Renormalization-group method for simple operator problems in quantum mechanics

I. L. Egusquiza* and M. A. Valle Basagoiti†
Department of Theoretical Physics, University of the Basque Country, 644 P.K., 48080 Bilbao, Spain
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The failure of conventional perturbation theory due to secularity is considered with renormalization-group tecniques in two operator problems. Specifically, some results concerning the quantum anharmonic oscillator and quantum parametric resonance are obtained with a rather modest effort in comparison to other methods. [S1050-2947(98)05803-X]

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I. INTRODUCTION AND MOTIVATIONS

In many different contexts perturbation theory fails miserably because of the growth of higher-order terms, contrary to the basic perturbative assumption. This secularity is present in both classical and quantum theories, and pervades the motivation for the search for analytical methods to improve on perturbative expansions.

We will here analyze two simple quantum-mechanical problems where secularities appear and invalidate naive perturbation expansions

It is the renormalization-group (RG) method for global asymptotic analysis, as advocated by Goldenfeld and collaborators [1–3], which we extend to operator problems in this short note (see also [4] for a geometrical point of view of the RG method). The key idea of the RG method for global asymptotic analysis is the introduction of a time parameter, additional to the initial value point, in such a way that the perturbation expansion is valid in the vicinity of the introduced time parameter. The coupling constants, constants of motion, and/or initial conditions (depending on your viewpoint and background one or another of these descriptions will be more suitable) are turned into running constants, that is to say that these constants are suitably modified by the change of the introduced time parameter. On the other hand, the solution itself cannot depend on the additional, new time parameter, so derivation with respect to the latter of the perturbative solution will impose evolution equations for the running constants. These equations are then solved for the running constants, and on substitution in the perturbative expansion, together with the choice that the time parameter is constantly updated to be time itself, we obtain an improved solution.

This method has the clear advantage over the elementary multiple-scale perturbation analysis [5,6] that no *a priori* determination of the scales that appear in the problem is necessary, and a naive perturbation expansion is enough as a starting point (it has to be pointed out, though, that in some variants of the multiple scales analysis the functional form of the secondary scales is fixed a *posteriori*, according to a consistency condition—see, for example [7]). Many examples and illustrations of this advantage of the RG method can be found in the works of Goldenfeld and collaborators.

The multiple scales method itself has been applied to operator problems [8,9], but not the RG method, and this paper shows the application of the RG method to operator problems. We shall choose the quantum anharmonic oscillator and the phenomenon of quantum parametric resonance as our case studies, because of the important role they have traditionally had as theoretical laboratories for new perturbative methods, and because of their paradigmatic character in the context of cosmological (p)reheating, in which we are interested (for an application of the RG method in this context, see [10]). Our results are comparable to all other methods, and since they are obtained with a modest effort, we think that the RG method is highly competitive in the operator context as well. There are a number of areas where its usefulness might be proved, such as quantum optics, but we leave that for further work.

II. RG ANALYSIS OF THE QUANTUM ANHARMONIC OSCILLATOR

The first problem we shall first address within the RG method is the quantum anharmonic oscillator. It is described by the classical action

$$S=\int dt\left(\frac{1}{2}m\dot{q}^2-\frac{1}{2}m\omega^2q^2-\frac{1}{4}\lambda mq^4\right), \tag{2.1}$$

from which follows the nonlinear motion equation

$$\frac{d^2q}{dt^2}+\omega^2q+\lambda q^3=0\ , \tag{2.2}$$

known as Duffing's equation. Also, it is easily seen that the same equation governs the quantum dynamics, now with $q(t)$ understood as the position operator in the Heisenberg picture.

Let us perform a simple perturbation expansion in the λ coupling constant, $q=q_0+\lambda q_1+O(\lambda^2)$. The solution to order 0 is simply

$$q_0=\sqrt{\frac{\hbar}{2m\omega}}(\beta e^{-i\omega t}+\beta^\dagger e^{i\omega t}). \tag{2.3}$$

In this expression $\beta^\dagger$ and β are creation and annihilation operators for the oscillator problem. The first-order equation

*Electronic address: wtpegegi@lg.ehu.es
†Electronic address: wtpvabam@lg.ehu.es

presents resonance and, therefore, secular terms. Writing down just the singular (i.e., secular) part of q_1, which we call $q_{1,s}$,

$$q_{1,s}=\left(\frac{\hbar}{2m\omega}\right)^{3/2}\frac{i(t-\tau)}{2\omega}\{[\beta(\beta^\dagger)^2+\beta^\dagger\beta\beta^\dagger+(\beta^\dagger)^2\beta]e^{i\omega t} -(\beta^2\beta^\dagger+\beta\beta^\dagger\beta+\beta^\dagger\beta^2)e^{-i\omega t}\}. \tag{2.4}$$

Let us now allow a dependence in τ of the pair of operators β and $\beta^\dagger$. Imposing the RG condition $dq/d\tau=0$, we obtain

$$\frac{d\beta}{d\tau}+\frac{i\lambda\hbar}{4m\omega^2}(\beta^2\beta^\dagger+\beta\beta^\dagger\beta+\beta^\dagger\beta^2)=O(\lambda^2), \tag{2.5}$$

and the Hermitian conjugate thereof. We notice that $\mathcal{N}=\beta^\dagger\beta$ and $[\beta,\beta^\dagger]$ are constants under the flow of τ, which allows us to solve these equations in the form

$$\beta(\tau)=\beta(0)e^{-3i\lambda\hbar\mathcal{N}\tau/(4m\omega^2)},$$
$$\beta^\dagger(\tau)=e^{3i\lambda\hbar\mathcal{N}\tau/(4m\omega^2)}\beta^\dagger(0), \tag{2.6}$$

which, on being substituted in the perturbative expansion of q, together with the change $\tau\rightarrow t$, gives us

$$q(t)=\sqrt{\frac{\hbar}{2m\omega}}[e^{-i\omega t}\beta(0)e^{-3i\lambda\hbar\mathcal{N}\tau/(4m\omega^2)} +e^{i\omega t}e^{3i\lambda\hbar\mathcal{N}\tau/(4m\omega^2)}\beta^\dagger(0)]. \tag{2.7}$$

We have thus obtained an asymptotic expression for this operator. On computing $\langle n-1|q(t)|n\rangle$, we see that the energy difference between levels comes out as $E_n-E_{n-1}=\hbar\omega[1+(3\lambda\hbar n)/(4m\omega^3)+O(\lambda^2)]$, consistent with all previous computations of this quantity.

It has to be observed that our result is identical to the one obtained by Bender and Bettencourt [8], as is only to be expected, given the equivalence of the multiple-scale method and the RG methods for a wide class of differential equations, to which the (classical) Duffing equation belongs. On the other hand, note the simplicity of our approach, where no *a priori* scale has to be assumed.

In order to stress this latter point, let us consider the second-order computation for this problem. The source term for q_2 is given by $-(q_0^2q_1+q_0q_1q_0+q_1q_0^2)$, where we have to consider the full q_1 and not just the singular part. In this source term there will be terms that will give rise to secularities of the form $e^{\pm 3i\omega t}(t-\tau)$ and $e^{\pm i\omega t}(t-\tau)^2$. These we shall be able to ignore, because the renormalization to first order takes care of them. As a matter of fact, this is precisely what the statement of perturbative renormalizability amounts to in our case: that no divergences of a different form arise in the process of renormalization, that is to say, that all divergent (secular) terms can be taken care of by renormalization of the terms $\beta e^{-i\omega t}$ and $\beta^\dagger e^{i\omega t}$.

Another (simple) technicality in the problem at hand is that, since we have checked that $[\beta,\beta^\dagger]$ is constant to order λ^2, we can use the commutator in the λ and λ^2 terms, thus making the computation somewhat easier.

This results in the following expression for the secular relevant part of q_2, $q_{2,\mathrm{sr}}$:

$$q_{2,\mathrm{sr}}=\frac{-3il^5(t-\tau)}{16\omega^3}[(5\mathcal{N}^2-1)\beta^\dagger e^{i\omega t}-\beta(5\mathcal{N}^2-1)e^{-i\omega t}], \tag{2.8}$$

whence the improved RG equation reads

$$\frac{d\beta}{d\tau}+\frac{3i\lambda\hbar}{4m\omega^2}\beta\mathcal{N}-\frac{3i\lambda^2\hbar^2}{64m^2\omega^5}\beta(5\mathcal{N}^2-1)=O(\lambda^3), \tag{2.9}$$

thus giving us

$$\beta(\tau)=\beta(0)\exp\left(\frac{-3i\lambda\hbar\mathcal{N}\tau}{(4m\omega^2)}+\frac{3i\lambda^2h^2(5\mathcal{N}^2-1)\tau}{64m^2\omega^5}\right), \tag{2.10}$$

and, as a consequence,

$$E_n-E_{n-1}=\hbar\omega[1+(3\lambda\hbar n)/(4m\omega^3) -3\lambda^2\hbar^2(5n^2-1)/(64m^2\omega^6)+O(\lambda^3)]. \tag{2.11}$$

III. QUANTUM PARAMETRIC RESONANCE

As a last example of the usefulness of the RG method for quantum-mechanical problems, we shall now illustrate its application to the phenomenon of quantum parametric resonance. Consider then the following Hamiltonian:

$$H=\frac{1}{2m}P^2+\frac{1}{2}m\omega_0^2[A+2q\cos(\omega_0t)]X^2, \tag{3.1}$$

where A, q, and ω_0 are constants. The evolution of any given state is computed by acting on it with the evolution operator $U(t,t_0)$, which satisfies

$$i\hbar\frac{\partial U}{\partial t}(t,t_0)=H(t)U(t,t_0), \tag{3.2}$$

with $U(t_0,t_0)=I$, the identity operator.

Let us divide the Hamiltonian into an unperturbed and a perturbation part:

$$H=H_0+H_1=\left(\frac{1}{2m}P^2+\frac{1}{8}m\omega_0^2X^2\right) +\left\{\frac{1}{2}m\omega_0^2\left[\left(A-\frac{1}{4}\right)+2q\cos(\omega_0t)\right]X^2\right\}. \tag{3.3}$$

The reason for this decomposition lies in our previous knowledge that resonance will definitely set in if A is equal to 1/4, but this is not essential for the final results.

The evolution operator can be written as

$$U(t,t_0)=e^{-i(t-t_0)H_0/\hbar}U_I(t,t_0), \tag{3.4}$$

in such a way that the interaction picture evolution operator obeys the following equation:

$$i\hbar\frac{\partial U_I}{\partial t}(t,t_0)=H_I(t)U_I(t,t_0), \tag{3.5}$$

and in our case

$$H_I=\frac{1}{2}m\omega_0^2\left[\left(A-\frac{1}{4}\right)+2q\ \cos(\omega_0 t)\right]X_I^2$$
$$=\frac{\hbar\omega_0}{2}\left[\left(A-\frac{1}{4}\right)+2q\ \cos(\omega_0 t)\right](e^{-i\omega_0(t-t_0)/2}a$$
$$+e^{i\omega_0(t-t_0)/2}a^\dagger)^2. \tag{3.6}$$

The constant operators a and $a^\dagger$ are the annihilation and creation operators at time t_0.

We now perform the usual perturbative expansion for the interaction picture evolution operator, restricting ourselves to the Born formula, $U_I=1-(i/\hbar)\int_{t_0}^t ds\ H_I(s)$. However, this leads to secular terms, and in order to eliminate them, we shall rather use this approximation close to the time $t=\tau$, by using the initial condition $U_I(\tau,t_0)=\alpha(\tau)$, such that

$$U_I(t,t_0)=\alpha(\tau)-\frac{i}{\hbar}\int_\tau^t ds\ H_I(s)\alpha(\tau)$$
$$+(\text{higher order terms}). \tag{3.7}$$

Retaining only the secular terms, we obtain

$$U_I(t,t_0)=\alpha(\tau)-\frac{i\omega_0}{2}(t-\tau)\left[\left(A-\frac{1}{4}\right)(aa^\dagger+a^\dagger a)\right.$$
$$\left.+q(e^{i\omega_0 t_0}a^2+e^{-i\omega_0 t_0}(a^\dagger)^2)\right]\alpha(\tau). \tag{3.8}$$

For the sake of simplicity, let us set $t_0=0$, without any loss of generality. We know that U_I cannot depend on the choice of τ, and we are thus led to the RG equation to first order

$$\frac{\partial\alpha}{\partial\tau}+\frac{i\omega_0}{2}\left[\left(A-\frac{1}{4}\right)(aa^\dagger+a^\dagger a)+q[a^2+(a^\dagger)^2]\right]\alpha(\tau)=0\ , \tag{3.9}$$

which, on being solved, provides us with an improved expression for the interaction picture evolution operator, $U_I(t,0)=\exp(-itH_{\text{eff}}/\hbar)$, where H_{eff} is the large time asymptotic effective Hamiltonian read off directly from the RG equation:

$$H_{\text{eff}}=\frac{\hbar\omega_0}{2}\left[\left(A-\frac{1}{4}\right)(aa^\dagger+a^\dagger a)+q[a^2+(a^\dagger)^2]\right]$$
$$=2\left[\frac{1}{2m}\left(A-\frac{1}{4}-q\right)P^2+\frac{1}{8}m\omega_0^2\left(A-\frac{1}{4}+q\right)X^2\right]. \tag{3.10}$$

This is the first important result of our computation: we have resummed the effect of the variable frequency into an effective large-time constant Hamiltonian. If it happens that, for small positive q, $\frac{1}{4}-q<A<\frac{1}{4}+q$, this effective Hamiltonian corresponds to an upside-down harmonic oscillator, thus marking the principal instability band (to the order we have computed).

It now behooves us to compute the creation of particles due to this instability. In order to do this, we shall first write down the integral kernel that corresponds to U_I in the position representation, $K(x,t;x',t')$, using standard results for quadratic Hamiltonians [11]. Let $\gamma=\omega_0\sqrt{q^2-(A-1/4)^2}$ and $\varphi=\sqrt{(q-1/4+A)/(q+1/4-A)}$. The integral kernel $K(x,t;x',0):=\langle x|U_I(t,0)|x'\rangle$ is computed to be (asymptotically)

$$K(x,t;x',0)=\left(\frac{im\omega_0\varphi}{4\pi\hbar\ \sinh(\gamma t)}\right)^{1/2}\exp\left(\frac{-im\omega_0\varphi}{4\hbar\ \sinh(\gamma t)}[(x^2+x'^2)\cosh(\gamma t)-2xx']\right). \tag{3.11}$$

It is now feasible to compute the asymptotic value of $\langle n|U(t,0)|0\rangle$ through simple tabulated integrals, and we can calculate the transition probability from the ground state to even states:

$$P_{0\to 2l}(t)=\frac{(2l)!}{(l!)^2 2^{2l}}\frac{2\varphi}{\sqrt{4\varphi^2+(1+\varphi^2)^2\ \sinh^2(\gamma t)}}\left(\frac{(1+\varphi^2)^2\ \sinh^2(\gamma t)}{4\varphi^2+(1+\varphi^2)^2\ \sinh^2(\gamma t)}\right)^l. \tag{3.12}$$

It is easy to check that unitarity is preserved. An analogous computation leads to the rate of particle production,

$$\mathcal{N}(t)=\frac{(1+\varphi^2)^2}{4\varphi^2}\sinh^2(\gamma t). \tag{3.13}$$

These results can be compared with the computations of Shtanov *et al.* [12], and coincide completely for the specific case at hand. Shtanov *et al.* arrive at this result through Bogolyubov transformations (to identify the function giving particle creation) and Krylov-Bogolyubov averaging (to perform the asymptotic analysis). This coincidence comes as no surprise given the first-order equivalence of Krylov-Bogolyubov averaging to two-timing (a particular instance of the multiple-scale method) for a wide class of differential equations [6], and the (again first-order) equivalence of the multiple-scale and RG methods for many instances of equations.

IV. CONCLUSIONS

We have performed several operator computations in quantum mechanics using the RG method for global asymptotic analysis. This method has the serious advantage that unitarity is built in, and that computations are simpler and more direct than in other techniques for asymptotic analysis. These ideas should be useful in a wide realm of applications. In the special case of quantum parametric resonance we derive explicitly an (asymptotic) effective Hamiltonian, which is an upside-down harmonic oscillator whenever the system is in the instability region: the instability associated with the parametric resonance is turned into the unboundedness of the interaction Hamiltonian, thus demonstrating the basic equivalence (asymptotically) of such different systems. Even so, unitarity is preserved throughout our computation, and the asymptotic results we obtain are well behaved with respect to this fundamental property of quantum mechanical evolution. We have performed an analysis of the first instability band only for the quantum Mathieu equation. However, it is possible within this method to examine the whole forbidden or allowed band structure of this model, following in the quantum context the study carried out for the classical case by Goldenfeld and collaborators. As a matter of fact, for quadratic Hamiltonians the whole instability analysis can be reduced to classical mechanics, i.e., to classical Mathieu (or similar) equations. What we emphasize as novel in our results is the interpretation of instabilities as being due to effective large time upside-down harmonic oscillator Hamiltonians. Furthermore, analogous analyses can be carried for nonquadratic Hamiltonians, even time dependent, where the quantum-mechanical character of the problem would show itself to its fullest.

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