z-1/2)}{2\Gamma(z)}a^{-z+1/2}+\frac{2\pi^{z}}{\Gamma(z)}a^{-z/2+1/4}\sum_{n=1}^{\infty}n^{z-1/2}K_{z-1/2}(2\pi n a^{1/2}). \tag{A2}$$

The sum in n in the above two equations is manifestly convergent since the Bessel functions behave asymptotically as $z\to\infty$ as [39]

$$K_\nu(z)\sim\left(\frac{\pi}{2z}\right)^{1/2}e^{-z}. \tag{A3}$$

This also provides an analytic continuation for the generalized Epstein-Hurwitz function through $\zeta_G(z,a)=2\zeta_E(z,a)+a^{-z}$ and allows us to evaluate the first two terms in the Laurent series near the simple poles at $z=1/2-k$, $k=0,1,2,\ldots$, which we require in our evaluation of the free energy. After some manipulation of Eq. (A2), we have, for $z\to1/2-k$,

$$\zeta_G(z,a)\sim\frac{\Gamma(k+1/2)}{\pi^{1/2}\Gamma(k+1)}\frac{a^k}{(z-1/2+k)}\{1+(z-1/2+k)\times[\mathcal{S}(a,k)-2\beta(2k)-\ln a]+O((z-1/2+k)^2)\}, \tag{A4}$$

where

$$\beta(x)=\frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right)\right] \tag{A5}$$

and $\psi(z)$ is the digamma function [39], and also we define

$$\mathcal{S}(a,k)\equiv4(-1)^k\pi^{-k}a^{-k/2}\Gamma(k+1)\sum_{n=1}^{\infty}n^{-k}K_k(2\pi n a^{1/2}). \tag{A6}$$

2. Small-parameter expansion

The small-a expansion for $\zeta_{EH}(z,a)$ is best obtained by using a Mellin-Barnes integral representation, as given by Elizalde *et al.* [40],

$$\zeta_{EH}(z,a)=\frac{1}{2\pi i}\int_c dt\,B(t,z-t)\zeta(2z-2t)a^{-t},\qquad 0<c<z-1/2, \tag{A7}$$

where $B(x,y)\equiv\Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function. The above also provides an integral representation, through the relation $\zeta_G(z,a)=2\zeta_{EH}(z,a)+a^{-z}$, for the generalized Epstein function. Closing the contour to the left and evaluating residues, one obtains

$$\zeta_{EH}(z,a)=\sum_{n=0}^{\infty}\frac{\zeta(2z+2n)\Gamma(z+n)}{\Gamma(z)\Gamma(n+1)}(-a)^n \tag{A8}$$

and for the derivative

$$\zeta'_{EH}(z,a)=\sum_{n=0}^{\infty}\{2\zeta'(2z+2n)+[\psi(z+n)-\psi(z)]\times\zeta(2z+2n)\}\frac{\Gamma(z+n)}{\Gamma(z)\Gamma(n+1)}(-a)^n. \tag{A9}$$

We need to analytically continue these results to $z<1/2$, in particular to $z=-q$, where $q=0,1,2,3,\ldots$. We then have

$$\zeta_{EH}(-q,a)=-\frac{1}{2}a^q, \tag{A10}$$

so that

$$\zeta_G(-q,a)=0. \tag{A11}$$

We also obtain for the derivative, evaluated at nonpositive integers,

$$\zeta'_{EH}(-q,a)=2\Gamma(q+1)\sum_{n=0}^{q}\frac{\zeta'(2n-2q)}{\Gamma(q-n+1)\Gamma(n+1)}a^n+\frac{1}{2}[\gamma+\psi(q+1)]a^q+(-1)^{q+1}\Gamma(q+1)\times\sum_{n=q+1}^{\infty}\frac{\zeta(2n-2q)\Gamma(n-q)}{\Gamma(n+1)}(-a)^n. \tag{A12}$$

For the free energy given by Eq. (45), we require $\zeta'_G(-1,\overline{\rho})$ and $\mathcal{S}(\overline{\rho},2)$. The former can be obtained directly from Eq. (A12) and is

$$\zeta'_G(-1,\overline{\rho})=-\frac{1}{\pi^2}\zeta(3)+[1-\ln(4\pi^2\overline{\rho})]\overline{\rho}+2\sum_{n=2}^{\infty}\frac{\zeta(2n-2)}{n(n-1)}(-\overline{\rho})^n. \tag{A13}$$

A similar Mellin transform treatment to the one above for $\zeta_{EH}(z,a)$ can be made for $\mathcal{S}(a,k)$ and we simply give the result for $\mathcal{S}(\overline{\rho},2)$, which is

$$\mathcal{S}(\bar{\rho},2)=\frac{2}{45\bar{\rho}^{2}}-\frac{2}{3\bar{\rho}}+\ln(\bar{\rho}/4)-\frac{3}{2}+\gamma+\frac{4}{\pi^{1/2}}\times\sum_{n=1}^{\infty}(-1)^{n}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\zeta(2n+1)\bar{\rho}^{n}. \tag{A14}$$

3. Large-parameter expansion

For the large parameter expansion, we need to return to the analytic continuation for $\zeta_G(z,a)$, given by Eq. (A2) and $\zeta_G(z,a)=2\zeta_{EH}(z,a)+a^{-z}$. Use of the leading large-parameter scaling of the Bessel functions as given in Eq. (A3) readily allows us to deduce the $a\gg 1$ behavior of $\zeta_G(z,a)$, which is

$$\zeta_G(z,a)=\pi^{1/2}\frac{\Gamma(z-1/2)}{\Gamma(z)}a^{-z+1/2}+\frac{2\pi^{z}a^{-z/2}}{\Gamma(z)}\times\exp(-2\pi a^{1/2})[1+O(a^{-1})]. \tag{A15}$$

We can also deduce the large-parameter expansion for the z derivative from Eq. (A1):

$$\begin{aligned}\zeta_G'(z,a)&=\pi^{1/2}\frac{\Gamma(z-1/2)}{\Gamma(z)}[\psi(z-1/2)-\psi(z)-\ln a]a^{-z+1/2}\\&\quad-\pi^{1/2}\frac{\psi(z)}{\Gamma(z)}\sum_{n=1}^{\infty}\int_0^{\infty}dt\,t^{z-3/2}\exp\left(-at-\frac{\pi^2n^2}{t}\right)\\&\quad+\frac{\pi^{1/2}}{\Gamma(z)}\sum_{n=1}^{\infty}\int_0^{\infty}dt\,t^{z-3/2}\ln t\,\exp\left(-at-\frac{\pi^2n^2}{t}\right).\end{aligned} \tag{A16}$$

For the specific case of the free-energy density of Eq. (45), we require the $\bar{\rho}\gg 1$ expansion for $\zeta_G'(-1,\bar{\rho})$, which is

$$\begin{aligned}\zeta_G'(-1,\bar{\rho})&=-\frac{4\pi}{3}\bar{\rho}^{3/2}-\frac{4}{\pi}\bar{\rho}^{3/4}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}K_{3/2}(2\pi n\bar{\rho}^{1/2})\\&=-\frac{4\pi}{3}\bar{\rho}^{3/2}-\frac{2}{\pi}\bar{\rho}^{1/2}\exp(-2\pi\bar{\rho}^{1/2})\left[1+O\left(\frac{1}{\bar{\rho}}\right)\right].\end{aligned} \tag{A17}$$

The large-$\bar{\rho}$ expansion of $\mathcal{S}(\bar{\rho},2)$ is similarly obtained and is

$$\mathcal{S}(\bar{\rho},2)=\frac{4}{\pi^{2}\bar{\rho}^{5/4}}\exp(-2\pi\bar{\rho}^{1/2})\left[1+O\left(\frac{1}{\bar{\rho}}\right)\right]. \tag{A18}$$

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