Quantum stochastic motion in complex space

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We show that a quantum system may be associated with a backward stochastic process in complex configuration space when the so-called weak value of the position operator is interpreted as a conditional expectation value. The quantum-mechanical expectation values of the position, momentum, angular momentum, and energy are shown to be the weighted averages of the corresponding quantities for the stochastic process. Moreover, the stochastic trajectory is shown to reduce to the correct classical trajectory in the limit where the de Broglie wavelength vanishes. [S1050-2947(98)03003-0]

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I. INTRODUCTION

Since the establishment of a correspondence between the Schrödinger equation and a stochastic process in real configuration space [1,2], many efforts have been made to develop a full stochastic interpretation of quantum mechanics based on [1]. While the uncertainty relation [3–5] and the interference phenomena [6,7] can be properly interpreted within Nelson's theory, there is a fundamental problem with [1]. The problem arises when there are nodal surfaces in the amplitude of the wave function. The particle cannot move across the nodal surface. This means that the spaces separated by the nodal surfaces are mutually exclusive regions for the particle. Accordingly, the different parts of the wave function separated by the nodal surfaces should evolve independently. However, in general, this is not true.

Recently, a stochastic approach that is free from the difficulty of [1] has been proposed [8]. It is based on the idea [9] that views the so-called weak value in the weak measurement theory [10,11] as a conditional expectation value. (Hereafter, we shall refer to the weak value of the weak measurement of a quantity as the weak value of that quantity.) This view allows one to calculate from a backward stochastic differential equation in a complex configuration space the motion of a particle corresponding to an ensemble of physical systems prepared in a state $|\psi\rangle$ (preselected state) and conditioned to be in a position eigenstate at a later time (postselected state). The *final* condition for the differential equation is the position eigenvalue of the postselected state. The solution of this equation is referred to as *weak trajectory*. The problem of [1] is evaded by the fact that, in complex space, the nodal surfaces of the wave function in real space may be circumvented. The conditional expectation value of the weak trajectories is the weak value of the position operator. The real part of the weak trajectory is interpreted as the trajectory of a particle in real configuration space. This interpretation is justified by the reduction of the weak trajectories to their classical counterparts in the limit where the de Broglie wavelength of the particle vanishes.

Here we would like to investigate in more detail the recently proposed theory [8]. We shall proceed further to show that the quantum mechanical expectation values of the position, momentum, angular momentum, and energy can be interpreted as the weighted averages of the corresponding quantities for the stochastic process. The position-momentum uncertainty relation can be interpreted as the product of the mean-square deviations of the position and momentum of the stochastic process. This stochastic interpretation is extended to the relativistic single-particle system satisfying the Klein-Gordon equation. The extension is achieved by rewriting the Klein-Gordon equation as a stationary-state Schrödinger equation in a four-dimensional Euclidean space. The fluctuation in time is shown to vanish in the nonrelativistic limit.

The contents of this article are organized as follows. In Sec. II the weak measurement theory and the idea that views the weak value as the conditional expectation value in probability theory are reviewed. The conditional probability density, corresponding to a weak position measurement on a quantum system preselected in a state $|\psi\rangle$ and postselected in a position eigenstate, is shown to satisfy the Fokker-Planck equation with an imaginary diffusion coefficient $\nu = i\hbar/2m$, where m is the mass of the particle. In Sec. III a stochastic interpretation of quantum mechanics is introduced by associating a backward stochastic process in a complex configuration space with a quantum system. The correspondence between the stochastic process and the underlying quantum system is established by showing that the backward stochastic differential equation is equivalent to the Fokker-Planck equation. The quantum-mechanical expectation value of a function of the position operator is shown to be the weighted average of this function for the stochastic process. In Sec. IV the momentum, angular momentum, and energy of the stochastic process are defined in analogously to classical mechanics. Their weighted averages are shown to be equal to the quantum-mechanical expectation values of the corresponding operators. The uncertainty relation follows automatically by interpreting it as the product of the mean-square deviations of the position and momentum of the stochastic process. In Sec. V the stochastic interpretation developed in the previous sections is extended to the relativistic case. In Sec. VI further possible developments of this stochastic interpretation are discussed.

II. WEAK VALUE AND WEAK CONDITIONAL PROBABILITY

In quantum mechanics, a precision measurement of an observable **A** of a system is followed by the collapse of the system, caused by interaction between it and the measuring apparatus, from its initial state to an eigenstate of **A** whose eigenvalue is the measured value [12]. A complete collapse may, however, be avoided at the cost of losing precision. Aharonov, Albert, and Vaidman [10,11] explicitly took a time-dependent interaction $H(t)$ between the system and the apparatus [12] into account and used the uncertainty principle as applied to the momentum and position of the ''pointer'' of the apparatus to show that, by sacrificing the accuracy of the measurement, the system can be made to be disturbed as little, or weakly, as possible in a measurement. They argued that in such a weak measurement of, say, **A**, an accurate and meaningful result, called a weak value for **A**, is nevertheless obtained when an ensemble average is taken. The uncertainty in each individual weak measurement will of course be large. Specifically, for an ensemble of physical systems preselected at the state $|\psi\rangle$ at time $t=0$ and postselected in the state $|B\rangle$ at a later time t_f, the weak value for **A** in weak measurement made at time t, at $0\leqslant t\leqslant t_f$, is

$$\langle\mathbf{A}\rangle^{weak}\equiv\frac{\langle B|\exp(-i\int_t^{t_f}H(t)\ dt)\mathbf{A}\ \exp(-i\int_0^t H(t)\ dt)|\psi\rangle}{\langle B|\exp(-i\int_0^{t_f}H(t)\ dt)|\psi\rangle}. \tag{1}$$

This weak value is a complex quantity, whose real and imaginary parts correspond to the mean shifts in the position and momentum of the pointer, respectively. In what follows we shall apply the weak measurement theory to make measurements, always understood to be weak, in which the initial state is sufficiently undisturbed to retain its identity.

It was pointed out by Steinberg [9] that the weak value can be interpreted as a conditional expectation value in probability theory. Since this notion is crucial to the main theme of this work, we briefly discuss it. In probability theory, the conditional probability that **A** has value A given **B** has value B is

$$P(A|B)\equiv P(A\text{ and }B)/P(B). \tag{2}$$

In quantum mechanics, the probability that a physical system initially in the state $|\psi\rangle$ found subsequently in the state $|B\rangle$ is $\langle\psi|\mathrm{Proj}(B)|\psi\rangle$, where

$$\mathrm{Proj}(B)=\exp(i\textstyle\int Hdt/\hbar)|B\rangle\langle B|\ \exp(-i\int Hdt/\hbar)$$

is the projection operator. That is, to get the quantum equivalence $\langle\psi|\mathrm{Proj}(B)|\psi\rangle$ of the probability $P(B)$, the projection operator $\mathrm{Proj}(B)$ takes the place of the condition B in probability theory. Similarly, the condition A and B is to be replaced by the time-ordered product $\mathrm{Proj}(B)\mathrm{Proj}(A)$. The quantum equivalence of $P(A|B)$, denoted by $\overline{P(A|B)}$, which is viewed by Steinberg [9] as the conditional probability of the measurement **A** yielding the value A on an ensemble of physical systems initially prepared in the state $|\psi\rangle$ and constrained to be in the state $|B\rangle$ at a later time, is therefore

$$\begin{aligned}\overline{P(A|B)}&\\ &\equiv\frac{\langle\psi|\mathrm{Proj}(B)\mathrm{Proj}(A)|\psi\rangle}{\langle\psi|\mathrm{Proj}(B)|\psi\rangle}\\ &=\frac{\langle B|\exp(-i\int_t^{t_f}H(t)\ dt)|A\rangle\langle A|\exp(-i\int_0^t H(t)\ dt)|\psi\rangle}{\langle B|\exp(-i\int_0^{t_f}H(t)\ dt)|\psi\rangle}.\end{aligned} \tag{3}$$

Since $\langle\psi|\mathrm{Proj}(A)|\psi\rangle$ is the probability that **A** has value A when $|A\rangle$ is an eigenstate of **A**, Eq. (3) yields the conditional expectation value of **A**,

$$\langle\mathbf{A}\rangle_B\equiv\sum_A A\overline{P(A|B)}=\langle\mathbf{A}\rangle^{weak}, \tag{4}$$

where the sum extends over all the eigenvalues of **A**. The quantum-mechanical expectation value of **A** is the weighted average of its weak value,

$$\langle\mathbf{A}\rangle=\sum_B |\langle B|A\rangle|^2\langle\mathbf{A}\rangle^{weak}. \tag{5}$$

Since $\overline{P(A|B)}$ is in general complex when the two operators **A** and **B** do not commute, we shall refer to it as the *weak conditional probability* to distinguish it from a real-valued probability.

Now consider the position measurement on a physical system preselected in the state $|\psi\rangle$ at $t=0$, where **A** is the position operator $\hat{x}$ and the postselected state $|B\rangle$ at $t=t_f$ is $|x_f\rangle$. For simplicity we shall consider a single-particle system, although the following argument also applies to a many-particle system. From Eq. (3) the weak conditional transition probability density corresponding to a position measurement at $t_f\geqslant t\geqslant 0$ is

$$\overline{P(x,t|x_f,t_f)}=\frac{K(x_f,t_f;x,t)\psi(x,t)}{\psi(x_f,t_f)}, \tag{6}$$

where $K(x_f,t_f;x,t)$ is the quantum propagator. Note that given a quantum distribution at t_f, this probability density yields a distribution at t and that the two relations

$$\lim_{t\to t_f}\overline{P(x,t|x_f,t_f)}=\delta(x-x_f),\qquad \int dx\ \overline{P(x,t|x_f,t_f)}=1$$

are satisfied. Writing the wave function $\psi(x,t)$ as

$$\psi(x,t)=\sqrt{\rho(x,t)}e^{iS(x,t)/\hbar} \tag{7}$$

and making use of the Schrödinger equation for a particle with charge e and mass m in a vector potential $\vec{A}$, it is straightforward to show by direct substitution that the weak conditional probability density satisfies the backward Fokker-Planck equation

$$\begin{aligned}\frac{\partial}{\partial t}\overline{P(x,t|x_f,t_f)}+\vec{\nabla}\cdot[v_-(x,t)\overline{P(x,t|x_f,t_f)}]&\\ +\nu\nabla^2\overline{P(x,t|x_f,t_f)}=0,&\end{aligned} \tag{8}$$

where $\nu=i\hbar/2m$ is an imaginary diffusion coefficient and v_- is a backward drift velocity given by

$$v_-(x,t)=\frac{1}{m}\vec{\nabla}S(x,t)-\frac{e}{mc}\vec{A}(x,t)-2\nu\frac{\vec{\nabla}\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}}, \tag{9}$$

where c is the velocity of light. We note that in Eq. (8) both $v_-(x,t)$ and $\overline{P(x,t|x_f,t_f)}$ are complex functions of *real* variables.

The weak value of $\hat{x}$ is in general complex and is a function of time. It traces out a path, which ends at the point x_f, for $t \leqslant t_f$ in a complex space. This path can be calculated from $v_-(x,t)$ and $\overline{P(x,t|x_f,t_f)}$, namely,

$$\frac{d}{dt}\langle\hat{x}\rangle^{weak}=\frac{d}{dt}\int dx\; x\; \overline{P(x,t|x_f,t_f)}$$
$$=\int dx\; x\frac{\partial}{\partial t}\overline{P(x,t|x_f,t_f)}$$
$$=\int dx\; v_-(x,t)\overline{P(x,t|x_f,t_f)}. \qquad (10)$$

According to the weak measurement theory [10,11], the real part of this path is the ensemble-averaged trajectory of the particles preselected in the state $|\psi\rangle$ and postselected in the state $|x_f\rangle$. Differentiating Eq. (10) with respect to t and making use of the Schrödinger equation, we have

$$m\frac{d^2}{dt^2}\langle\hat{x}\rangle^{weak}=\int dx\left(-\vec{\nabla}V+e\vec{E}+\frac{e}{c}v_-\times\vec{B}-\nu\frac{e}{c}\vec{\nabla}\times\vec{B}\right)\overline{P(x,t|x_f,t_f)}, \qquad (11)$$

where V is the non-electromagnetic potential and $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, respectively. For linear systems, the real and imaginary parts of Eq. (11) decouple and the real part of $\langle\hat{x}\rangle^{weak}$ satisfies the classical equation of motion with the *final* conditions

$$\mathrm{Re}\langle\hat{x}\rangle^{weak}_{t_f}=x_f, \quad \frac{d}{dt}\mathrm{Re}\langle\hat{x}\rangle^{weak}_{t_f}=\mathrm{Re}v_-(x_f,t_f),$$

where Re( ) is the real part of ( ). The ensemble-averaged motion of the particle follows a classical path. For nonlinear system, the ensemble-averaged motion of the particle in general does not follow the classical path. Neverthless, the classical path can be obtained in the limit $\nu\rightarrow 0$ (the physical meaning of this limit will be discussed later). To see this, we note that in this limit all the quantities become real and the Fokker-Planck equation reduces to the Liouville equation, whose solution is a delta function [13]. Let us denote this delta function as $\delta(x-x_c(t))$; then

$$\langle\hat{x}\rangle^{weak}\rightarrow x_c(t),$$

$$v_-(x,t)\rightarrow v_c(x_c,t)=\frac{1}{m}\vec{\nabla}S(x_c,t)-\frac{e}{mc}\vec{A}(x_c,t), \qquad (12)$$

and Eq. (11) reduces to the classical equation of motion of a charged particle,

$$m\frac{d^2x_c}{dt^2}=-\vec{\nabla}V+e\vec{E}+\frac{e}{c}v_c\times\vec{B}, \qquad (13)$$

with the *final* conditions

$$x_c(t_f)=x_f, \quad v_c(t_f)=\frac{1}{m}\vec{\nabla}S(x_f,t_f)-\frac{e}{mc}\vec{A}(x_f,t_f).$$

We now discuss the physical meaning of the limit $\nu\rightarrow 0$. Since $\nu=i\hbar/2m$, the condition $\hbar/m\rightarrow 0$ necessarily implies that $\hbar/m$ is small compared to a certain characteristic quantity of the system of the same dimension, namely, (length)^2/time). A natural candidate for this characteristic quantity is lv, where l is a characteristic length of the system to be specified later and v is the velocity of the particle. The limiting condition is therefore $\hbar/mlv\rightarrow 0$, or $\lambda_d/2\pi l\rightarrow 0$, where λ_d is the de Broglie wavelength of the particle. Since in this limit $v_-(x,t)$ has to reduce to the classical velocity of the particle such that the correct classical trajectory can be obtained, it follows that the imaginary part of $v_-(x,t)$ has to vanish in the limit $\lambda_d/2\pi l\rightarrow 0$, namely, $(\hbar/m)(\vec{\nabla}\sqrt{\rho}/\sqrt{\rho})\ll v$ or, equivalently, $\lambda_d(\vec{\nabla}\sqrt{\rho}/\sqrt{\rho})\ll 1$. That is, in the limit $\lambda_d/2\pi l\rightarrow 0$ *the amplitude of the wave function does not vary appreciably in the space of one de Broglie wavelength*. We refer to this as the classical limit. Under this condition the wave nature of the particle becomes negligible. Apparently, the characteristic length l is the length over which the amplitude of the wave function changes significantly. We want to emphasize the importance of the condition specified above because the condition $\hbar/m\rightarrow 0$ does not necessarily lead to the vanishing of the imaginary part of $v_-(x,t)$ due to the dependence of $\rho(x,t)$ on $\hbar/m$. A complex final condition for Eq. (13) would lead to a complex solution $x_c(t)$ whose real part alone will not satisfy the classical equation of motion for nonlinear system.

III. STOCHASTIC INTERPRETATION OF QUANTUM MECHANICS

Equation (8) is reminiscent of the backward stochastic differential equation

$$d^*\xi(t)=v_-(\xi,t)dt+d^*W(t), \qquad (14)$$

constrained by the *final* condition $\xi(t_f)=x_f$, where $d^*\xi(t)=\xi(t)-\xi(t-dt)$ and $d^*W(t)=W(t)-W(t-dt)$ is a Brownian-type displacement with the same diffusion coefficient ν used in Eq. (8). Equations (8) and (14) are identical for real ν [13] (negative ν corresponds to a forward process). In the present case, with ν being purely imaginary, the two equations are defined in different configuration spaces. Equation (8) is defined in a real configuration space, while Eq. (14) is defined in a complex configuration space. However, we assert that as far as the weak value of the position operator is concerned, Eq. (14) is equivalent to Eq. (8), provided that v_- is an analytic function in complex configuration space. Specifically, the weak value of $\hat{x}^n$ equals the conditional expectation value of $\xi^n(t)$ for all non-negative integers n [14]. Before showing that they are equivalent let us first show that the stochastic trajectory of Eq. (14) reduces to the classical trajectory in the classical limit.

An application of the Ito calculus [13] on the conditional expectation value of Eq. (14) yields

$$d^2E(\xi(t))=dE(v_-(\xi,t))dt=E\left(\frac{\partial v_-}{\partial t}+v_-\cdot\vec{\nabla}v_--\frac{i\hbar}{2m}\nabla^2v_-\right)(dt)^2. \qquad (15)$$

Substituting Eq. (9) into Eq. (15) and making use of the Schrödinger equation, we have after some manipulation

$$m\frac{d^2E(\xi(t))}{dt^2}=E\left(-\vec{\nabla}V+e\vec{E}+\frac{e}{c}v_-\times\vec{B}-\frac{i\hbar}{2m}\frac{e}{c}\vec{\nabla}\times\vec{B}\right), \tag{16}$$

where V, $\vec{E}$, and $\vec{B}$ are defined in Eq. (11). Since $|d^*W| \propto\sqrt{\hbar/m}$, in the classical limit, the stochastic motion reduce to a deterministic motion, v_- reduces to v_c, all the different $\xi(t)$ in the ensemble reduces to a single trajectory in real configuration space, and Eq. (16) reduces to Eq. (13) with the same final conditions. This shows that the real part of every single weak trajectory in the ensemble reduces to the same classical trajectory, while the imaginary part of the weak trajectory vanishes in the classical limit.

We now return to the assertion that Eq. (14) is equivalent to Eq. (8). The task is to show that the conditional expectation value $E(\xi^n)$ equals $\langle\hat{x}^n\rangle^{weak}$ for all non-negative integers n. In [8] this equality is shown for the cases that $v_-(x,t)$ is an analytic function of x. Here we give a general argument independent of the analytic property of $v_-(x,t)$ on x. Our argument is based on the theorem of Poincaré: If a differential equation depends holomorphically on a parameter and the boundary conditions are independent of that parameter, then the solutions of the equation are holomorphic functions of the parameter. Taking the diffusion coefficient ν as a parameter, the theorem of Poincaré asserts that the solution of Eq. (8) is an analytic function of ν. Thus $\langle\hat{x}^n\rangle^{weak}$ is an analytic function of ν. In Eq. (14) $|d^*W|\propto|\sqrt{2\nu}|$; it seems that this equation has a branch cut in the complex ν plane. However, a branch cut in the complex ν plane only results in an overall sign change on d^*W. For a Gaussian-type random noise, an overall sign change does not result in any difference to the solution. The solution of Eq. (14) is an analytic function of ν and so is $E(\xi^n)$. As pointed out above, Eq. (14) is equivalent to Eq. (8) for real ν. This means that $E(\xi^n)=\langle\hat{x}^n\rangle^{weak}$ for real ν. The fact that both are analytic functions of ν ensures that they are equal for all ν. This establishes the equivalence between Eqs. (14) and (8). Specifically, if f is an analytic function of space, then at $t\leqslant t_f$ the weak value of $f(\hat{x},t)$ equals the conditional expectation value of $f(\xi,t)$ over all the weak trajectories constrained to be at the final point x_f at the time t_f. That is, at $t\leqslant t_f$

$$\langle f(\hat{x},t)\rangle^{weak}=\int dx\ f(x,t)\overline{P}(x,t|x_f,t_f)$$
$$=\int dz\ f(z,t)P(z,t|x_f,t_f), \tag{17}$$

where $P(z,t|x_f,t_f)$ is the conditional probability density for the backward stochastic process of Eq. (14) in the complex space z.

This equivalence enables us to associate a backward stochastic process in complex configuration space to an ensemble of quantum systems preselected in a state $|\psi\rangle$ and postselected in a position eigenstate. As both the real and imaginary parts of $\langle\hat{x}^n\rangle^{weak}$ have well-defined physical meanings based on the weak measurement theory [10,11], so does the real and imaginary parts of $E(\xi^n)$. Since $E(\xi^n)$ equals the weak value of the operator $\hat{x}^n$ we refer to the solution of Eq. (14) as the *weak trajectory* of a particle initially prepared in the state $|\psi\rangle$ and constrained to be at x_f at time t_f. According to the weak measurement theory [10,11], the real part of the conditional expectation value of the weak trajectory is the ensemble-averaged trajectory of the particle in real configuration space. We note that this ensemble-averaged trajectory of the particle is the ensemble average of the real part of the weak trajectory. Also, as shown before, the real part of every single weak trajectory in the ensemble reduces to the same classical trajectory in the classical limit. In view of these physical meanings for the real part of the weak trajectory, we therefore interpret the real part of a *single* weak trajectory derived from Eq. (14) as the trajectory of a single particle in real configuration space for $0\leqslant t\leqslant t_f$.

The quantum-mechanical expectation value of $f(\hat{x},t)$ at the time $t\leqslant t_f$ can be interpreted as the weighted average of $f(z,t)$ for the backward stochastic process that has the final spatial distribution $\rho(x_f,t_f)$, where $\rho(x,t)$ is the spatial distribution of the quantum system. To see this, let us denote the spatial distribution of the stochastic process in the complex space as $\varrho(z,t)$. Then $\varrho(z,t_f)=\rho(x_f,t_f)$ and for $t\leqslant t_f$

$$\varrho(z,t)=\int dx_f\ P(z,t|x_f,t_f)\rho(x_f,t_f). \tag{18}$$

Using Eqs. (5), (17), and (18), the quantum-mechanical expectation value of $f(\hat{x},t)$ at $t\leqslant t_f$ is

$$\langle f(\hat{x},t)\rangle=\int dx_f\ \langle f(\hat{x},t)\rangle^{weak}\rho(x_f,t_f)$$
$$=\int\int dx_f\ dz\ f(z,t)P(z,t|x_f,t_f)\rho(x_f,t_f)$$
$$=\int dz\ f(z,t)\varrho(z,t). \tag{19}$$

IV. MOMENTUM, ANGULAR MOMENTUM, ENERGY, AND UNCERTAINTY RELATION

We have seen in the preceding section that, as far as the function of the position operator is concerned, the backward stochastic process Eq. (14) is equivalent to the quantum system. We now show that this equivalence can be extended to the momentum, angular momentum, and energy and the uncertainty relation holds for the stochastic process. To do this, we have to find sensible definitions of these quantities for the stochastic process. Also, since the stochastic process is defined in the configuration space, it is necessary to relate the operators of these quantities to the position operator. For simplicity we shall consider the case that the vector potential $\vec{A}$ is zero. From Eq. (1) the weak values of the momentum, angular momentum, and energy can be expressed in terms of the weak values of functions of position operator

$$\langle\hat{p}\rangle^{weak}=\int dx\ mv_-(x,t)\overline{P}(x,t|x_f,t_f)=\langle mv_-(\hat{x},t)\rangle^{weak}, \tag{20}$$

$$\langle \hat{x}\times\hat{p}\rangle^{weak}=\int dx\,[x\times mv_-(x,t)]\overline{P(x,t|x_f,t_f)}$$
$$=\langle \hat{x}\times mv_-(\hat{x},t)\rangle^{weak}, \tag{21}$$

$$\langle \hat{p}^2\rangle^{weak}=\int dx\,[m^2v_-^2(x,t)$$
$$-i\hbar m\vec{\nabla}\cdot v_-(x,t)]\overline{P(x,t|x_f,t_x)}$$
$$=\langle m^2v_-^2(\hat{x},t)-i\hbar m(\vec{\nabla}\cdot v_-)(\hat{x},t)\rangle^{weak}, \tag{22}$$

$$\langle \hat{H}\rangle^{weak}=\int dx\left[\frac{1}{2}mv_-^2(x,t)-\frac{i\hbar}{2}\vec{\nabla}\cdot v_-(x,t)\right.$$
$$\left.+V(x,t)\right]\overline{P(x,t|x_f,t_f)}=\left\langle\frac{1}{2}mv_-^2(\hat{x},t)\right.$$
$$\left.-\frac{i\hbar}{2}(\vec{\nabla}\cdot v_-)(\hat{x},t)+V(\hat{x},t)\right\rangle^{weak}. \tag{23}$$

The next step is to find the counterparts of the above quantities for the stochastic process. We base our consideration on classical mechanics. For deterministic motion, the velocity of a particle at a point on its trajectory is defined as

$$v(x,t)=\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t},$$

where $x(t)$ is the trajectory of the particle. For stochastic motion, the above definition cannot be directly applied to a single weak trajectory due to the existence of the random term. Nevertheless, an ensemble-averaged velocity at every point of the weak trajectory can be defined. Using Eq. (14) the ensemble-averaged velocity entering the point ξ at the time t is

$$\lim_{\Delta t\to 0}E\left(\frac{\Delta\xi}{\Delta t}\right)=v_-(\xi,t). \tag{24}$$

This suggests that the momentum and angular momentum at a point ξ on a weak trajectory can be defined as

$$p(\xi,t)=mv_-(\xi,t), \tag{25}$$

$$l(\xi,t)=\xi\times p(\xi,t). \tag{26}$$

The kinetic energy at every point on a weak trajectory can be defined in an analogous way [1]. Noting that it involves the square of $\Delta\xi$ and $\Delta W^2\propto\Delta t$, the contribution from the random term has to be treated properly. An integration of Eq. (14) yields

$$\xi(t-\Delta t)=\xi(t)+\int_t^{t-\Delta t}dr\;v_-(\xi(r),r)+W(t-\Delta t)-W(t). \tag{27}$$

This means that

$$\Delta\xi=\xi(t)-\xi(t-\Delta t)$$
$$=-\int_t^{t-\Delta t}dr\;v_-(\xi(r),r)+W(t)-W(t-\Delta t). \tag{28}$$

Substituting Eq. (27) into the integral of Eq. (28), expanding v_- with respect to the point $\xi(t)$ to the order Δt, and making use of the Ito calculus [13], Eq. (28) can be expressed as

$$\Delta\xi_k=v_{-k}(\xi,t)\Delta t-\vec{\nabla}\cdot\left(v_{-k}\int_t^{t-\Delta t}dr\,[W(t)-W(r)]\right)$$
$$+\Delta W_k. \tag{29}$$

This leads to the result [1]

$$\lim_{\Delta t\to 0}E\left(\left[\frac{\Delta\xi}{\Delta t}\right]^2\right)=v_-^2(\xi,t)-\frac{i\hbar}{m}\vec{\nabla}\cdot v_-(\xi,t)+\lim_{\Delta t\to 0}i\frac{3\hbar}{2m\Delta t}. \tag{30}$$

The singular term in Eq. (30) is a constant that is the same for all weak trajectories. It can be removed from the energy by a proper choice of the zero point for the energy. This suggests that the kinetic energy can be defined as

$$\frac{p^2}{2m}=\frac{1}{2}mv_-^2-\frac{i\hbar}{2}\vec{\nabla}\cdot v_-,$$

or equivalently

$$p^2(\xi,t)=m^2v_-^2(\xi,t)-i\hbar m\vec{\nabla}\cdot v_-(\xi,t), \tag{31}$$

$$E(\xi,t)=\frac{1}{2}mv_-^2(\xi,t)-\frac{i\hbar}{2}\vec{\nabla}\cdot v_-(\xi,t)+V(\xi,t). \tag{32}$$

With the above definitions of momentum, angular momentum, and energy for the stochastic process, it follows from Eq. (17) that the conditional expectation values of these quantities for the stochastic process are the weak values of the corresponding quantities for the quantum mechanics. Similarly, by Eq. (19), the weighted averages of these quantities for the stochastic process with the final spatial distribution $\rho(x_f,t_f)$ are the quantum-mechanical expectation values of the corresponding quantities. It also follows from Eq. (19) that the product of the mean-square deviations of position and momentum for the stochastic process is the position-momentum uncertainty relation of quantum mechanics.

V. RELATIVISTIC CASE

We have seen that a backward stochastic process can be associated with a nonrelativistic quantum system. It is possible to extend this stochastic interpretation to a single-particle quantum system satisfying the Klein-Gordon equation. The key to the extension is to note that the Klein-Gordon equation can be written as a stationary-state Schrödinger equation in a four-dimensional Euclidean space [15]. The transformation between the Minkowski space and the Euclidean space is

$$(x^1,x^2,x^3,x^4)=(x,y,z,ict),$$

$$(p_1,p_2,p_3,p_4)=\left(p_x,p_y,p_z,\frac{iE}{c}\right).$$

With these transformations, a time operator ($\hat{x}^4$) is introduced analogously to the position operator. The conjugate four-momentum operators are

$$\hat{p}_\mu=\frac{\hbar}{i}\frac{\partial}{\partial x^\mu},\quad \mu=1,2,3,4.$$

Writing the Klein-Gordon equation as

$$\frac{1}{2m}\left(\hat{p}^\mu-\frac{e}{c}A^\mu\right)\left(\hat{p}_\mu-\frac{e}{c}A_\mu\right)\psi=-\frac{mc^2}{2}\psi,$$

where $a^\mu=a_\mu$ in four-dimensional Euclidean space, the solution ψ can be viewed as one of the stationary-state solutions of the four-dimensional Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial\tau}=\frac{\hbar^2}{2m}\left(\frac{\hbar}{i}\vec{\nabla}-\frac{e}{c}A\right)^2,$$

with

$$\Psi(x,\tau)=e^{-i(mc^2/2\hbar)\tau}\psi(x).$$

Here τ plays the role of time. We shall see that in the classical limit τ is the proper time of the particle. Writing ψ as

$$\sqrt{\rho(x)}e^{iS(x)/\hbar},$$

Ψ can be expressed in the same form as Eq. (7),

$$\Psi(x,\tau)=\sqrt{\rho(x)}e^{i[mc^2\tau/2+S(x)]/\hbar}.$$

All the formulations and interpretations developed in the previous sections follow immediately.

To see the physical meaning of the parameter τ let us consider the free particle case. The wave function of this state is

$$\Psi(x,\tau)=e^{i(mc^2/2\hbar)\tau}e^{i(p_\mu x^\mu/\hbar)}.$$

The backward stochastic process corresponding to an ensemble of physical systems preselected in this state and postselected in the four-position eigenstate at τ_f is

$$d^*\Xi_\mu(\tau)=\frac{p_\mu}{m}d\tau+d^*W_\mu, \tag{33}$$

constrained by the final condition $\Xi(\tau_f)=x_f$. The random term of the spatial component vanishes in the classical limit, as discussed in Sec. II. What about the random term of the time component? Noting that $x_4=ict$ and $|d^*W_4|\propto\sqrt{\hbar/m}$, the fluctuation in time vanishes in the limit $\hbar/mc^2=\lambda_c/2\pi c\to 0$, where λ_c is the Compton wavelength of the particle. The dimension of λ_c/c indicates that the limiting condition is satisfied if it is small compared to the quantity l/v, where l is the size of the system and v is the velocity of the particle. The limiting condition is therefore $(\lambda_c/2\pi l)(v/c)\to 0$. The fluctuation in time vanishes in the nonrelativistic limit. In the relativistic case $(\lambda_c/2\pi l)(v/c)\approx\lambda_c/2\pi l$ and the fluctuation in time is important when the size of the system is of the order of a Compton wavelength. This time fluctuation may be related to the vacuum fluctuation (virtual particle-antiparticle production and annihilation). In the classical limit the random term vanishes and the time component of Eq. (33) becomes

$$dt=\frac{E}{mc^2}d\tau.$$

Since $E=mc^2/\sqrt{1-(v/c)^2}$ we have $dt=d\tau/\sqrt{1-(v/c)^2}$. This shows that τ is the proper time of the particle.

VI. SUMMARY AND DISCUSSION

We have proposed a stochastic interpretation of quantum mechanics in which a quantum system is associated with a backward stochastic process in complex configuration space. The equivalence between the stochastic process and the quantum mechanics is established through the following results. (i) For the stochastic trajectories in the complex space, which we call weak trajectories, constrained to be at the final position x_f at time t_f, we showed that the conditional expectation values of their moments are equal to the weak values of moments of the position operator of the quantum system. This leads to the equality (17) between the weak value of a function of the position operator $f(\hat{x},t)$ and the conditional expectation value of $f(z,t)$ for the stochastic process. (ii) Using Eqs. (5) and (17), for $t\leqslant t_f$, the quantum-mechanical expectation value of $f(\hat{x},t)$ is shown to be the weighted average of $f(z,t)$ for the stochastic process that has the spatial distribution $\rho(x_f,t_f)$ at $t=t_f$, where $\rho(x,t)$ is the spatial distribution of the quantum system. This equality is expressed in Eq. (19). (iii) The weak values and expectation values of the momentum, angular momentum, and energy for the quantum system are shown to be the conditional expectation values and weighted averages of the corresponding quantities, respectively, for the stochastic process. The position-momentum uncertainty relation follows automatically with the interpretation that it is the product of the mean-square deviations of position and momentum for the stochastic process.

The real part of the weak trajectory is interpreted as the trajectory of a particle in real configuration space. This interpretation is justified, on the one hand, by the equality (17) between the position operator and the weak trajectory and, on the other hand, by the reduction of a *single* weak trajectory to the correct classical trajectory in the classical limit. Finally, the stochastic interpretation is extended to the relativistic single-particle system satisfying the Klein-Gordon equation by treating the Klein-Gordon equation as a stationary-state Schrödinger equation in four-dimensional Euclidean space. The fluctuation in time is shown to vanish in the nonrelativistic limit.

A question needs to be answered: Why is it the backward instead of the forward stochastic process that corresponds to the quantum system considered? The reason that it is the backward process is because the weak value depends not only on the preselected state but also on the postselected state. One has to go backward in time from the postselection to determine the weak value of a physical quantity. Another question is whether our interpretation could be extended to the weak values and expectation values of the other observables in addition to the operators discussed in Secs. III and IV. The answer is yes because all the measurements of the physical observables are essentially position measurements. One could in principle relate the operator of an observable to a corresponding function of position operator as done in Sec. IV for the momentum, angular momentum, and energy operators.

The stochastic interpretation presented here does not have the shortcoming of the previous theory [1] as discussed in Sec. I. It opens up the possibility of a full stochastic interpretation of quantum mechanics. One could proceed further to develop a theory of stochastic mechanics in complex space in an analogous way to [1]. This would involve the consideration of a stochastic process in complex space satisfying a certain dynamical law. The drift velocity of this stochastic process will be determined by this dynamical law. With a properly chosen dynamics the quantum mechanics would be the real-space manifestation of the stochastic mechanics in complex space. Work along this line is currently under way.

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