Using a quantum computer to investigate quantum chaos

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(Received 12 May 1997)

We show that the quantum baker's map, a prototypical map invented for theoretical studies of quantum chaos, has an efficient realization in terms of quantum gates. Chaos in the quantum baker's map could be investigated experimentally on a quantum computer based on only three quantum bits.
[S1050-2947(98)08102-5]

PACS number(s): 03.67.Lx, 03.65.Sq, 05.45.+b, 89.70.+c

Since the discovery that a quantum computer can in principle factor large integers in polynomial time [1,2], quantum information has become a major theoretical and experimental research topic, focusing on properties, applications, generation, and preservation of highly entangled quantum states [3]. Although it is not clear if a full-scale quantum computer will ever be realized [4,5], experiments with quantum gates are being performed at present [6–9]. It is important to devise applications for early quantum computers that are incapable of large-scale computations such as factoring.

Early quantum computers appear to be well suited to study the dynamics of simple quantum maps. The quantum baker's map [10], one of the simplest quantum maps used in quantum chaos research, has been extensively studied in recent years [11–16]. Hannay *et al.* [17] have proposed a realization of the baker's map in classical optics, which allows them to understand its quantization in terms of the relationship between ray and wave optics. Here we describe a genuine quantum system whose dynamics is governed by the quantum baker's map. As a consequence of recent progress in the field of quantum computing [6–9], an experimental realization of the quantum baker's map seems possible in the near future.

Any unitary operator can be approximated by a sequence of simple quantum gates [18–20]. The main result of this paper is that the quantum baker's map can be realized in terms of quantum gates in a particularly simple and efficient way. Similar to the quantum Fourier transform, simulating the quantum baker's map on a quantum computer is exponentially faster than a simulation on a classical computer.

The quantum baker's map displays behavior of fundamental interest even for a Hilbert space of small dimension. Numerical simulations [13] in $D=16$ dimensional Hilbert space suggest that a rudimentary quantum computer based on as few as three quantum bits (qubits) (i.e., three two-state systems spanning $D=8$ dimensional Hilbert space) could be used to study chaos in the quantum baker's map. In particular, it may be possible to find experimental evidence for hypersensitivity to perturbation, a proposed information-theoretical characterization of quantum chaos [13,21–23].

The classical baker's transformation [24] maps the unit square $0\leqslant q,p\leqslant 1$ onto itself according to

$$(q,p)\mapsto\begin{cases}(2q,\,\tfrac{1}{2}p) & \text{if } 0\leqslant q\leqslant \tfrac{1}{2}\\ [2q-1,\tfrac{1}{2}(p+1)] & \text{if } \tfrac{1}{2}<q\leqslant 1.\end{cases} \quad (1)$$

This corresponds to compressing the unit square in the p direction and stretching it in the q direction, while preserving the area, then cutting it vertically, and finally stacking the right part on top of the left part, in analogy to the way a baker kneads dough.

To define the quantum baker's map [10], we quantize the unit square following [11,25]. To represent the unit square in D-dimensional Hilbert space, we start with unitary ''displacement'' operators $\hat{U}$ and $\hat{V}$, which produce displacements in the ''momentum'' and ''position'' directions, respectively, and obey the commutation relation [25]

$$\hat{U}\hat{V}=\hat{V}\hat{U}\epsilon, \quad (2)$$

where $\epsilon^D=1$. We choose $\epsilon=e^{2\pi i/D}$. We further assume that $\hat{V}^D=\hat{U}^D=1$, i.e., periodic boundary conditions. It follows [11,25] that the operators $\hat{U}$ and $\hat{V}$ can be written as

$$\hat{U}=e^{2\pi i\hat{q}},\quad \hat{V}=e^{-2\pi i\hat{p}}. \quad (3)$$

The position and momentum operators $\hat{q}$ and $\hat{p}$ both have eigenvalues j/D, $j=0,\ldots,D-1$.

In the following, we restrict the discussion to the case $D=2^L$, i.e., the dimension of Hilbert space is a power of 2. For consistency of units, let the quantum scale on ''phase space'' be $2\pi\hbar=1/D=2^{-L}$. A transformation between the position basis $\{|q_j\rangle\}$ and the momentum basis $\{|p_j\rangle\}$ is effected by the discrete Fourier transform F'_L, defined by the matrix elements

$$(F'_L)_{kj}=\langle p_k|q_j\rangle=\sqrt{2\pi\hbar}\,e^{-ip_kq_j/\hbar}=\frac{1}{\sqrt{D}}e^{-2\pi ikj/D}. \quad (4)$$

There is no unique way to quantize a classical map. Here we adopt the quantized baker's map introduced by Balazs and Voros [10] and defined by the matrix

$$T'=F'^{-1}_L\begin{pmatrix}F'_{L-1} & 0\\ 0 & F'_{L-1}\end{pmatrix}, \quad (5)$$

where the matrix elements are to be understood relative to the position basis $\{|q_j\rangle\}$. Saraceno [11] has introduced a quantum baker's map with stronger symmetry properties by using antiperiodic boundary conditions, but in this article we restrict the discussion to periodic boundary conditions as used in [10].

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The discrete Fourier transform used in the definition of the quantum baker's map (5) plays a crucial role in quantum computation and can be easily realized as a quantum network using simple quantum gates. The following discussion of the quantum Fourier transform follows [2] closely. The $D=2^L$ dimensional Hilbert space modeling the unit square can be realized as the product space of L qubits (i.e., L two-state systems) in such a way that

$$|q_j\rangle=|j_{L-1}\rangle\otimes|j_{L-2}\rangle\otimes\cdots\otimes|j_0\rangle, \tag{6}$$

where $j=\Sigma j_k 2^k$, $j_k\in\{0,1\}$ $(k=0,\ldots,L-1)$, and each qubit has basis states $|0\rangle$ and $|1\rangle$.

To construct the quantum Fourier transform, two basic unitary operations or *quantum gates* are needed: the gate A_m acting on the mth qubit and defined in the basis $\{|0\rangle,|1\rangle\}$ by the matrix

$$A_m=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{7}$$

and the gate B_{mn} operating on the mth and nth qubits $(m<n)$ and defined by

$$B_{mn}|j_{L-1}\rangle\otimes\cdots\otimes|j_0\rangle=e^{i\phi_{mn}}|j_{L-1}\rangle\otimes\cdots\otimes|j_0\rangle, \tag{8}$$

where

$$\phi_{mn}=\begin{cases}\pi/2^{n-m} & \text{if } j_m=j_n=1\\ 0 & \text{otherwise.}\end{cases} \tag{9}$$

In addition we define the gate S_{mn} that swaps the qubits m and n.

The discrete Fourier transform F_L can now be expressed in terms of the three types of gates as

$$F_L=S\times(A_0B_{01}\cdots B_{0,L-1})\times\cdots(A_{L-3}B_{L-3,L-2}B_{L-3,L-1})\times(A_{L-2}B_{L-2,L-1})\times(A_{L-1}), \tag{10}$$

where

$$S=\begin{cases}S_{0,L-1}S_{1,L-2}\cdots S_{L/2-1,L/2} & \text{for } L \text{ even}\\ S_{0,L-1}S_{1,L-2}\cdots S_{(L-3)/2,(L+1)/2} & \text{for } L \text{ odd}\end{cases} \tag{11}$$

reverses the order of the qubits. The quantum baker's map (5) is then given by

$$T=F_L^{-1}(I\otimes F_{L-1}), \tag{12}$$

where F_{L-1} acts on the $L-1$ least significant qubits and I is the identity operator acting on the most significant qubit. The gates corresponding to the bit-reversal operator S can be omitted if the qubits in the tensor product (6) are relabeled after each execution of F_L or F_{L-1}.

In $D=8=2^3$ dimensional Hilbert space, one iteration of the quantum baker's map is performed by the short sequence of gates

$$T=S_{02}A_0B_{01}^\dagger B_{02}^\dagger A_1B_{12}^\dagger A_2S_{01}A_0B_{01}A_1. \tag{13}$$

This implementation of the quantum baker's map can be viewed in two complementary ways. On the one hand, it shows that the quantum baker's map can be efficiently simulated on a quantum computer. A 30-qubit quantum computer could perform simulations that are virtually impossible on present-day classical computers. On the other hand, an iteration of the gate sequence (12) on an L-qubit quantum computer is a physical realization of the quantum baker's map. This opens up the possibility of an experimental investigation of chaos in a physical system in a purely quantum regime.

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