Interference in hyperbolic space

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The interference in a phase space algorithm of Schleich and Wheeler [Nature **326**, 574 (1987)] is extended to the hyperbolic space underlying the group SU(1,1). The extension involves introducing the notion of weighted areas. Analytic expressions for the asymptotic forms for overlaps between the eigenstates of the generators of su(1,1) thus obtained are found to be in excellent agreement with the numerical results. [S1050-2947(98)08602-8]

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I. INTRODUCTION

A central problem in quantum mechanics is the calculation of the overlap, that is, the scalar product between two quantum states. Wheeler, Schleich, and co-workers [1–5] have given an elegant geometrical algorithm to find the semiclassical limits of the overlap integral. According to this algorithm, a state is represented in an x-p phase space by a PBS (Planck-Bohr-Sommerfeld) band with a total area of 2π ($\hbar=1$). The overlap between bands of two quantum states $|\Psi\rangle$ and $|\Phi\rangle$ gives intuitive interpretation of the probability amplitude $\langle\Psi|\Phi\rangle$, i.e., in the case of more than one overlap the contributing amplitudes have to be combined, the phase difference determined by an area bounded by the two bands. Thus interference features arise. As a simple example, if we describe a harmonic oscillator state $|n\rangle$ in an x-p phase space as an annulus centered on the origin and the position state $|x\rangle$ as a long strip located at x (Fig. 1), then the probability of finding the particle at position x can be approximated as

$$W_{nx}=|\langle x|n\rangle|^2$$
$$=\left|\left(\frac{1}{\pi}\right)^{-1/4}(2^n n!)^{-1/2}H_n(x)e^{-x^2/2}\right|^2 \quad (1)$$
$$\approx\left|\left(\frac{A^{\diamond}_{nx}}{2\pi}\right)^{1/2}(e)^{-i(S_{nx}-\pi/4)}+\left(\frac{A^{\diamond}_{nx}}{2\pi}\right)^{1/2}(e)^{i(S_{nx}-\pi/4)}\right|^2. \quad (2)$$

Here $1/2\pi$ can be regarded as classical probability density and $A^{\diamond}_{nx}$ is the area of the diamond shaped region of the overlap between the two PBS bands, while S_{nx} is the area bounded by two Kramer trajectories associated with states $|n\rangle$ and $|x\rangle$. Obviously, formula (2) predicts oscillation when the strip band moves from left to right.

Phase-space-interference (PSI) approaches have been applied, for example, to oscillations in transition amplitudes for Frank-Condon transitions [5], the photon number and the phase distribution [2,6–8] for squeezed states, the interference fringes exhibited by superposition states [9], etc. Attempts have also been made to apply such approaches to extended phase spaces [10,13,14].

Lassig and Milburn [10] discussed the semiclassical limits of the angular-momentum marginal probability distributions by applying the PSI approach to a compact spherical phase space; a two-sphere embedded in a three-dimensional Euclidean space, (J_x,J_y,J_z) with

$$J^2=J_x^2+J_y^2+J_z^2. \quad (3)$$

This is the classical phase-space representation for a particle sliding, without friction, on a surface of constant positive curvature. An angular-momentum state, say eigenstate $|j,m\rangle_z$ of $\hat{J}_z$, is represented by a PBS band centered on a Kramer trajectory with $J_z=m$. The radius of the sphere is $\sqrt{j(j+1)}$, which is the square root of the eigenvalues of the Casimir invariant of the SU(2) group

$$\hat{J}^2=\hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2. \quad (4)$$

The groups SU(2) and SU(1,1) are of great interest in many branches of physics [15–17]. In particular, it is widely used in the study of nonclassical properties of light in quantum optics [18–21]. Thus, for instance, passive interferometers can be characterized by an SU(2) symmetry, while active interferometers involve the group SU(1,1) [21]. Motivated by Ref. [10], it is natural to ask the question of

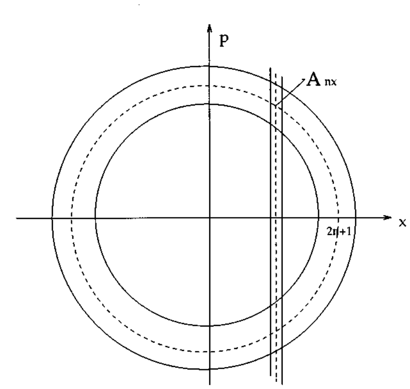

FIG. 1. Phase representation of a harmonic state $|n\rangle$ and $|x\rangle$ as PBS bands. Kramer trajectories corresponding to the two states are depicted by dashed lines.

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whether such an interference in a phase-space approach can be extended to the SU(1,1) case. The primary aim of this paper is to answer this question in the affirmative.

The su(2) generators can be naturally associated with constants of motion for geodesic motion on a surface of constant positive curvature, which can be visualized as a sphere embedded in a three-dimensional Euclidean space. The generators of su(1,1), on the other hand, may be associated with geodesic motion on a surface of constant negative curvature, i.e., a hyperbolic surface [11]. Such a surface may be visualized by a global embedding in a space endowed with a Minkowski metric. In this paper we will adapt the interference in a phase space algorithm to the case of a hyperbolic phase space and thus obtain asymptotic expressions for the overlap of eigenstates of the generators of su(1,1). In doing this we show a crucial difference between the hyperbolic case and the spherical case. In the spherical case, the appropriate phase space has the same geometry as the configuration space, i.e., a sphere. However, in the hyperbolic case the appropriate phase-space is not represented by the same embedded sheet as the configuration space.

In this paper, we use an interference in the phase-space method to derive approximate results for the overlap between eigenstates of two generators of su(1,1) algebra. In Sec. II, the appropriate phase-space arena for applying the interference in the phase-space algorithm is identified by considering the motion of a particle sliding without friction on a surface of constant negative curvature. The exact quantum calculation is given in Sec. III. In Sec. IV, by introducing a two-dimensional hyperbolic space embedded in a Minkowski three-dimensional space and replacing the *areas* in Eq. (2) by corresponding *weighted areas* for the noncompact group SU(1,1), we find the asymptotic form of the exact result. The results obtained by two approaches coincide well with each other and are discussed in the last section.

II. CLASSICAL MOTION ON A HYPERBOLIC SURFACE

The classical description of motion on surfaces of constant negative curvature has been extensively discussed by Balazs and Voros [11]. We summarize their approach, with a minor change of notation. Consider a classical point particle of unit mass sliding without friction on a hyperbolic sheet defined by

$$\mathbf{x}\cdot\mathbf{x}=(x^1)^2+(x^2)^2-(x^3)^2=-1, \tag{5}$$

where the dot product is defined with the Minkowski, nonpositive definite Minkowski metric g_{ij} with signature $(+1,+1,-1)$. This defines two disconnected hyperbolic sheets intersecting the x^3 axes at two points ± 1. We shall only consider the upper hyperbolic sheet with $x^3>1$. This hyperbolic sheet will be referred to as the configuration manifold.

We now determine the dynamics on the configuration manifold. The Lagrangian of a free particle moving on this sheet is defined by the kinetic energy, which in Minkowski coordinates takes the form

$$\mathcal{L}=\tfrac{1}{2}\,\dot{\mathbf{x}}^2, \tag{6}$$

where the overdot denotes differentiation with respect to time. Of course we must also include the constraint in Eq. (5). The Euler-Langrange equations then take the form

$$\ddot{\mathbf{x}}=C\mathbf{x}, \tag{7}$$

where $C=2E$ with E the conserved kinetic energy, which is of course positive. The solutions to these equations define geodesics on the hyperbolic surface. In analogy with the conserved components of angular momentum for motion on a sphere we have the following constants of motion:

$$K^1=(x_2\dot{x_3}-x_3\dot{x_2}),$$

$$K^2=(x_3\dot{x_1}-x_1\dot{x_3}),$$

$$K^3=-(x_1\dot{x_2}-x_2\dot{x_1}).$$

These functions form the components of the Minkowski three-vector.

The Hamiltonian description is found in the usual way. The canonical momenta are defined by $p_1=\dot{x}^1, p_2=\dot{x}^2, p_3=-\dot{x}^3$. The corresponding contravariant three-vector then satisfies

$$\mathbf{p}\cdot\mathbf{p}=2E. \tag{8}$$

Thus the components of momenta lie on a different hyperbolic sheet to that defined by Eq. (5) for the configuration manifold. In the Hamiltonian formulation the three constants of motion take the form

$$K_1=-(x^2p_3+x^3p_2), \tag{9}$$

$$K_2=x^3p_1+x^1p_3, \tag{10}$$

$$K_3=x^1p_2-x^2p_1, \tag{11}$$

which are the covariant components of a three-vector $\mathbf{K}$, which satisfies

$$\mathbf{K}\cdot\mathbf{K}=K_1^2+K_2^2-K_3^2=2E. \tag{12}$$

Thus $\mathbf{K}$ lies on the same hyperbolic sheet as $\mathbf{p}$. Note, however, that $\mathbf{p}\cdot\mathbf{K}=0$.

Inspection shows that K_3 simply corresponds to the component of angular momentum around the x_3 axis. Thus this function is the generator of rotations around the x^3 axis. The functions K_1 and K_2 generate displacements along the x^1,x^2 axes, but constrained to lie on the hyperbolic sheet. In the language of special relativity, these correspond to boosts in a $1+2$ space time. The corresponding group is SU(1,1) and we thus expect the generators of the transformations to form the corresponding Lie algebra. Using the method of Lie algebras of vector fields [12], we can construct a differential operator from any function $F(p_i,x^i)$ on phase space by

$$\hat{F}=\frac{\partial F}{\partial p_i}\frac{\partial}{\partial x_i}-\frac{\partial F}{\partial x^i}\frac{\partial}{\partial p_i}. \tag{13}$$

Thus to each of the functions K_i we can associate a differential operator $\hat{K}_1,\hat{K}_2,\hat{K}_3$. The Lie algebra of the resulting operators is found to be

$$[\hat{K}_1,\hat{K}_2]=\hat{K}_3,$$

$$[\hat{K}_2,\hat{K}_3]=-\hat{K}_1,$$

$$[\hat{K}_3,\hat{K}_1]=-\hat{K}_2,$$

which is the algebra of su(1,1), as expected.

III. QUANTUM CALCULATION

We begin the discussion by deriving the exact expression for the overlap between eigenstates of the operators $\hat{K}_3=\frac{1}{4}(a^\dagger a+aa^\dagger)$ and $\hat{K}_1=(i/4)(a^2-a^{\dagger 2})$. This is the problem of the photon number distribution for eigenstate of $\hat{K}_1$, for the photon number state $|n\rangle$ is actually the eigenstate of $\hat{K}_3$ with eigenvalue $(2n+1)/4$.

Denote $|\mu\rangle$ as the eigenstate of $G=(1/2i)(a^2-a^{\dagger 2})$ [22,23]. Then, we have $K_1|\mu\rangle=-(\mu/2)|\mu\rangle$. The position representation of state $|\mu\rangle$ is

$$\langle x|\mu\rangle_j=\frac{1}{\sqrt{2\pi}}x_j^{i\mu-1/2} \quad (j:+,-), \tag{14}$$

where the generalized function $x_j^\lambda(j:+,-)$ are defined by

$$x_+^\lambda=\begin{cases}x^\lambda & \text{for } x>0\\ 0 & \text{otherwise,}\end{cases} \tag{15}$$

$$x_-^\lambda=\begin{cases}0 & \text{for } x>0\\ |x|^\lambda & \text{otherwise.}\end{cases} \tag{16}$$

States from different classes are orthogonal: ${}_-\langle\mu|\mu\rangle_+=0$; while states within one class are orthonormal with δ-function normalization

$${}_j\langle\mu|\mu'\rangle_j=\delta(\mu-\mu') \quad (j:+,-). \tag{17}$$

Making use of expressions (1), (14), and the properties of Hermite polynomials [24],

$$\int_0^\infty e^{-2\alpha x^2}x^\nu H_{2m}(x)dx=(-1)^m 2^{2m-3/2-\nu/2}\frac{\Gamma((\nu+1)/2)\Gamma(m+1/2)}{\sqrt{\pi}\alpha^{\nu+1/2}}F\left(-m,\frac{\nu+1}{2},\frac{1}{2},\frac{1}{2\alpha}\right),$$

$$\int_0^\infty e^{-2\alpha x^2}x^\nu H_{2m+1}(x)dx=(-1)^m 2^{2m-\nu/2}\frac{\Gamma((\nu+1)/2)\Gamma(m/2)}{\sqrt{\pi}\alpha^{(\nu+1)/2}}F\left(-m,\frac{\nu+1}{2},\frac{3}{2},\frac{1}{2\alpha}\right), \tag{18}$$

we calculate the overlap $\langle n|\mu\rangle_j$ $(j:+,-)$ as follows:

$$\langle n|\mu\rangle_j=\int \langle n|x\rangle dx\langle x|\mu\rangle_j \quad (j:+,-)$$
$$=\begin{cases}(-1)^{n/2}(2\pi)^{-5/4}(2)^{i\mu/2}(n!/2^n)^{-1/2}\Gamma((2i\mu+1)/4)\Gamma((n+1)/2)F(-n/2,(2i\mu+1)/4,\frac{1}{2},2) & (n=2m)\\ (-1)^{(n-1)/2}(2\pi^5)^{-1/4}(2)^{i\mu/2}(n!/2^n)^{-1/2}\Gamma((2i\mu+3)/4)\Gamma((n+1)/2)F(-n/2,(2i\mu+3)/4,\frac{3}{2},2) & (n=2m+1),\end{cases} \tag{19}$$

where $F(a,b,c,z)$ is the hypergeometric function. For even number state $(n=2m)$, the transition probability is

$$|\langle 2m|\mu\rangle|^2=\frac{1}{\sqrt{32\pi^3}}\frac{(2m-1)!!}{(2m)!!}\left|\Gamma\left(\frac{1+2i\mu}{4}\right)F\left(-m,\frac{1+2i\mu}{4},\frac{1}{2},2\right)\right|^2. \tag{20}$$

With the help of mathematical relations [24]

$$F(a,b,2b,z)=(1-z)^{-a/2}F\left(\frac{a}{2},b-\frac{a}{2},b+\frac{1}{2},\frac{z^2}{4z-4}\right) \tag{21}$$

and

$$F(a,b,c,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{22}$$

we can reduce Eq. (20) at $\mu=0$ to

$$|\langle 2m|\mu\rangle|^2(\mu=0)=\left|\frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(3/4+m/2)\Gamma(1/2-m/2)}\right|^2. \tag{23}$$

Formula (23) implies that, for $\mu=0$, the overlap between the eigenstates $|\mu\rangle$ of K_1 and the number states $|4k+2\rangle$ $k=0,1,2,\ldots$ vanishes. Expression (20) is plotted in Fig. 5 by solid line $(n=60,n=8$ respectively).

IV. INTERFERENCE ALGORITHM

Our objective is to apply the method of interference in phase space to calculate the overlap between the eigenstates

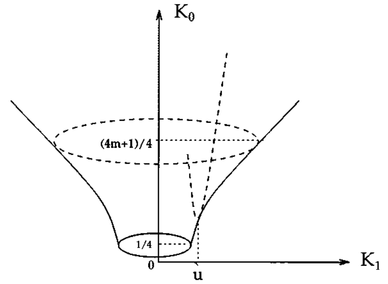

FIG. 2. The structure of the hyperbolic space considered. The region on which the interference algorithm can be applied is limited to the $K_0 \geqslant \frac{1}{4}$ part. The Kramer trajectories of states $|n\rangle$ and $|\mu\rangle$ are depicted by dashed lines.

of the operators $\hat{K}_i$. We first observe that we may define a representation of three generators of su(1,1) in terms of the Bose creation and annihilation operators by

$$\hat{K}_3 = \frac{1}{4}(a^\dagger a + a a^\dagger), \quad \hat{K}_1 = \frac{i}{4}(a^2 - a^{\dagger 2})$$

$$\hat{K}_2 = \frac{-1}{4}(a^2 + a^{\dagger 2}), \tag{24}$$

The commutation relations are

$$[\hat{K}_1, \hat{K}_2] = -i\hat{K}_3, \quad [\hat{K}_2, \hat{K}_3] = i\hat{K}_1, \quad [\hat{K}_3, \hat{K}_1] = i\hat{K}_2. \tag{25}$$

These are identical, up to multiplication by i, to the classical operators corresponding to the invariance transformations of a particle moving freely on a surface of constant negative curvature. It is not difficult to verify that for this representation the quadratic Casimir operator for SU(1,1)

$$\hat{K}^2 = \hat{K}_3^2 - \hat{K}_1^2 - \hat{K}_2^2 \tag{26}$$

is invariant. The Casimir invariant (26) reduces to the number

$$\hat{K}_1^2 + \hat{K}_2^2 - \hat{K}_3^2 = \frac{3}{16}. \tag{27}$$

This immediately suggests that the appropriate phase-space arena for the interference in phase-space method is the two-dimensional hyperbolic sheet defined by Eq. (12), with E fixed. For instance, the eigenstate of operator $\hat{K}_3$, $|n\rangle$, can be depicted as a cone centered along the K_0 axis with its base intersecting the K_0 axis at $K_3 = (2n+1)/4$. The Kramer trajectory associated with this state $|n\rangle$ is the intersection of plane $K_3 = (2n+1)/4$ and hyperboloid (27). We define the PBS band corresponding to this Kramer trajectory to be centered on the trajectory and lying between $K_3 = 2n/4$ and $K_3 = (2n+2)/4$. Similarly, the Kramer trajectory of state $|\mu\rangle$ $(\hat{K}_1|\mu\rangle = -\mu/2|\mu\rangle \equiv u|\mu\rangle)$ is the intersection of the surface $K_1 = u$ and the hyperbolic phase-space surface. (See Fig. 2.)

Before applying the interference algorithm on the hyperbolic space, one should note that as there is no negative number state, the two-dimensional hyperbolic space must be restricted to $K_3 \geqslant \frac{1}{4}$ part of Eq. (27). This is of course a peculiarity of the single-mode representation of su(1,1) with which we are working. If we were to use a two-mode Bose representation for su(1,1) this restriction would not be necessary.

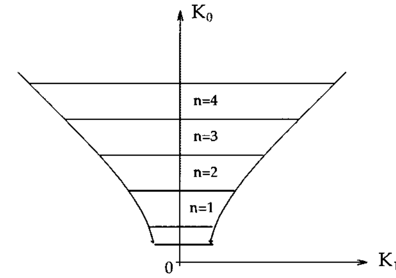

FIG. 3. The PBS bands representing different number states.

Another characteristic of the hyperbolic space is that the geometric area of a PBS band representing the number state $|n\rangle$ will increase with the increment of quantum number n. Because of this, Eq. (2) cannot be used directly to obtain the asymptotic form of $|\langle n|\mu\rangle|^2$. This can be solved by introducing the concept of *weighted area*, and changing the two *area* $A^\diamond$ and S in formula (2) into *weighted area*. The weight function $P(K_3)$ is defined as

$$P(K_3) 2\pi r dl = P(K_3') 2\pi r' dl', \tag{28}$$

where $dl = dK_3\sqrt{1+(dr/dK_3)^2}, dK_3 = dK_3'$. It is very easy to verify that

$$P(K_3) = \frac{c}{\sqrt{r^2 + K_3^2}} = \frac{c}{\sqrt{2K_3^2 + 3/16}}. \tag{29}$$

Constant c can be determined by assuming the phase area of each PBS band of state $|n\rangle (n \neq 0)$ be π. [This assumption is related to fact that K_3, K_1, and K_2 are quadratic operators and are coincident with Eq. (23) (see Fig. 3).] So, we have $c = 1$ and

$$P(K_3) = \frac{1}{\sqrt{2K_3^2 + 3/16}}. \tag{30}$$

Thus the asymptotic form based on the interference approach can be written as

$$W_{n\mu} = |\langle n|\mu\rangle|^2 = 4\frac{A_{n\mu}^{\diamond w}}{\pi}\cos^2\left(S_{n\mu}^{w} - \frac{\pi}{4}\right), \tag{31}$$

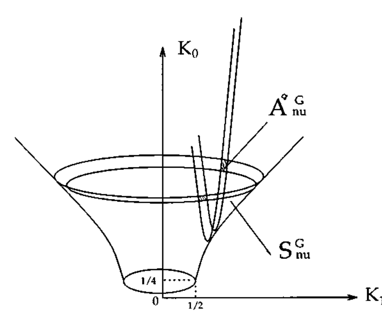

FIG. 4. Two PBS bands corresponding to states $|n\rangle$ and $|\mu\rangle$.

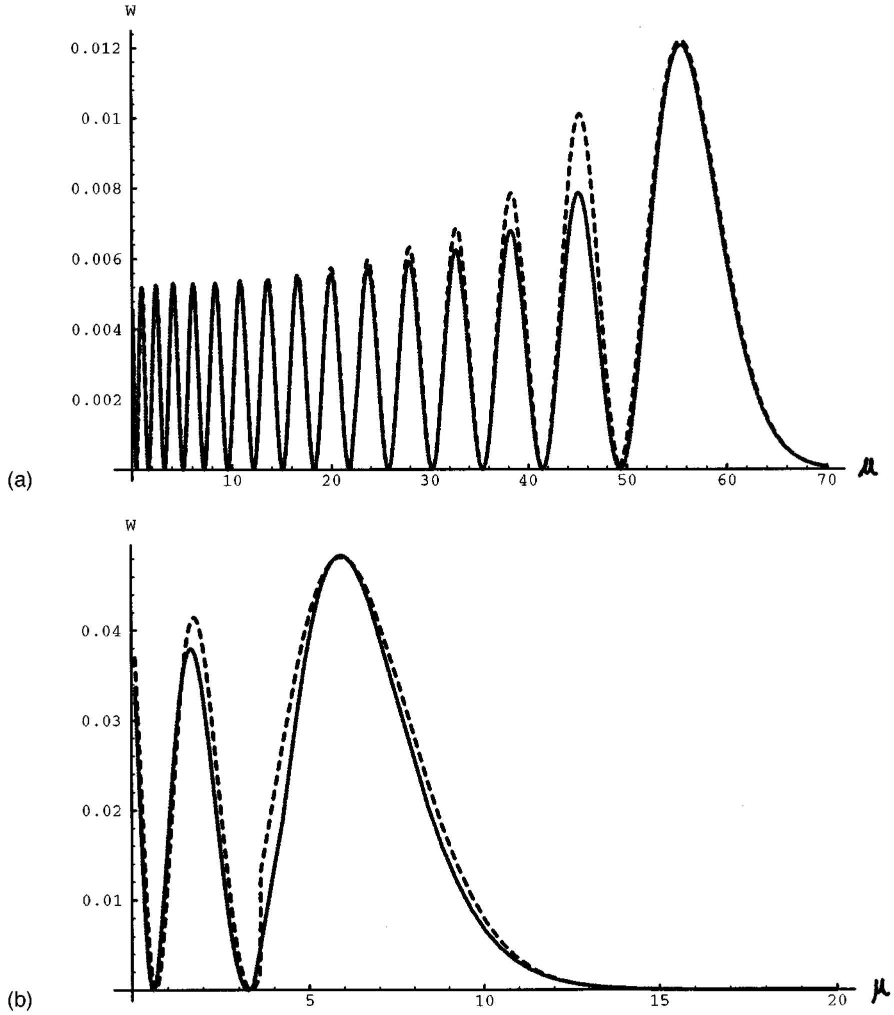

FIG. 5. The photon number distribution for $\mu(\mu=-2u)$, for different states $|n\rangle$ (a) $n=60$, (b) $n=8$. The solid line is the exact expression obtained from quantum calculation, and the dashed line is the result using interference approach.

where $A_{n\mu}^{\diamond w}$ means *weighted area* corresponding to *geometric area* $A_{n\mu}^{\diamond g}$ represented by the shaded regions in Fig. 4. To be more precise, when $n=2m$, we have

$$A_{n\mu}^{\diamond w}(n=2m)=\int_{m}^{m+1/2}P(K_3)dK_3\sqrt{1+\left(\frac{\partial K_2}{\partial K_3}\right)^2}\frac{dK_1}{\sin\varphi}\,\delta(K_1-u) \tag{32}$$

$$=\int_{m}^{m+1/2}P(K_3)dK_3\sqrt{\frac{K_3^2+\dfrac{3}{16}}{K_3^2+\dfrac{3}{16}-u^2}}\sqrt{\frac{2K_3^2+\dfrac{3}{16}-u^2}{k_3^2+\dfrac{3}{16}-u^2}}, \tag{33}$$

where $\varphi=\tan^{-1}\sqrt{(K_3^2+3/16-u^2)}/u^2$ and $u=-\mu/2$. The delta function in the integrand arises because of Eq. (17) [3].

For the same reasons as above, $S_{n\mu}^{w}$ in Eq. (31) is taken to be the *weighted area* bounded by two Kramer trajectories of states $|\mu\rangle$ and $|n\rangle$ and can be expressed as

$$S_{n\mu}^{w}(n=2m)=\int P(K_3)r\varphi dK_0\sqrt{1+\left(\frac{dr}{dK_3}\right)^2}=\begin{cases}\displaystyle\int_{\sqrt{u^2-3/16}}^{m+1/4}\varphi dK_3, & \mu\geqslant\dfrac{1}{2}\\ \displaystyle\int_{1/4}^{m+1/4}\varphi dK_3, & 0\geqslant\mu<\dfrac{1}{2}.\end{cases} \tag{34}$$

The above expressions can be uniformed into one:

$$S^{w}_{n\mu}(n=2m)\approx\int_{1+4\mu^2}^{4m+1}dz\,\frac{1}{4}\left(\phi+\frac{d\phi}{dz}\Delta z\right)$$
$$=\frac{1}{4}\left[z\tan^{-1}\sqrt{\frac{z^2-1-4\mu^2}{4\mu^2}}+\tan^{-1}\sqrt{\frac{z^2-1-4\mu^2}{4z^2\mu^2}}-2|\mu|\sinh^{-1}\sqrt{\frac{z^2-1-4\mu^2}{1+4\mu^2}}\right]_{z=4m+1}, \tag{35}$$

where

$$\phi=\tan^{-1}\sqrt{\frac{z^2-1-4\mu^2}{4\mu}}\left(u=-\frac{\mu}{2}\right), \tag{36}$$

$$\Delta z=z_1-z_2=\frac{4}{z_1+z_2}\approx\frac{2}{z}(z_1^2\equiv r^2+1;z_2^2\equiv r^2-3). \tag{37}$$

Substituting Eqs. (33) and (35) into Eq. (31), we get the asymptotic expression for the photon number distribution for state $|\mu\rangle$. The results obtained by the interference approach (supplemented with an Airy function as described in the Appendix) are compared with the results using exact calculation in Fig. 5. Obviously, the agreement is quite good.

In this paper we have shown that the interference in the phase-space algorithm may be adapted to the case of a hyperbolic phase space. This enables us to obtain good semiclassical results for the inner product of eigenstates of the generators of the group su(1,1), which are the canonical momenta operators for this phase space. Classical motion on surfaces of constant negative curvature with periodic boundary conditions is strongly chaotic, and in fact ergodic. The quantum description of the motion has been used as a prototype for understanding semiclassical quantization of classically ergodic systems [11]. The interference in phase-space methods described in this paper may well be useful in providing an alternative description of the quantum motion in such systems.

APPENDIX

Equations (2) and (31) cannot be applied in the region where there is only one area of overlap. In this region, the asymptotic form of the scalar product can be expressed by the Airy function.

Recall that a WKB wave function for a periodic system has a phase factor of form $\cos[\int_x^{x_0}p(x)dx-\pi/4]$, where $p(x)$ is the momentum as a function of position along a curve of constant energy, and x_0 is the classical turning point. It satisfies the second-order differential equation

$$\frac{d^2u}{dx^2}+p^2(x)u=0. \tag{A1}$$

Near the turning point x_0, where $p(x_0)=0$, one can Taylor expand $p^2(x)$ at x_0 as $p^2(x)\approx 2pp'|_{x=x_0}(x-x_0)$ and approximate the above differential equation to

$$\frac{d^2u}{dx^2}+a(x-x_0)\approx 0 \quad (a=pp'|_{x_0}). \tag{A2}$$

This equation has the solution of Airy function

$$u(x)\approx \mathrm{Ai}[a^{1/3}(x-x_0)]. \tag{A3}$$

In hyperbolic space, the phase factor in Eq. (31) can be written as $\cos[\int_{2\mu}^{x_0}f(x)dx-\pi/4]$ and the corresponding differential equation should be

$$\frac{d^2y}{dx^2}+f^2(x)y=0, \tag{A4}$$

where $x=2\mu, f(x)=\frac{1}{4}\tan^{-1}(\sqrt{z_0^2+3-x^2}/x)(z_0=4m+1)$. Similarly, we Taylor expand $f(x^2)$ at the region near $x_0=\sqrt{z_0^2+3}$. Making use of the relations $\tan^{-1}\alpha\approx\alpha$, we obtain

$$f^2(x)\approx 2ff'|_{x=x_0}(x-x_0)=\frac{x-\sqrt{z_0^2+3}}{8x_0}. \tag{A5}$$

Thus, near the turning point, the phase factor $\cos^2\phi$ is replaced by

$$\mathrm{Ai}^2\{[8\sqrt{(4m+1)^2+3}]^{-1/3}[2\mu-\sqrt{(4m+1)^2+3}]\}.$$

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