Unified treatment of some Casimir energies and Lamb shifts: A dielectric between two ideal conductors

Martin Schaden,[1,*] Larry Spruch,[1,2,3] and Fei Zhou[1,4]
[1]*Physics Department, New York University, 4 Washington Place, New York, New York 10003*
[2]*Max-Planck-Institut für Kernphysik, Postfach 103980, D-69029 Heidelberg, Germany*
[3]*Physikalisches Institut der Universität Heidelberg, Philosophenweg 12, D-69117 Heidelberg, Germany*
[4]*Lehman Brothers, 3 World Financial Center, New York, New York 10285*
(Received 22 August 1997)

Lamb shifts and Casimir energies, often thought of as long-range and short-range effects, respectively, and studied separately, are each a manifestation of quantum electrodynamics and can be studied together. We do so for a dielectric medium between two parallel ideal conducting plates. We extract the usual Casimir energy and a bulk Lamb shift by studying the quantum fluctuations of the radiation field in the dielectric. We derive finite expressions for the Casimir energy and the bulk Lamb shift valid for any permittivity satisfying the Kramers-Krönig relation; some of the Casimir shifts obtained are simpler in form than any in the literature. We separate the divergent and finite contributions to the bulk Lamb shift. For the dilute nonrelativistic gas we show that the divergent contribution to the bulk Lamb shift defines the bare electron mass in terms of the physically observed free-electron mass. Although we lack a physical interpretation for the necessary subtraction in the case of an arbitrary dielectric, it is natural to interpret the finite part of the bulk energy as a ''Lamb shift'' in this case too. We show that the derived finite bulk Lamb shift and Casimir energies in the limit of a dilute homogeneous gas are consistent with earlier results for a single atom between two ideal conductors, and for an atom near one of the walls. As an application, the radiative contribution to the interaction energy of a single electron with uniform probability density between two ideal walls is obtained. [S1050-2947(98)01602-3]

PACS number(s): 31.30.Jv, 12.20.Ds, 11.10.Gh

I. INTRODUCTION

Casimir energies and Lamb shifts are among the most interesting and best studied effects of quantum electrodynamics (QED). Each has been calculated many times, in a number of ways, for a variety of circumstances. Despite their common QED origin, the Casimir energy is often taken to be a long-range effect and the Lamb shift is often taken to be a short-range effect, and, to our knowledge, the two have never been evaluated in the course of a single calculation.

Furthermore, even in situations for which only the Casimir effect was considered, the formalism used was often more sophisticated and thus less transparent than was required. Electromagnetic eigenfrequencies and eigenmodes were often explicitly evaluated and the systems considered were often rather complicated. (With regard to the last point, the effects of a dielectric medium with permittivity ε_3 between two ideal plane parallel walls have sometimes been analyzed by studying three dielectric media with permittivities ε_1, ε_3, and ε_2, setting $\varepsilon_1=\varepsilon_2=\infty$ only at the end of the calculation. Although the result is more general, the technical complications of such a treatment tend to considerably obscure the physical picture.) We here restrict our attention to the relatively simple case for which a Casimir energy and a Lamb shift both contribute to the quantum electrodynamic energy, a homogeneous dielectric material with frequency-dependent permittivity $\varepsilon(\omega^2)$ and permeability $\mu=1$ between two ideal uncharged plane parallel conductors; the limit in which the dielectric is a dilute gas, which we also consider, is probably the simplest case.

We derive expressions for the Casimir energy as well as the bulk Lamb shift in terms of the permittivity $\varepsilon(\omega^2)$ of the material itself, evaluating the energy of the fluctuating radiation field in such a material. This macroscopic approach has the advantage of not requiring a microscopic description of $\varepsilon(\omega^2)$, which—at least in principle—is a measurable quantity. Our expressions (3.5) for the Casimir energy, as well as (3.17) for the Lamb shift between two materials with the same electron number density, give the contribution of the radiation field to the energy. This approach is therefore independent of any approximations used in a theoretical calculation of $\varepsilon(\omega^2)$ and is thus equally valid in a relativistic and nonrelativistic description of the material and furthermore applies also to materials with $\varepsilon(\omega^2)$ not necessarily close to unity.

Technically we evaluate the energy of the radiation field as a sum over the energies of the individual modes. This sum is transformed into an integral expression using the generalized argument theorem—see Eq. (2.9) below—thus eliminating the need to determine the energies of the individual modes. We stress the importance of the analytic properties of $\varepsilon(\omega^2)$ implied by the Kramers-Krönig relation, which in principle restricts this derivation to the idealized case of materials with arbitrarily narrow (stable) resonances. The final expressions (3.5) and (3.17), however, do not depend on this idealization and are valid for realistic materials. They depend only on the permittivity $\varepsilon(-\xi^2)$ at real ξ, which is a regular, real function for any physical substance, and the restriction we impose to derive these expressions can be dropped. (This

*Electronic address: ms68@scires.nyu.edu

can also be explicitly shown in a more rigorous, but less transparent path integral derivation of our results [1].)

We emphasize the common conceptual origin of the Lamb shift and the Casimir effect as the finite renormalized energy density and energy per unit area of the fluctuating radiation field. We show that infinities of the unrenormalized energy density are associated with a moment of the discontinuity of $\varepsilon(\omega)$ on the positive real axis. In the case of a dilute gas, this infinity can be interpreted as a redefinition of the electron mass and we therefore subtract the divergent contribution to the Lamb shift proportional to the second moment of the discontinuity of $\varepsilon(\omega)$ on the positive real axis.

We finally explicitly show that the *finite* expressions we derive reproduce the well-known results for the Casimir energy and Lamb shift in special cases. In particular, we verify Bethe's original expression for the Lamb shift in a nonrelativistic description of the case where the dielectric is a dilute gas [2]. The interaction energy with the walls of a dilute gas between two ideal walls, studied here, is shown to be consistent with the interaction energies of a single atom at a fixed distance from an ideal wall [3] and of a single atom at a fixed location between two ideal walls [4]. We also obtain the interaction energy with the walls of a single electron with a uniform probability density between the conductors. We provide estimates of the range of validity of the approximations obtained.

The Casimir effect, it might be noted, had its origins in an experiment on the stability of lyophobic colloids, which showed that the atom-atom interaction at large distances r did not obey the van der Waals $1/r^6$ law, but varied as $1/r^7$ [5]. Theorists were led to study the effects of electromagnetic fluctuations; these effects could explain the $1/r^7$ law [6], and also led to new results for the interaction of parallel ideal walls [7] and of an atom and an ideal wall [3]. Not long thereafter there were theoretical studies of a number of interactions involving one or two dielectric walls [8]. Over the years, a few very good experiments were performed, largely on the force between a pair of dielectric walls [9], but the primary effort was by theorists. We list three reviews dealing primarily with the above material [10–12]. More recently there have been studies of the interaction energy of an electron and a dielectric wall [13], and interactions involving sets of walls [14]. In the latter study the sets were a fixed distance apart, with the number of walls in each set and the thickness and permittivity of each wall arbitrary. The interaction between the sets was obtained, as was the interaction with the sets of an atom or electron at a fixed location between the sets; these results encompass essentially all known interactions of walls and of atoms and electrons with walls. A very recent result is the experimental study of the force between a conducting sphere of radius R and a conducting wall, where the minimum distance a between the sphere and the wall satisfies $a \ll R$ [15]; that force can be expressed in terms of the force per unit area between two parallel conducting walls [9].

II. THE ENERGY DENSITY OF THE FLUCTUATING ELECTRIC FIELD IN A DIELECTRIC

The problem to be considered is that of two ideal conducting plates parallel to the (x,y) plane, at $z=0$ and $z=l$, with a uniform dielectric with permittivity $\varepsilon(\omega^2)$ between them. In Coulomb gauge the fluctuating electromagnetic fields between the plates are determined by the vector potential $\mathbf{A}$. With $\mathbf{x}_\perp=(x,y,0)$, $\mathbf{k}_\perp=(k_x,k_y,0)$, $\mathbf{A}_\perp=(A_x,A_y,0)$, and $\hat{\mathbf{e}}_z$ the unit vector parallel to the z axis, $\mathbf{A}$ is a linear superposition of the functions

$$\mathbf{A}(\mathbf{k}_\perp,k_n;\mathbf{x})=[\mathbf{A}_\perp(\mathbf{k}_\perp,n)\sin(k_n z) + \hat{\mathbf{e}}_z A_z(\mathbf{k}_\perp,n)\cos(k_n z)]e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}. \quad (2.1)$$

For plates $L\times L$ with $L\gg l$, the components of the wave vector $\mathbf{k}_\perp$ in the transverse direction can be taken to be continuous, but the boundary conditions for ideal conducting plates imply that the component k_n takes only discrete values

$$k_n l=n\pi, \quad n=0,1,2,\ldots \quad (2.2)$$

and the Coulomb gauge condition, $\nabla\cdot\mathbf{A}=0$, reduces to

$$i\mathbf{k}_\perp\cdot\mathbf{A}_\perp(\mathbf{k}_\perp,n)-k_n A_z(\mathbf{k}_\perp,n)=0. \quad (2.3)$$

For a given $\mathbf{k}_\perp$ and $n\neq 0$, two of the three amplitudes A_x,A_y,A_z are independent and there are two states of polarization, but $k_n=0$ for $n=0$ and there is then only one polarization state. Since $\mathbf{A}$ satisfies the wave equation, the frequency ω_n of the nth mode is formally a solution of the dispersion relation

$$\omega_n^2\varepsilon(\omega_n^2)-\omega_\perp^2=c^2k_n^2, \quad (2.4)$$

where

$$\omega_\perp=c|\mathbf{k}_\perp|. \quad (2.5)$$

The implicit solutions ω_n are thus functions of l and ε.

It should be emphasized, however, that solutions ω_n to Eq. (2.4) in principle exist only for the rather unphysical case of a substance with stable (zero-width) excited states, that is, for a discontinuity of $\varepsilon(\omega^2)$ on the positive real ω^2 axis

$$\sigma(\omega)=\mathrm{Disc}\,\varepsilon(\omega^2)=\frac{1}{2i}\lim_{\eta\to 0^+}[\varepsilon(\omega^2+i\eta)-\varepsilon(\omega^2-i\eta)] =\mathrm{Im}\;\varepsilon(\omega^2+i\eta), \quad (2.6)$$

described by a discrete sum of delta functions. In the physical case of resonances with nonzero width, the Kramers-Krönig relation between real and imaginary parts of $\varepsilon(\omega^2)$ required by causality,

$$\varepsilon(\omega^2)=1+\frac{2}{\pi}\int_0^\infty d\omega'\,\frac{\omega'\sigma(\omega')}{\omega'^2-\omega^2}, \quad (2.7)$$

can be shown to imply (see Appendix) that the imaginary part of $\omega^2\varepsilon(\omega^2)$ does not vanish for any ω with $\mathrm{Re}\;\omega>0$. Although in principle solutions to Eq. (2.4) thus cannot be found for a real substance, approximate real solutions ω_n to Eq. (2.4) exist for sufficiently narrow resonances. The following treatment of the Casimir effect for a substance with very narrow resonances is thus an idealization in much the same spirit as the plates are idealized as perfect conductors. (Conceptually, the zero-width idealization is the more treacherous one, since a rather good approximation in the

study of analytic functions can lead to quite bad results; intuitively, the zero-width idealization should be an excellent approximation, since the widths of the atomic levels are so small compared to the energy differences between levels.) The results obtained are nevertheless correct for any dielectric permittivity $\varepsilon(\omega^2)$ satisfying Eq. (2.7), since our final expressions remain valid for resonance spectra with segments that are *dense*, and segments that contain sharp resonances, as when band structures are present. As noted above, our final results can also be obtained directly in the framework of a path integral derivation of the Casimir effect. That the complete neglect of the width of the resonance is a meaningful approximation is perhaps less surprising if we recognize that there is no dissipation of energy when dealing with electromagnetic fluctuations for the ground state of the system.

The conventional derivation of the Casimir effect as arising from the dependence of the zero point energy $\mathcal{E}(\varepsilon,l)$ due to electromagnetic fluctuations

$$\mathcal{E}(\varepsilon,l)=L^2\int\frac{d^2k_\perp}{(2\pi)^2}\left[\frac{1}{2}\hbar\omega_0+2\sum_{n=1}^{\infty}\frac{1}{2}\hbar\omega_n\right] \tag{2.8}$$

would seem to be valid only in the ideal case of arbitrarily sharp and discrete resonances, but from the discussion above is of more general validity. From normal ordering, the energy of a vacuum fluctuation of frequency ω is $\hbar\omega/2$ and in Eq. (2.8) the two independent states of polarization for $n>0$ have been taken into account. Note also that even in the idealized case of arbitrarily sharp resonances Eq. (2.4) generally has several solutions for given n and $\mathbf{k}_\perp$. The sum in Eq. (2.8) should be understood as extending over all these $\omega_n^{(i)}$.

The divergent sum (2.8) can formally be evaluated without explicit knowledge of the solutions to Eq. (2.4) by using the generalized argument theorem, which expresses Eq. (2.8) as a contour integral in the complex ω plane. To do so, we need a contour $\mathcal{C}$ and a holomorphic function $F(\omega)$ whose roots within $\mathcal{C}$ are all the ω_n (and only the ω_n). In the context of Casimir effect studies, the generalized argument theorem, apparently first used in related cases by van Kampen *et al.* [16], gives

$$\sum_n{}'\,\omega_n=\frac{1}{2\pi i}\oint_{\mathcal{C}}\omega d\omega\frac{\partial}{\partial\omega}\ln F(\omega)=-\frac{1}{2\pi i}\oint_{\mathcal{C}}d\omega\ln F(\omega), \tag{2.9}$$

where the sum extends over all zeros ω_n of F within $\mathcal{C}$, with the solutions for $n=0$ weighted by 1/2. The last expression in Eq. (2.9) is the result of an integration by parts, using the fact that $F(\omega)$ is holomorphic. [Without the ω factor, the first integral of Eq. (2.9) would just give the difference between the number of zeros and the number of poles of $F(\omega)$ in $\mathcal{C}$; this is the usual argument theorem.]

A function often used for F in studying the current situation is

$$\exp(2ik_zl)-1, \tag{2.10}$$

where

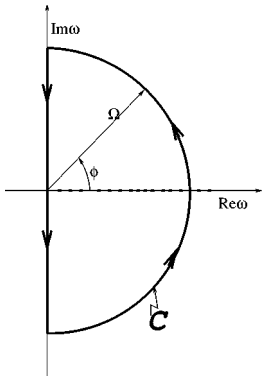

FIG. 1. The contour $\mathcal{C}$ in the complex ω plane used to evaluate the Casimir energy by the generalized argument theorem. Also indicated schematically are the zeros on the positive real ω axis of the holomorphic functions used in the generalized argument theorem (for $\omega_\perp\neq0$).

$$k_z=[\omega^2\varepsilon(\omega^2)-\omega_\perp^2]^{1/2}/c. \tag{2.11}$$

However, the function in Eq. (2.10) is not holomorphic; in addition to the desired roots at $k_z=k_n$, it has a branch point in ω at any solution ω_0 of

$$\omega_0^2\varepsilon(\omega_0^2)-\omega_\perp^2=0. \tag{2.12}$$

Since any ω_0 (and in particular one lying within $\mathcal{C}$) is independent of l, the contribution of such a branch point to the Casimir force between the plates vanishes—the force being the derivative with respect to l—and the branch point is simply ignored. We are, however, also interested in the ε dependence of the fluctuating electromagnetic energy in the presence of the dielectric and therefore will use the holomorphic function

$$F_1(\omega)=\frac{\sin k_zl}{k_zl}\equiv F_1(\omega^2) \tag{2.13}$$

to obtain the contribution from all ω_n with $n\neq0$ and, separately,

$$F_0(\omega^2)=k_z^2c^2=\omega^2\varepsilon(\omega^2)-\omega_\perp^2 \tag{2.14}$$

for the contribution from the ω_0.

The contour $\mathcal{C}$, depicted in Fig. 1, runs down the imaginary ω axis from $i\Omega$ to $-i\Omega$ and is closed by a semicircle of radius Ω in the $\mathrm{Re}\omega>0$ plane. A finite value for Ω regularizes the divergent expression (2.8) and we are ultimately interested in the ε and l dependence of Eq. (2.8) when Ω tends to infinity. Note that the statement that F_1 in Eq. (2.13) and F_0 in Eq. (2.14) are functions of ω^2 depends crucially on the fact that ε satisfies Eq. (2.7) and is thus itself a function of ω^2 only. A simple nonrelativistic damped harmonic oscillator model of a gas, for example, gives $\varepsilon(\omega)=1+[\omega_{\mathrm{pl}}^2/(\omega_r^2-\omega^2-i\omega\gamma_r)]$, where ω_{pl} is the plasma frequency, ω_r is the resonant frequency, and γ_r is the damping factor. This form of $\varepsilon(\omega^2)$ is a function of ω^2 and satisfies Eq. (2.7) if and only if $\Gamma\sim0$.

Separating the contributions to the contour integral into integrations along the imaginary axis (Im) and the semicircle (S), we thus have

$$\sum{}' \omega_n = I_{\rm Im}(\Omega) + I_S(\Omega). \tag{2.15}$$

With $\omega = i\xi$ in $I_{\rm Im}$ and $\omega = \Omega \exp(i\phi)$ in I_S, Eq. (2.9) gives

$$I_{\rm Im}(\Omega) = -\frac{1}{2\pi}\int_{-\Omega}^{\Omega} \xi\, d\xi \frac{\partial}{\partial \xi}\left(\frac{1}{2}\ln F_0(-\xi^2) + \ln F_1(-\xi^2)\right), \tag{2.16}$$

$$I_S(\Omega) = \frac{\Omega}{2\pi i}\int_{-\pi/2}^{\pi/2} e^{i\phi}\, d\phi \frac{\partial}{\partial \phi}\left(\frac{1}{2}\ln F_0(\Omega e^{i\phi}) + \ln F_1(\Omega e^{i\phi})\right). \tag{2.17}$$

Consider first the integral $I_{\rm Im}$ of Eq. (2.16). For imaginary arguments, $k_z(\omega^2 = -\xi^2)$ defined by Eq. (2.11) is better written as $iK_z(\xi^2)$, where

$$K_z(\xi^2) = +[\xi^2 \varepsilon(-\xi^2) + \omega_\perp^2]^{1/2}/c. \tag{2.18}$$

[Since F_0 and F_1 are both holomorphic functions, we are free to choose either sign of the root in Eq. (2.18).] $K_z(\xi^2)$ of Eq. (2.18) is a real and positive function because physically acceptable $\varepsilon(-\xi^2)$ satisfy Eq. (2.7) with a positive discontinuity $\sigma(\omega^2) > 0$ and are therefore real and greater than unity.

With

$$F_0(-\xi^2) = -K_z^2 c^2 \tag{2.19}$$

and

$$F_1(-\xi^2) = \frac{\sin(iK_z l)}{iK_z l} = \frac{e^{K_z l}(1 - e^{-2K_z l})}{2K_z l} \tag{2.20}$$

we find

$$\tfrac{1}{2}\ln F_0(-\xi^2) + \ln F_1(-\xi^2) = K_z l + \ln[1 - e^{-2K_z l}], \tag{2.21}$$

where we have dropped a term that is independent of ξ and that therefore vanishes under $\partial/\partial\xi$. Inserting Eq. (2.21) in to Eq. (2.16), we integrate by parts, noting that $\varepsilon(-\Omega^2) \to 1$ and therefore $K_z(\Omega^2) \to \Omega/c$ for large $\Omega \gg \omega_\perp$, to arrive at

$$I_{\rm Im}(\Omega) = \frac{1}{\pi}\int_0^\Omega d\xi\, \{\ln[1 - e^{-2K_z l}] + lK_z\} - \frac{\Omega^2 l}{\pi c}. \tag{2.22}$$

The integral over the first term in Eq. (2.22) is finite in the limit $\Omega \to \infty$ and, as we shall see, is related to the Casimir effect.

To evaluate $I_S(\Omega)$, we note that $\omega = \Omega e^{i\phi}$ on the arc, with Ω very large. For fixed $\omega_\perp \ll \Omega$ and $k_z \sim (\Omega/c)e^{i\phi}$,

$$F_0 \sim \omega^2 = (\Omega e^{i\phi})^2, \tag{2.23}$$

and

$$F_1 \sim \frac{\sin[(\Omega l/c)e^{i\phi}]}{(\Omega l/c)e^{i\phi}}; \tag{2.24}$$

it follows that

$$\frac{\partial}{\partial \phi}\left[\frac{1}{2}\ln F_0 + \ln F_1\right] \sim i\frac{\Omega l}{c} e^{i\phi} \cot(\Omega l e^{i\phi}/c), \tag{2.25}$$

dropping terms that vanish in the limit $\Omega \to \infty$. The ϕ integration in Eq. (2.17) can readily be performed upon observing that

$$\cot(\Omega l e^{i\phi}/c) \overset{\Omega\sim\infty}{\to} \begin{cases} -i & \text{for } 0 < \phi < \pi/2 \\ i & \text{for } -\pi/2 < \phi < 0 \end{cases} \tag{2.26}$$

and gives

$$I_S(\Omega \sim \infty) = \frac{\Omega^2 l}{\pi c}. \tag{2.27}$$

Summing Eqs. (2.22) and (2.27) one obtains for the regularized zero-point energy at given $\omega_\perp = c|\mathbf{k}_\perp|$ for large values of Ω:

$$\mathcal{E}^R(\varepsilon, l, \omega_\perp^2; \Omega) = \sum{}' \hbar\omega_n = \frac{\hbar}{\pi}\int_0^\Omega d\xi \ln[1 - e^{-2K_z l}] + \frac{l\hbar}{\pi}\int_0^\Omega d\xi K_z. \tag{2.28}$$

The first contribution to $\mathcal{E}^R$ in Eq. (2.28) is finite in the limit $\Omega \to \infty$ (and gives a finite energy per unit area after integration over transverse degrees of freedom) and vanishes when $l \to \infty$. [For a vacuum between the ideal walls, i.e., $\varepsilon(\omega^2) = 1$, this is just the l dependent Casimir pressure of one transverse mode. It is modified by the medium for $\varepsilon(\omega^2) \neq 1$.] For a dilute gas, we interpret this change in the energy density of the medium as partially arising from an l-dependence of the Lamb shift of the individual atomic states due to the presence of the walls.

The second contribution to Eq. (2.28), which is quadratically divergent in Ω, is linearly dependent on the separation l of the ideal walls. It leads to an l-independent constant pressure between the walls. This pressure is balanced by the pressure exerted by the vacuum outside the walls only if the medium has permittivity $\varepsilon(\omega^2) = 1$. For $\varepsilon = 1$ this bulk contribution to the total energy is given by the vacuum energy of the radiation field. It is therefore natural to extract from this term the change in energy density due to the radiation field for $\varepsilon(\omega^2) \neq 1$. For a dilute gas this energy should be the Lamb shift of the individual atoms in the absence of the ideal walls. In the next section we extract the finite physical effect from the regularized expression (2.28) in the limit $\Omega \to \infty$.

III. RENORMALIZATION OF THE FLUCTUATION ENERGY

Only energy differences are of physical interest. Casimir effects are just the difference between the energy with ideal walls at a separation l and the energy in the same region with no walls present. Similarly, in deriving the Lamb shift one compares the interaction energy of the radiation field with the corresponding interaction energy of ''free'' electrons (mass renormalization). In the present analysis of both effects it is necessary to make both subtractions consistently.

We propose to do this in two steps.

We obtain the Casimir energy $\mathcal{E}_{\rm Cas}(\varepsilon,l)$ by extracting the bulk energy from the total energy. To do so, we first consider the regularized energy of the radiation field in a box with ideal walls and of fixed dimensions $L\times L\times L$ and permittivity $\varepsilon(\omega^2)$, with an additional ideal wall placed a distance $l \ll L$ from one of the sides of the box. We then subtract from that energy the energy of the radiation field for the same box with the same $\varepsilon(\omega^2)$ but without the additional wall [17]. In the limit of large Ω we then have

$$\Delta\mathcal{E}^R(\varepsilon,l,\omega_\perp^2)=\lim_{\Omega\to\infty}[\mathcal{E}^R(\varepsilon,l,\omega_\perp^2;\Omega)+\mathcal{E}^R(\varepsilon,L-l,\omega_\perp^2;\Omega)-\mathcal{E}^R(\varepsilon,L,\omega_\perp^2;\Omega)]. \tag{3.1}$$

The limit in Eq. (3.1) is finite, since the second term in Eq. (2.28), the bulk energy term directly proportional to the separation of the plates, drops out, as do contributions to Eq. (3.1) from the first term in Eq. (2.28) that decay exponentially with L in the limit of large L. Integrating $\Delta\mathcal{E}^R$ over the transverse modes gives the standard result for the Casimir energy

$$\begin{aligned}\mathcal{E}_{\rm Cas}(\varepsilon,l)&=L^2\int\frac{d^2k_\perp}{(2\pi)^2}\Delta\mathcal{E}^R=L^2\int\frac{d(\omega_\perp)^2}{4\pi c^2}\Delta\mathcal{E}^R\\&=L^2\frac{\hbar}{(2\pi c)^2}\int_0^\infty d(\omega_\perp^2)\int_0^\infty d\xi\ln[1-e^{-2lK_z}]\\&=L^2\frac{\hbar}{2(\pi c)^2}\int_0^\infty d\xi\ \xi^2\varepsilon(-\xi^2)\int_1^\infty dp\\&\quad\times p\ln[1-e^{-2l\xi p\sqrt{\varepsilon(-\xi^2)}/c}].\end{aligned} \tag{3.2}$$

The last expression for $\mathcal{E}_{\rm Cas}$ in Eq. (3.2) is obtained by changing variables [18] from $\omega_\perp^2$ to $p^2=[\omega_\perp^2/\xi^2\varepsilon(-\xi^2)]+1$. Setting $p'=2l\xi p\sqrt{\varepsilon(-\xi^2)}/c$, Eq. (3.2) becomes

$$\mathcal{E}_{\rm Cas}(\varepsilon,l)=\frac{L^2\hbar}{8(\pi l)^2}\int_0^\infty d\xi\ I(\xi)=-\frac{L^2\hbar}{8(\pi l)^2}\int_0^\infty d\xi\ \xi\frac{\partial I}{\partial\xi}, \tag{3.3}$$

where

$$I(\xi)=\int_{2l\xi\sqrt{\varepsilon(-\xi^2)}/c}^\infty p'dp'\ln[1-e^{-p'}]; \tag{3.4}$$

the surface term of the partial integration leading to the second expression for $\mathcal{E}_{\rm Cas}$ in Eq. (3.3) vanishes. Evaluating $\partial I/\partial\xi$ one obtains the Casimir energy as an integral over ξ only:

$$\mathcal{E}_{\rm Cas}(\varepsilon,l)=L^2\frac{\hbar}{(2\pi c)^2}\int_0^\infty d\xi\ \xi\left(\frac{d}{d\xi}[\xi^2\varepsilon(-\xi^2)]\right)\times\ln[1-e^{-2l\xi\sqrt{\varepsilon(-\xi^2)}/c}]. \tag{3.5}$$

This relatively simple form for the Casimir energy seems not to have been given previously. At this point we should perhaps clarify our earlier remark that Eq. (3.5) can be taken to be nonperturbative and relativistically valid if $\varepsilon(\omega)$ is known. To be formally exact one in principle has to know the dependence of $\varepsilon(\omega)$ on the separation l [19].

Before proceeding to the second subtraction, we note that the behavior of $\varepsilon(-\xi^2)$ in the limit of small ξ completely determines the asymptotic form of the Casimir energy $\mathcal{E}_{\rm Cas}$ at large l. If $\varepsilon(0)$ is finite, as in most realistic situations, one can replace $\varepsilon(-\xi^2)$ by $\varepsilon(0)$ for l sufficiently large. One can then evaluate the integral in Eq. (3.5), and one finds that the asymptotic Casimir energy

$$\mathcal{E}_{\rm Cas}(\varepsilon,l)\overset{l\sim\infty}{\to}-\frac{\pi^2\hbar L^2c/\sqrt{\varepsilon(0)}}{720l^3} \tag{3.6}$$

is determined by the speed, $c/\sqrt{\varepsilon(0)}$, of long-wavelength excitations, a physically reasonable result. [For $\varepsilon(\omega^2)=1$, Eq. (3.6) is exact for all separations and is nothing other than the familiar Casimir energy for two ideal walls in a vacuum.] To determine the domain of validity of Eq. (3.6) one begins by noting that the maximum value of ξ that contributes significantly to the integral in Eq. (3.5) is that for which the exponent is about equal to unity, that is, $\xi_{\rm max}$ is the solution of $2l\xi_{\rm max}\sqrt{\varepsilon(-\xi_{\rm max}^2)}/c=1$. Furthermore one can approximate the monotonically decreasing function $\varepsilon(-\xi^2)$ by $\varepsilon(0)$ (greater than unity) if $\hbar\xi\ll\hbar\omega_1$, where $\hbar\omega_1$ is the lowest excitation energy of the medium, which for the present purposes we can take to be the lowest excitation energy of an atom in the medium. We can therefore replace $\varepsilon^{1/2}(-\xi^2)$ by $\varepsilon^{1/2}(0)$ if $\xi_{\rm max}<\omega_1$. Taking a typical value for $\hbar\omega_1$ to be $(e^2/10a_0)$, where a_0 is the Bohr radius, the requirements on l for Eq. (3.6) to be valid are consistent if, crudely,

$$l\gtrsim\frac{5a_0}{(e^2/\hbar c)\sqrt{\varepsilon(0)}}\approx\frac{10^3a_0}{\sqrt{\varepsilon(0)}}. \tag{3.7}$$

In the second step we extract the dependence of the energy density of the large box on the permittivity $\varepsilon(\omega^2)$ of the material it contains. We therefore compare the energy of the box with permittivity $\varepsilon_A(\omega^2)$ with that of the same box with permittivity $\varepsilon_B(\omega^2)$, for large L. Neglecting the first term in Eq. (2.28) since it vanishes for $L\to\infty$, this regularized energy difference at fixed $\omega_\perp=c|\mathbf{k}_\perp|$ is given by the difference in the second term of Eq. (2.28). With K_z defined in Eq. (2.18), we then have

$$\begin{aligned}\Delta\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B,\omega_\perp)&=\lim_{\Omega\to\infty}[\mathcal{E}^R(\varepsilon_A,L,\omega_\perp;\Omega)-\mathcal{E}^R(\varepsilon_B,L,\omega_\perp;\Omega)]\\&=\frac{L\hbar}{\pi c}\lim_{\Omega\to\infty}\int_0^\Omega d\xi\ [\sqrt{\xi^2\varepsilon_A(-\xi^2)+\omega_\perp^2}-\sqrt{\xi^2\varepsilon_B(-\xi^2)+\omega_\perp^2}].\end{aligned} \tag{3.8}$$

For large ξ Eq. (2.7) gives

$$\varepsilon(-\xi^2)=1+\frac{2}{\pi\xi^2}\int_0^\infty d\omega'\ \omega'\sigma(\omega')+O(\xi^{-4})\ , \tag{3.9}$$

where $\sigma(\omega')$ is the discontinuity of ε defined in Eq. (2.6). The limit in Eq. (3.8) therefore exists if and only if

$$\omega_{\rm pl}^2(A)=\frac{2}{\pi}\int_0^\infty d\omega'\ \omega'\sigma_A(\omega')=\frac{2}{\pi}\int_0^\infty d\omega'\ \omega'\sigma_B(\omega')$$
$$=\omega_{\rm pl}^2(B). \tag{3.10}$$

As we will see shortly, Eq. (3.10) is guaranteed by the well-known Thomas-Reiche-Kuhn sum rule [20] for the plasma frequency $\omega_{\rm pl}$ if the two substances under consideration have the same electron number density. A simple and natural choice for $\varepsilon_B(\omega^2)$ is the permittivity of an excited state of the system; the electron number densities are then clearly the same. We will henceforth assume this choice to have been made, and we can therefore set $\Omega=\infty$.

Even though we are comparing substances with the same electron density, the integration over the transverse frequencies of the modes still generally diverges. To extract and interpret this divergence, consider the regularized difference of the energy densities,

$$\frac{\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B;\Lambda)}{L^3}\equiv\frac{1}{4\pi c^2 L}\int_0^{\Lambda^2} d(\omega_\perp^2)\ \Delta\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B,\omega_\perp^2), \tag{3.11}$$

where Λ^2 is a simple cutoff for the transverse frequencies $\omega_\perp^2$. Introducing a second cutoff in addition to Ω is potentially dangerous and limits have to be taken with some care. Note in this respect that one must perform the ξ integration in Eq. (3.11) with the cutoff Λ still in place, because Eq. (2.28) is correct only for $\omega_\perp^2\ll\Omega^2$. The cutoff Ω in the total energy also restricts the transverse frequencies, as is clear from the contour integration. Our simplified cutoff procedure only makes sense for $\Lambda\ll\Omega$, which implies that the limit $\Lambda\to\infty$ can only be taken *after* the limit $\Omega\to\infty$. This ordering of the limits is important and gives a divergent contribution to the energy density, which for a dilute gas will be shown to renormalize the electron mass. A naive reversal in the order of the limits on the other hand would give an erroneous (and apparently finite) result for the difference (3.11) in the energy densities.

The integrations over ξ and $\omega_\perp^2$ can be separated by using the identity

$$x_A^{1/2}-x_B^{1/2}=-\frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}[e^{-\lambda x_A}-e^{-\lambda x_B}]. \tag{3.12}$$

The integration over $\omega_\perp^2$ is trivial and gives

$$\frac{\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B;\Lambda)}{L^3}=-\frac{\hbar}{8\pi^2c^3\sqrt{\pi}}\lim_{\Lambda\to0}\int_0^\infty\frac{d\lambda}{\lambda^{5/2}}(1-e^{-\lambda\Lambda^2})$$
$$\times\int_0^\infty d\xi[e^{-\lambda\xi^2\varepsilon_A(-\xi^2)}-e^{-\lambda\xi^2\varepsilon_B(-\xi^2)}]. \tag{3.13}$$

Suppressing the ξ dependence of ε_A and ε_B to simplify notation, we write the term in square brackets of Eq. (3.13) as

$$e^{-\lambda\xi^2\varepsilon_A}-e^{-\lambda\xi^2\varepsilon_B}=[e^{-\lambda\xi^2\varepsilon_A}-e^{-\lambda\xi^2\varepsilon_B}+\lambda\xi^2(\varepsilon_A-\varepsilon_B)]$$
$$-\lambda\xi^2(\varepsilon_A-\varepsilon_B), \tag{3.14}$$

to isolate the divergent part of Eq. (3.11). Setting

$$\frac{\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B;\Lambda)}{L^3}=\frac{\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B)}{L^3}+\frac{\mathcal{E}_{AB}^{\rm div}(\varepsilon_A,\varepsilon_B;\Lambda)}{L^3}, \tag{3.15}$$

where

$$\frac{\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B)}{L^3}=-\frac{\hbar}{8\pi^2c^3\sqrt{\pi}}\lim_{\Lambda\to0}\int_0^\infty\frac{d\lambda}{\lambda^{5/2}}(1-e^{-\lambda\Lambda^2})\int_0^\infty d\xi[\{e^{-\lambda\xi^2\varepsilon_A(-\xi^2)}+\lambda\xi^2\varepsilon_A(-\xi^2)\}-\{A\to B\}]; \tag{3.16}$$

the limit $\Lambda\to\infty$ of the integrand in Eq. (3.16) can be taken, since the term in square brackets is proportional to λ^2 for $\lambda\sim0$. We can further simplify $\mathcal{E}_{AB}$ by integrating by parts over λ. The surface term vanishes, and one obtains

$$\frac{\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B)}{L^3}=-\frac{\hbar}{12\pi^2c^3\sqrt{\pi}}\int_0^\infty d\xi\xi^2\left[\left\{\varepsilon_A(-\xi^2)\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}(1-e^{-\lambda\xi^2\varepsilon_A(-\xi^2)})\right\}-\{A\to B\}\right]$$
$$=\frac{\hbar}{6\pi^2c^3}\int_0^\infty d\xi\xi^3[\varepsilon_B^{3/2}(-\xi^2)-\varepsilon_A^{3/2}(-\xi^2)], \tag{3.17}$$

where Eq. (3.12) was used for the λ integration. We will later discuss the physical interpretation of $\mathcal{E}_{AB}$ as the difference of the bulk Lamb shifts of states A and B.

The linearly divergent part in Λ of Eq. (3.15) arises from the second term in Eq. (3.14). It gives

$$\frac{\mathcal{E}_{AB}^{\rm div}(\varepsilon_A,\varepsilon_B;\Lambda)}{L^3}=\frac{\hbar}{8\pi^2c^3\sqrt{\pi}}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}(1-e^{-\lambda\Lambda^2})\int_0^\infty d\xi\xi^2[\varepsilon_A(-\xi^2)-\varepsilon_B(-\xi^2)]=\frac{\hbar\Lambda}{4\pi^2c^3}\int_0^\infty d\xi\xi^2[\varepsilon_B(-\xi^2)-\varepsilon_A(-\xi^2)], \tag{3.18}$$

where we again made use of Eq. (3.12). Since Eq. (3.10) holds, the divergent contribution (3.18) is proportional to a moment of the difference $\sigma_A-\sigma_B$ in the discontinuities,

$$\int_0^\infty d\xi\ \xi^2[\varepsilon_B(-\xi^2)-\varepsilon_A(-\xi^2)] = \int_0^\infty d\omega'\,\omega'^2[\sigma_A(\omega')-\sigma_B(\omega')], \tag{3.19}$$

which we will interpret shortly. $\mathcal{E}_{AB}^{\rm div}$ in the limit of large Λ thus becomes

$$\mathcal{E}_{AB}^{\rm div}(\varepsilon_A,\varepsilon_B;\Lambda)/L^3=\frac{\hbar\Lambda}{4\pi^2c^3}\int_0^\infty d\omega'\,\omega'^2[\sigma_B(\omega')-\sigma_A(\omega')]. \tag{3.20}$$

In our derivation of the energy density of the fluctuating field and the extraction of the divergence (3.20) we have not assumed any special properties of the dielectric. We will, however, in the following show only that the divergent part (3.18) can be interpreted as a redefinition of the electron mass in the case where the permittivity $\varepsilon_A(\omega^2)$ is described by a discrete set of nonrelativistic oscillators with vanishing width. This should, in particular, be a valid approximation for a dilute gas, but probably also describes the permittivity of other materials quite well. The finite *result* we obtain with this restriction, however, remains valid when the spectrum of the oscillators becomes dense and therefore should apply to any substance whose permittivity satisfies Eq. (2.7).

For heuristic purposes, we begin by recalling that the classical expression for $\varepsilon_A(\omega)$ for a dilute gas of atoms in the state A is

$$\varepsilon_A(\omega)=1+\frac{4\pi e^2}{m}\sum_r\frac{N_{\rm El}(\omega_r;A)}{\omega_r^2-\omega^2-i\gamma_r\omega}, \tag{3.21}$$

where $N_{\rm El}(\omega_r;A)$ is the number of electrons per unit volume with frequency ω_r when the system is in state A. Here and later, Σ_r represents both a sum and an integral. With each energy level assumed to have zero width, we take each damping factor γ_r to be vanishingly small, set $\gamma_r\omega=\Gamma\sim 0$, and use

$$\lim_{\Gamma\to 0}{\rm Im}\frac{1}{\omega_r^2-\omega^2-i\Gamma}=\pi\delta(\omega_r^2-\omega^2)=\frac{\pi\delta(\omega_r-\omega)}{2\omega_r} \tag{3.22}$$

to find

$$\sigma_A(\omega)={\rm Im}\ \ \varepsilon_A(\omega)=\frac{2\pi^2e^2}{m}\sum_r\frac{N_{\rm El}(\omega_r;A)}{\omega_r}\delta(\omega_r-\omega). \tag{3.23}$$

Note that any analytic function $\sigma_A(\omega)$ can be written in this manner if one allows part of the spectrum of resonances to become dense. We thus obtain the classical state-independent result

$$\frac{2}{\pi}\int_0^\infty\omega\sigma_A(\omega)d\omega=\frac{4\pi e^2}{m}\sum_r N_{\rm El}(\omega_r;A)=\frac{4\pi e^2}{m}N_{\rm El}=\omega_{\rm pl}^2, \tag{3.24}$$

where $N_{\rm El}$ is the number of electrons per unit volume and $\omega_{\rm pl}$ is the plasma frequency. The quantum analog of Eq. (3.23) is obtained by replacing ω_r by $\omega_{rA}=(E_r-E_A)/\hbar$ (where A denotes the state under consideration and E_r and E_A are quantum energies of the dilute gas), and $N_{\rm El}(\omega_r;A)$ by the oscillator strength f_{rA}. Equation (3.24) then follows immediately on using the Thomas-Reiche-Kuhn [20] sum rule $\Sigma_r f_{rA}=N_{\rm El}(A)$, obtained using commutator relations.

For the case of interest, the second moment of the discontinuity in Eq. (3.20), we use Eq. (3.23) to obtain the classical state-dependent result

$$\int_0^\infty\omega^2\sigma_A(\omega)d\omega=\frac{2\pi^2e^2}{m}\sum_r\omega_r N_{\rm El}(\omega_r;A). \tag{3.25}$$

We first evaluate the sum semiclassically by noting that the ground-state energy of a harmonic oscillator of frequency ω_r is $(3/2)\hbar\omega_r$ and that the kinetic energy $p_r^2/2m$ of the oscillator is half that, so that $\omega_r=(4/3\hbar)(p_r^2/2m)$. We thereby obtain

$$\int_0^\infty\omega^2\sigma_A(\omega)d\omega=\frac{8\pi^2e^2}{3m\hbar}\left(\frac{E_{\rm kin}}{V}\right)_A, \tag{3.26}$$

where

$$\left(\frac{E_{\rm kin}}{V}\right)_A=\sum_r\frac{p_r^2}{2m}N_{\rm El}(\omega_r;A) \tag{3.27}$$

is the kinetic energy ($E_{\rm kin}$) per unit volume (V) in the state A. Proceeding somewhat more formally, we again replace ω_r by ω_{rA} and $N_{\rm El}(\omega_r;A)$ by f_{rA}, and use commutator relations to arrive at the same result in a quantum format, namely,

$$\int_0^\infty\omega^2\sigma_A(\omega)d\omega=\frac{8\pi^2e^2}{3m\hbar L^3}\left\langle A\left|\sum_i\frac{p_i^2}{2m}\right|A\right\rangle, \tag{3.28}$$

where the sum extends over all the electrons.

Nonrelativistically, the energy density difference between states $|A\rangle$ and $|B\rangle$ as given by Eq. (3.15) can therefore be rewritten as

$$\frac{\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B;\Lambda)}{L^3}=\frac{\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B)}{L^3}+\left\langle A\left|\frac{\delta H(\Lambda)}{L^3}\right|A\right\rangle-\left\langle B\left|\frac{\delta H(\Lambda)}{L^3}\right|B\right\rangle, \tag{3.29}$$

where

$$\delta H(\Lambda)=-\frac{2e^2\Lambda}{3c^3m}\sum_i\frac{\mathbf{p}_i^2}{2m}. \tag{3.30}$$

The operator δH is interpreted as a cutoff-dependent definition of the bare electron mass m_0 of the Hamiltonian without radiative corrections [21]

$$m_0(\Lambda)=\frac{m}{1+2e^2\Lambda/3c^3m}. \tag{3.31}$$

Since the physically measured mass m of a free electron includes all radiative corrections, the energy density difference between states $|A\rangle$ and $|B\rangle$ due to the radiation field is given by (3.17).

If a nonrelativistic approximation for the permittivities in Eq. (3.17) is used, this expression for the difference in energy densities of states $|A\rangle$ and $|B\rangle$ diverges logarithmically. The logarithmic divergence of the integral in Eq. (3.17) can be traced to the failure of nonrelativistic kinematics to describe the permittivities at high frequencies. The integral in Eq. (3.17) should be finite for *physical* permittivities that take into account the relativistic phase space at high-energy transfers. However, since the divergence induced by the nonrelativistic description of the permittivities is only logarithmic, we will resort to the usual *ad hoc* remedy of introducing a cutoff for the frequency integral of the order of $\xi_{\max}=mc^2/\hbar$. As will be seen in the next section, where we assume the dielectric to be a dilute gas of atoms, the use of the cutoff allows a simple comparison of Eq. (3.17) with nonrelativistic estimates of the Lamb shift in the literature. We will also suggest that Eq. (3.17) can itself be interpreted as a bulk Lamb shift.

IV. SPECIAL LIMITING CASES

To simplify the discussion, we will use subscripts M, $\mathcal{D}$, At, and El to denote a metallic (ideal) surface, a dielectric that is well approximated as a dilute gas of atoms, an atom, and an electron, respectively. To gain some insight into the physical meaning of $\mathcal{E}_{\rm Cas}(\varepsilon,l)$ of Eq. (3.5) and of $\mathcal{E}_{AB}(\varepsilon_A,\varepsilon_B)$ of Eq. (3.17), let us consider a dielectric that is a dilute gas of atoms in a single state A. If the gas is sufficiently dilute its permittivity ε_A will be adequately described in terms of the atomic polarizability α_A of the individual atoms by

$$\varepsilon_A(-\xi^2)\approx 1+4\pi N_{\rm At}(A)\alpha_A(-\xi^2) \tag{4.1}$$

to first order in the atomic number density $N_{\rm At}(A)$. In some situations, the atomic number density $N_{\rm At}$ may explicitly depend on the separation l of the ideal mirrors, as, for instance, when the number of atoms is fixed.

A. The Lamb shift for a dilute gas

For our model atom, a nonrelativistic oscillator with zero-width excited states, the polarizability is

$$\alpha_A(-\xi^2)\approx\frac{e^2}{m}\sum_r\frac{f_{rA}}{\omega_{rA}^2+\xi^2}, \tag{4.2}$$

where, as before, $\omega_{rA}=(E_r-E_A)/\hbar$ is the frequency associated with the energy difference to the state $|r\rangle$ and the f_{rA} are the oscillator strengths with

$$\sum_r f_{rA}=Z_A, \tag{4.3}$$

the number of electrons of the atom. To first order in the atomic number density $N_{\rm At}(A)$, Eq. (4.1) gives

$$\varepsilon_A^{3/2}(-\xi^2)=1+6\pi N_{\rm At}(A)\alpha_A(-\xi^2). \tag{4.4}$$

By Eq. (3.17) the difference in energy density of a gas of atoms prepared such that all are in state $|A\rangle$ or all are in state $|B\rangle$ becomes

$$\mathcal{E}_{AB}/L^3\sim-\frac{\hbar e^2}{\pi mc^3}\int_0^\infty d\xi\ \xi^3\left(\left\{N_{\rm At}(A)\sum_r\frac{f_{rA}}{\omega_{rA}^2+\xi^2}\right\}-\{A\to B\}\right). \tag{4.5}$$

Since Z, $N_{\rm At}$, and the electron number density $N_{\rm El}$ are the same for states $|A\rangle$ and $|B\rangle$, that is,

$$Z_A=Z_B\equiv Z,\quad N_{\rm At}(A)=N_{\rm At}(B)\equiv N_{\rm At},$$
$$N_{\rm El}(A)=N_{\rm El}(B)\equiv N_{\rm El}=ZN_{\rm At}, \tag{4.6}$$

Eq. (4.5) is only logarithmically divergent. We cut off the frequency integral at $\xi_{\max}=mc^2/\hbar$, beyond which the nonrelativistic approximation for the polarizability (4.2) is anyhow unacceptable. With

$$\mathcal{N}_{\rm At}\equiv N_{\rm At}L^3 \tag{4.7}$$

the total number of atoms between the plates, Eq. (4.5) becomes

$$\frac{\mathcal{E}_{AB}}{\mathcal{N}_{\rm At}}\sim-\frac{\hbar e^2}{\pi mc^3}\left[\left\{\sum_r\omega_{rA}^2 f_{rA}\ln\frac{mc^2}{\hbar|\omega_{rA}|}\right\}-\{A\to B\}\right]. \tag{4.8}$$

The choice of a zero energy reference level is a matter of convention; the individual terms in curly brackets of Eq. (4.8) are usually interpreted as the finite nonrelativistic Lamb shifts for the individual states [2].

If there is only one atom, then $N_{\rm At}=1/L^3$, and since $\alpha(0)$ is of order a_0^3, Eq. (4.1) is then satisfied and so is Eq. (4.8). We also want to allow for a number of atoms. A necessary but not sufficient condition for the validity of Eq. (4.8) is that Eq. (4.1) be applicable, which can be taken to be

$$4\pi N_{\rm At}\alpha(0)\ll 1. \tag{4.9}$$

We should, however, demand not only that Eq. (4.1) be satisfied, which it is if Eq. (4.9) is valid, but that the atom-atom interaction energy (quadratic in $N_{\rm At}$) be small compared to the Lamb shift, which is of first order in $N_{\rm At}$ but is nevertheless a very small effect. To be precise, we demand that

$$\Sigma_{\text{At-At}}\ll\mathcal{E}_{AB}/L^3\sim 2\pi\hbar N_{\rm At}\nu_0, \tag{4.10}$$

where $2\pi\hbar\nu_0$ is a typical Lamb shift and $\Sigma_{\text{At-At}}$ is the average atom-atom interaction energy per unit volume. $\Sigma_{\text{At-At}}$ for a homogeneous gas of atoms with number density $N_{\rm At}$ is

$$\Sigma_{\text{At-At}}\approx N_{\rm At}^2\int_0^\infty 4\pi r^2 dr\,V_{\text{At-At}}(r), \tag{4.11}$$

where $V_{\text{At-At}}(r)$ is the interaction potential between two atoms, which for $r\gg a_0$ is the van der Waals potential due to the dipole-dipole interaction of the atoms, that is,

$$V_{\text{At-At}}(r \gg a_0) \sim V_{\text{vdW}} \approx \left(\frac{(ea_0)^2}{r^3} \right)^2 \frac{a_0}{e^2} = \frac{e^2 a_0^5}{r^6}. \quad (4.12)$$

For $r < Ca_0$ with C of the order of 3, the interaction between the atoms will be assumed repulsive and we thus obtain a crude estimate of the integral in Eq. (4.11) by setting the lower bound to $\sim Ca_0$ and using the van der Waals interaction (4.12). Thus, we demand that

$$\Sigma_{\text{At-At}} \approx N_{\text{At}}^2 \frac{4\pi e^2 a_0^2}{3C^3} \ll 2\pi\hbar N_{\text{At}} \nu_0. \quad (4.13)$$

If we arbitrarily choose ν_0 to be the Lamb shift frequency for the $2s \to 2p$ separation in hydrogen, that is, $\nu_0 \approx 10^9$ Hz, the Lamb shift (4.8) will be the leading effect for atomic densities given roughly by

$$N_{\text{At}} \ll 4 \times 10^{-8} C^3 / a_0^3, \quad (4.14)$$

implying fewer than $\sim 10^{19}$ atoms per cm^3 for $C = 3$, a typical number density for a gas at standard conditions. Since $\alpha(0)$ is typically of the order of a_0^3, the restriction (4.14) on N_{At} for the validity of Eq. (4.8) is generally far more stringent than Eq. (4.9). Note, however, that the second order effect due to the interaction between the atoms is a bulk effect that does not contribute to the Casimir energy, nor, more significantly, to the Casimir force.

B. Casimir energies: A dilute gas versus a vacuum

In addition to the Lamb shift just discussed, there is the Casimir energy (3.5). Using Eq. (4.1) and expanding Eq. (3.5) in the atomic number density N_{At} (we are considering a dilute gas), the change in the Casimir energy on inserting a dielectric between the plates where there had previously been a vacuum (i.e., $\varepsilon = 1$) is

$$\begin{aligned} \Delta\mathcal{E}_{\text{Cas}}(\alpha, l) &= \mathcal{E}_{\text{Cas}}(\varepsilon \approx 1 + 4\pi N_{\text{At}} \alpha, l) - \mathcal{E}_{\text{Cas}}(\varepsilon = 1, l) \\ &= N_{\text{At}} \frac{L^2 \hbar}{\pi c^2} \int_0^\infty d\xi \; \xi \frac{\partial}{\partial \xi} \{ \xi^2 \alpha(-\xi^2) \\ &\quad \times \ln[1 - e^{-2l\xi/c}] \}. \end{aligned} \quad (4.15)$$

(The difference, $\Delta\mathcal{E}_{\text{Cas}}$, is finite; there is no need to study two states $|A\rangle$ and $|B\rangle$, and we therefore omit subscripts denoting states in the following.) A partial integration gives the remarkably simple expression—one that seems not to have been given previously—

$$\Delta\mathcal{E}_{\text{Cas}}(\alpha, l) = -N_{\text{At}} \frac{L^2 \hbar}{\pi c^2} \int_0^\infty d\xi \; \xi^2 \alpha(-\xi^2) \ln[1 - e^{-2l\xi/c}] \quad (4.16)$$

for the change in the Casimir energy, compared to a vacuum, due to the presence of a dilute gas between the plates. The linear relationship (4.16) between the change of the Casimir energy and the atomic polarizability represents the independent interaction of each atom with the fields present in the absence of the atoms but in the presence of the walls. For large l we can replace $\alpha(-\xi^2)$ by $\alpha(0)$, assumed to be finite [in Sec. IV D we will consider a case for which $\alpha(\omega^2) \sim \infty$ for $\omega \sim 0$], and Eq. (4.16) gives

$$\begin{aligned} \Delta\mathcal{E}_{\text{Cas}}(\alpha, l \sim \infty) &= N_{\text{At}} \frac{\alpha(0) L^2 \hbar c}{4\pi l^3} \zeta(4) = N_{\text{At}} \frac{\alpha(0) \pi^3 L^2 \hbar c}{360 l^3} \\ &= V_{M\,\mathcal{D}\,M}(\alpha(0), l \sim \infty), \end{aligned} \quad (4.17)$$

where $\zeta(x)$ is the Riemann zeta function

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x}, \quad (4.18)$$

and $V_{M\,\mathcal{D}\,M}(\alpha(0), l \sim \infty)$ is the interaction energy of a dilute gas between two ideal mirrors with the conductors. The change in the asymptotic behavior of the Casimir effect (4.17) is, of course, more readily computed by directly expanding the asymptotic behavior of the Casimir energy (3.6) to first order in N_{At}.

C. Asymptotic Casimir energy of a dilute gas and the potential energy of a single atom

The asymptotic expression (4.17) for $V_{M\,\mathcal{D}\,M}(\alpha(0), l \sim \infty)$ can also be derived from the interaction energy $V_{M\,\text{At}\,M}(z)$ of a single atom between two metallic plates with the plates. We here follow Barton's notation, now locating the plates at $\tilde{z} = -l/2$ and $\tilde{z} = l/2$; the atom is at a fixed distance $\tilde{z}$ from the midpoint. The interaction of a single atom for large l is [4]

$$V_{M\,\text{At}\,M}(\tilde{z}, l \sim \infty) = \frac{\pi^3 \hbar c}{l^4} \left[\frac{1}{360} - \frac{3 - 2\cos^2(\pi \tilde{z}/l)}{8\cos^4(\pi \tilde{z}/l)} \right] \alpha(0). \quad (4.19)$$

[As throughout this paper, we here disregard the magnetic contribution to the interaction energy, given for $V_{M\,\text{At}\,M}(\tilde{z})$ by Barton [4].] Equation (4.19) is a valid expression for the potential energy of an atom between two ideal mirrors only for distances a from either wall that are large compared to c/ω_1, where ω_1 is the lowest resonance frequency of the atom. ($2a/c$ is the to-and-from time of flight of a photon between the atom and the wall, and $2\pi/\omega_1$ is the period of the atom; it follows that for $2a/c > 2\pi/\omega_1$ an atom located at a is in the retardation zone.) Closer to either wall the potential is not retarded and is proportional to $1/a^3$ instead of $1/a^4$ as in Eq. (4.19). (The $1/a^3$ dependence is the nonretarded interaction of a dipole with its mirror image.) Very close to either wall the interaction potential finally depends on details of the structure of the atom.

To lowest order in N_{At}, the interaction energy $V_{M\,\mathcal{D}\,M}(\alpha(0), l \sim \infty)$ of a dilute gas is obtained by integrating Eq. (4.19) over a homogeneous distribution of atoms of density N_{At}

$$V_{M\,\mathcal{D}\,M}(\alpha(0), l \sim \infty) = 2N_{\text{At}} L^2 \int_0^{l/2} d\tilde{z} \, V_{M\,\text{At}\,M}(\tilde{z}), \quad (4.20)$$

where we used the fact that the contributions from $-l/2$ to 0 and from 0 to $l/2$ are the same, and the interaction

$V_{M\,\mathrm{At}\,M}(\tilde{z})$ is the interaction valid for all $\tilde{z}$. To evaluate the integral in Eq. (4.20), we choose a distance a, which satisfies

$$c/\omega_1 \ll a \ll l. \tag{4.21}$$

We can then write

$$V_{M\,\mathcal{D}\,M}(\alpha(0), l\sim\infty) = V'_{M\,\mathcal{D}\,M} + 2V_{\mathcal{D}\,M}(\alpha(0), a), \tag{4.22}$$

where

$$V'_{M\,\mathcal{D}\,M} = 2N_{\mathrm{At}}L^2 \int_0^{l/2-a} d\tilde{z}\, V_{M\,\mathrm{At}\,M}(\tilde{z}) \tag{4.23}$$

and

$$V_{\mathcal{D}\,M}(\alpha(0), a) = N_{\mathrm{At}}L^2 \int_{l/2-a}^{l/2} d\tilde{z}\, V_{M\,\mathrm{At}\,M}(\tilde{z}) \tag{4.24}$$

is the interaction of the ''skin'' of gas of thickness a adjacent to one of the conductors. We require only the potential of an atom in the retardation region between the plates to evaluate Eq. (4.23). Using Eq. (4.19), we find

$$V'_{M\,\mathcal{D}\,M} = \frac{N_{\mathrm{At}}L^2\pi^3\hbar c}{l^3}\left[\frac{1-(2a/l)}{360} - \frac{\cos(\pi a/l)}{4\pi\sin^3(\pi a/l)}\right]\alpha(0). \tag{4.25}$$

Ignoring terms of order a/l and using Eq. (4.17), we arrive at

$$V_{M\,\mathcal{D}\,M}(\alpha(0), l\sim\infty) = \Delta\mathcal{E}_{\mathrm{Cas}}(\alpha(0), l\sim\infty) + \left[2V_{\mathcal{D}\,M}(\alpha(0), a) - N_{\mathrm{At}}\frac{\alpha(0)\hbar c L^2}{4\pi a^3}\right]. \tag{4.26}$$

Since Eq. (4.26) is valid for $l\sim\infty$ for any a satisfying Eq. (4.21), the term in square brackets must vanish, that is,

$$V_{\mathcal{D}\,M}(\alpha(0), a) = N_{\mathrm{At}}\frac{\alpha(0)\hbar c L^2}{8\pi a^3} \tag{4.27}$$

and $\Delta\mathcal{E}_{\mathrm{Cas}}(\alpha, l\sim\infty)$ is indeed equal to $V_{M\,\mathcal{D}\,M}(\alpha(0), l\sim\infty)$. The contribution to $V_{\mathcal{D}\,M}(\alpha(0), a)$ from atoms a distance between a and $a+da$ from the conductor is

$$dV_{\mathcal{D}\,M}(\alpha(0), a) = (N_{\mathrm{At}}L^2 da)\left[-\frac{3\alpha(0)\hbar c}{8\pi a^4}\right] = (N_{\mathrm{At}}L^2 da)V_{\mathrm{At}\,M}(\alpha(0), a). \tag{4.28}$$

$V_{\mathrm{At}\,M}(\alpha(0), a)$ is the well-known [3] retarded potential of an atom with a mirror for large a. [It may be interesting to note that in Lifshitz's seminal paper on Casimir effects involving dielectrics [8] he determined the interaction energy $V_{D\,D\,D}$ of three planar dielectrics, where the outer dielectrics of permittivity ε_1 and ε_2 extend to $\pm\infty$, and sandwich a slab of thickness l and permittivity ε_3. By setting $\varepsilon_3 = 1$ and taking ε_2 to be the permittivity of a dilute gas, he obtained $V_{\mathrm{At}\,D}$, and by letting $\varepsilon_1 \to \infty$, he also obtained $V_{\mathrm{At}\,M}$, without further subtleties. The present analysis is conceptually simpler in at least one regard in that we start with only one dielectric, having set $\varepsilon_1 = \varepsilon_2 = \infty$ at the outset. $V_{M\,D\,M}$ then follows on taking ε_3 to be the permittivity of a dilute gas. To determine $V_{\mathrm{At}\,M}$, however, we either have to argue that in this case one must neglect terms representing multiple reflections, that is, one must retain only the $n=1$ term in Eq. (4.17), or compare with the known potential $V_{M\,\mathrm{At}\,M}(\tilde{z})$ and subtract infinities. For illustrative purposes and to compare with the literature, we chose the latter method to extract $V_{\mathrm{At}\,M}$.]

We have assumed that the atoms are uniformly distributed. This is of course not quite the case since the atoms are attracted to the wall, but especially for large l it should be a reasonable assumption; $V_{\mathrm{At}\,M}$ is strong but not singular near the wall, and temperature effects help a bit. In any event, N_{At} will be reasonably uniform away from the wall, since we took N_{At} to be independent of l and for that case the number of atoms that attach themselves to the wall will saturate. Note that it is perfectly possible to maintain the value of N_{At} by introducing atoms as l increases. The experimentally perhaps more easily realizable situation in which the number of atoms $N_{\mathrm{At}}lL^2$ is constant can be analyzed by the procedure used in the following study of electrons between walls.

Feynman [22] pointed out [23] that one could obtain the Lamb shift for an atom by considering one such atom in a box with conducting walls, with the effective index of refraction then given by $n(\omega) = \varepsilon^{1/2}(\omega) \approx 1 + 2\pi\alpha(\omega)/L^3$, where L^3 is the volume of the box and $\alpha(\omega)$ is the polarizability of the atom. The discussion immediately above shows that the study of an atom in a box can give not only the Lamb shift but also the correction to the Casimir effect. Our earlier discussion was of course more general, for it considered a reasonably arbitrary $\varepsilon(\omega)$.

D. Interaction energy of an electron with one and two walls

Finally we point out that Eq. (4.16) in some cases also allows one to evaluate the change in the Casimir energy compared to a vacuum when $\alpha(-\xi^2)$ is singular at $\xi^2 = 0$, as, for instance, when there is an isolated electron between the plates with $\alpha_{\mathrm{El}}(-\xi^2) = e^2/(m\xi^2)$. The expansion (4.1) of the permittivity $\varepsilon(-\xi^2)$ around $\varepsilon = 1$ in this case is certainly not valid for $\xi^2 < 4\pi N_{\mathrm{El}}e^2/m$ and the arguments leading to Eq. (4.16) might seem to be questionable. However, for $N_{\mathrm{El}} \to 0$ the only frequencies for which the expansion does not apply are in the neighborhood of the origin. Since the integral in Eq. (4.15) is not singular at the origin, Eq. (4.16) remains valid (with N_{At} replaced by N_{El}) in the limit $N_{\mathrm{El}} \to 0$, which includes the single-electron case. Introducing a single electron therefore has an overall negligible effect on the field and effectively interacts with the field that would be there in its absence. The contribution to the total interaction energy from electromagnetic fields with extremely low frequency, for which the effect of introducing an electron is not negligible, vanishes due to the available phase space. The integral in Eq. (4.16) is readily performed and one obtains for the radiative contribution to the potential energy of a single electron with uniform probability density $[N_{\mathrm{El}} = 1/(L^2 l)]$ between two ideal conductors,

$$\Delta\mathcal{E}_{\rm Cas}(\alpha_{\rm El},l)|_{N_{\rm El}=1/(L^2l)}=\frac{\hbar e^2}{2\pi mcl^2}\zeta(2)=\frac{\pi\hbar e^2}{12mcl^2}. \tag{4.29}$$

[Note that Eq. (4.29) does not include the classical electromagnetic energy between the plates due to the presence of an electron, which is independent of m and $\hbar$ but many orders of magnitude larger than Eq. (4.29). A confirmation of $\Delta\mathcal{E}_{\rm Cas}(\alpha_{\rm El},l)|_{N_{\rm El}=1/(L^2l)}$ would therefore require a measurement of very high accuracy, for it would be necessary to determine the difference between the measured interaction and the classical Coulombic effects.]

One can check the validity of Eq. (4.29) by proceeding along the same lines as used in checking $V_{M\,\mathcal{D}\,M}(\alpha(0),l\sim\infty)$, that is, by confirming that

$$\Delta\mathcal{E}_{\rm Cas}(\alpha_{\rm El},l)|_{N_{\rm El}=1/(L^2l)}=\overline{V}_{M\ {\rm El}\ M}(l)=\frac{2}{l}\int_0^{l/2}d\tilde{z}\,V_{M\ {\rm El}\ M}(\tilde{z}), \tag{4.30}$$

where

$$V_{M\ {\rm El}\ M}(\tilde{z})=\frac{\pi\hbar e^2}{mcl^2}\left[\frac{1}{12}+\frac{1}{4\cos^2(\pi\tilde{z}/l)}\right] \tag{4.31}$$

is the retardation interaction of an electron at a fixed distance $\tilde{z}$ from a point midway between the two walls [24]. There is, however, the complicating factor that we could consider the case for which $N_{\rm At}$ is constant, but with one electron $N_{\rm El}$ varies as $1/l$. [One cannot sensibly study many electrons between the walls rather than just one, with $N_{\rm El}$ held independent of l by introducing electrons as l is increased, and maintain a uniform electron distribution; the electron-electron Coulomb repulsion and the attraction of the electrons by the walls would drive many electrons to the walls. Even with the one electron case under consideration the distribution will not truly be uniform, though $\overline{V}_{M\ {\rm El}\ M}(l)$ could still be of some use. $\overline{V}_{M\ {\rm El}\ M}(l)$ might be of greater practical use in a study of the effect of walls on an electron passing between walls for small deflections of the electron and for the electron incident with a probability density independent of its distance from the walls.] There is a further difference between the electron and atom cases, this difference being a simplifying factor; Eq. (4.31) is valid everywhere between the ideal walls, since the interaction with the electron is always retarded, for neither the ideal wall nor the electron have a natural period. Introducing a length $b\ll l$, we isolate the divergent contribution when the electron is close to either wall by writing

$$\overline{V}_{M\ {\rm El}\ M}(l)=\frac{2}{l}\left(v_{{\rm El}\ M}(b)+\int_0^{l/2-b}d\tilde{z}\,V_{M\ {\rm El}\ M}(\tilde{z})\right), \tag{4.32}$$

where

$$v_{{\rm El}\ M}(b)=\int_{l/2-b}^{l/2}d\tilde{z}\,V_{M\ {\rm El}\ M}(\tilde{z}); \tag{4.33}$$

we use the symbol v in Eq. (4.33) to indicate that this is not a potential. Performing the integral in Eq. (4.32) and neglecting terms that vanish in the limit $b/l\to 0$, one obtains

$$\overline{V}_{M\ {\rm El}\ M}(l)=\frac{2}{l}\left[\overline{v}_{{\rm El}\ M}(b)+\frac{\hbar e^2}{4\pi mcb}\right]+\Delta\mathcal{E}_{\rm Cas}(\alpha_{\rm El},l)|_{N_{\rm El}=1/(L^2l)}. \tag{4.34}$$

Demanding that $\overline{V}_{M\ {\rm El}\ M}(l)$ not depend on b for any b satisfying $b\ll l$, the term in square brackets in Eq. (4.34) is a constant C independent of b for sufficiently small b/l. We can now recover the interaction $V_{{\rm El}\ M}(b)$ of an electron at a distance b from one of the walls with that wall by noting that

$$\frac{d}{db}v_{{\rm El}\ M}(b)=-\frac{d}{db}\frac{\hbar e^2}{4\pi mcb}=\frac{\hbar e^2}{4\pi mcb^2}, \tag{4.35}$$

which follows on differentiating Eq. (4.34) with respect to b, and also, from Eq. (4.33), that

$$\frac{d}{db}v_{{\rm El}\ M}(b)=V_{M\ {\rm El}\ M}(l/2-b)\approx V_{{\rm El}\ M}(b); \tag{4.36}$$

the last step in Eq. (4.36) follows on recognizing that the interaction with the second wall is proportional to $1/l^2$, and therefore smaller than $V_{{\rm El}\ M}(b)$ by of order b^2/l^2. Comparing Eqs. (4.35) and (4.36) gives the well-known retarded interaction of an electron with a single wall

$$V_{{\rm El}\ M}(b)=\frac{\hbar e^2}{4\pi mcb^2}. \tag{4.37}$$

In the case of a dilute gas of atoms, the constant corresponding to C just defines the zero-energy level. It was subtracted by demanding that the interaction energy vanish for $l\to\infty$. As noted above, however, in the present case we are forced to consider only one electron, and the electron number density therefore decreases as $1/l$. Furthermore, the constant in fact diverges if the upper limit on the integral in Eq. (4.33) is $l/2$ and Eq. (4.31) is valid for all distances. However, the resulting (divergent) contribution proportional to $1/l$ to $\overline{V}_{M\ {\rm El}\ M}$ is of the same form as the Coulomb interaction of the electron with its image charge in the opposite wall. This (divergent) $1/l$ contribution to $\overline{V}_{M\ {\rm El}\ M}$ should thus be interpreted as a radiative correction to the Coulombic contribution to the interaction energy. The divergent constant C just redefines the charge of the electron in order $\hbar$ to the physical one. With the b independent (divergent) term in square brackets in Eq. (4.34) absorbed in the electron charge, $\overline{V}_{M\ {\rm El}\ M}$ and $\Delta\mathcal{E}_{\rm Cas}(\alpha_{\rm El},l)|_{N_{\rm El}=1/(L^2l)}$ each fall off as $1/l^2$ and are equal to one another.

V. DISCUSSION

The assumption that the walls are perfect conductors is an idealization, but, especially for large l, for which low-frequency contributions dominate, it is often a rather good approximation. In addition, results for conductors serve as checks on results for dielectrics; further, for l large so that $\varepsilon(-\xi^2)$ can be approximated by $\varepsilon(0)$, interaction energies

are relatively insensitive to variations in $\varepsilon(0)$, and by demanding that the interaction energies have the correct form for $\varepsilon(0)=\infty$ and for $\varepsilon(0)-1\ll 1$, one can easily guess at forms of the interaction [25] that are good to within about 10% over the entire range $1\leqslant\varepsilon(0)\leqslant\infty$.

We note that the bulk Lamb shift (3.17) can be applied to a dielectric in free space, for any reasonable smooth shape of the dielectric. If the volume of the dielectric is not too large, one would, however, have to know the Casimir energy, which is not simply proportional to the volume, in order to isolate the bulk Lamb shift.

A number of our results are in the literature, but some of the forms we obtain, see Eq. (3.5), for example, are simpler than any given previously, and some of our results have not been obtained before. These include simplified forms for some known Casimir energies, and Casimir energies for a dilute gas of atoms between plates and for an individual electron with uniform probability density between plates. But the primary contribution may well be placing the analysis on a somewhat firmer and perhaps also broader basis. The present unified derivation may thus be of some significance in itself. Further, we calculated the radiative corrections to the total energy in terms of the macroscopic permittivity $\varepsilon(\omega)$ of the material, which at least in principle, can be measured. Our finite results for the Casimir energy and bulk Lamb shift do not rely on a microscopic calculation of the permittivity and should be valid for the physical permittivity. A microscopic calculation would, however, reveal that the permittivity $\varepsilon(\omega)$ of the material itself depends on the separation l between the plates [19]. Only to interpret the divergent part of the bulk Lamb shift as a renormalization of the electron mass was a microscopic description of the dielectric in terms of atoms needed. We believe that the finite expressions for the radiative corrections to the total energy (3.5) and (3.17) are, however, valid beyond the dilute gas approximation—in the worst case, Eq. (3.17) *defines* the finite radiative bulk energy density up to a finite term proportional to the second moment of the discontinuity of $\varepsilon(\omega)$. We have seen that this term *vanishes in the dilute gas limit*. Without a rigorous *definition* of what is meant by the *radiative* contribution to the total energy in the general case, our choice for the finite bulk radiative contribution, while somewhat arbitrary, would seem to be a reasonable one; in particular, the limit as one approaches the dilute case is equal to the well known result for that case. In this respect, it is perhaps of some interest that we find the radiative bulk energy density to be related to the third power of the index of refraction in Eq. (3.17); this reflects the fact that the Lamb shift is proportional to c^{-3}.

We close by noting that the theory of quantized Maxwell fields in absorptive dielectrics has been the subject of a number of papers. These include Glauber and Lewenstein [26], Ref. [13], Barnett *et al.* [27], Kupiszewska and Mostowski [28], Barton [29], and Milonni [30].

Note added in proof. Equation (3.5) can be further simplified, to a derivative-free form.

ACKNOWLEDGMENTS

We are deeply grateful to G. Barton for reading the manuscript and making a number of useful remarks. We also want to thank L. Rosenberg for some helpful conversations at the early stages of this research. This work was supported in part by the National Science Foundation Grant No. PHY-9605218. L.S. was also supported by the Humboldt Foundation.

APPENDIX: ABSENCE OF COMPLEX SOLUTIONS TO THE DISPERSION RELATION [Eq. (2.4)]

Suppose Eq. (2.4) for some $b^2=\omega_\perp^2+c^2k_n^2\geqslant 0$ has a complex solution $\omega_n^2=z_0$ with $\mathrm{Re}\,z_0\geqslant 0$, $\mathrm{Im}\,z_0\neq 0$. Taking real and imaginary parts of Eq. (2.4) this would require, since b^2 is real, that

$$(\mathrm{Re}\,z_0)[\mathrm{Re}\,\varepsilon(z_0)]=b^2+(\mathrm{Im}\,z_0)[\mathrm{Im}\,\varepsilon(z_0)] \tag{A1}$$

and

$$(\mathrm{Im}\,z_0)[\mathrm{Re}\,\varepsilon(z_0)]=-(\mathrm{Re}\,z_0)[\mathrm{Im}\,\varepsilon(z_0)] \tag{A2}$$

be satisfied simultaneously. Taking the imaginary part of Eq. (2.7) one has in addition

$$\mathrm{Im}\,\varepsilon(z_0)=C^2(z_0)\mathrm{Im}\,z_0, \tag{A3}$$

where

$$C^2(z_0)=\frac{2}{\pi}\int_0^\infty d\omega'\,\frac{\omega'\sigma(\omega')}{|\omega'^2-z_0|^2}>0 \tag{A4}$$

is positive and real, because the discontinuity $\sigma(\omega')$ in Eq. (2.7), that is, $\mathrm{Im}\,\varepsilon(\omega^2)$, is positive and real. [If $\mathrm{Im}\,\varepsilon(\omega^2)$ were negative, a plane wave incident on the medium would grow exponentially.] Equation (A3) implies that $\mathrm{Im}\,z_0$ and $\mathrm{Im}\,\varepsilon(z_0)$ have the *same* sign for $\mathrm{Im}\,z_0\neq 0$. $\mathrm{Im}\,\varepsilon(z_0)$ therefore vanishes *only* for z_0 real. In general $\mathrm{Im}\,\varepsilon(z_0)$ vanishes only for negative real z_0, that is, on the imaginary axis of ω, since $C^2(z_0)$ can tend to infinity as z_0 approaches a point on the positive real axis. Inserting Eq. (A3) in Eq. (A2) and dividing by $\mathrm{Im}\,z_0\neq 0$ one obtains

$$\mathrm{Re}\,\varepsilon(z_0)=-(\mathrm{Re}\,z_0)C^2(z_0) \tag{A5}$$

for a solution to Eq. (2.4) with a nonvanishing imaginary part. Finally, using Eqs. (A3) and (A5) for the real and imaginary parts of $\varepsilon(z_0)$ we see that Eq. (A1) implies the contradiction

$$0\geqslant-(\mathrm{Re}\,z_0)^2C^2(z_0)=b^2+(\mathrm{Im}\,z_0)^2C^2(z_0)>0. \tag{A6}$$

A solution z_0 to Eq. (2.4) for a physical permittivity satisfying Eq. (2.7) with positive discontinuity $\sigma\geqslant 0$ is therefore real. Equation (A2) then requires that $\mathrm{Im}\,\varepsilon(z_0)=0$ for $z_0\neq 0$. These real solutions to Eq. (2.4) thus only exist in regions where σ vanishes. The discrete spectrum of zero-width resonances we consider is the simplest model that satisfies Eq. (2.7) and also allows us to calculate the zero-point energy in the conventional fashion. By letting the spectrum of resonances become dense *at the end of the calculation* we compute the Casimir energy and bulk Lamb shift of any system with a physical permittivity $\varepsilon(\omega^2)$.

[1] See, for example, S. Hawking, Commun. Math. Phys. **55**, 133 (1977); W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics*, Lecture Notes in Physics Vo1. 220 (Springer, Berlin, 1985), for a vacuum between the plates, and M. Schaden and L. Spruch (unpublished), for a dielectric between the plates.

[2] H. A. Bethe, Phys. Rev. **72**, 241 (1947); see also S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1962), p. 524ff.

[3] H. B. G. Casimir, J. Chim. Phys. Phys.-Chim. Biol. **46**, 407 (1949).

[4] G. Barton, Proc. R. Soc. London, Ser. A **410**, 141 (1987).

[5] E. J. W. Verwey and J. T. G. Overbeek, *Theory of the Stability of Lyophobic Colloids* (Elsevier, Amsterdam, 1948).

[6] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).

[7] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **60**, 793 (1948).

[8] E. M. Lifshitz, Zh. Éksp. Teor. Fiz. **29**, 94 (1955) [Sov. Phys. JETP **2**, 73 (1956)]; [see also L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Course of Theoretical Physics Vol. 2, 4th revised English ed. (Pergamon, New York, 1975)]; I. E. Dzyaloshinskii, E. M. Lifshitz, and L. Pitaevskii, Adv. Phys. **10**, 165 (1961).

[9] B. V. Derjaguin and I. I. Abriksova, Zh. Éksp. Teor. Fiz. **30**, 993 (1956) [Sov. Phys. JETP **3**, 819 (1957)]; M. J. Sparnaay, Physica (Amsterdam) **24**, 751 (1958); J. N. Israelachvili and D. Tabor, Proc. R. Soc. London, Ser. A **331**, 19 (1972).

[10] P. W. Milonni, *The Quantum Vacuum* (Academic, New York, 1993). See especially Chaps. 7 and 8.

[11] E. Hinds, in *Cavity Quantum Electrodynamics*, supplement to Advances in Atomic, Molecular, and Optical Physics, edited by P. R. Berman (Academic Press, New York, 1994), Chap. 1.

[12] L. Spruch, in *Long-Range Casimir Forces: Theory and Experiment in Multiparticle Dynamics*, edited by F. S. Levin and D. A. Micha (Plenum, New York, 1993), Chap. 1.

[13] Y. Tikochinsky and L. Spruch, Phys. Rev. A **48**, 4223 (1993).

[14] F. Zhou and L. Spruch, Phys. Rev. A **52**, 297 (1995).

[15] S. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997).

[16] N. G. van Kampen, B. R. A. Nijboer, and K. Schramm, Phys. Lett. **26A**, 307 (1968).

[17] See, for example, E. A. Power, *Introductory Quantum Electrodynamics* (Elsevier, Amsterdam, 1964), p. 131, or R. W. Robinett, *Quantum Mechanics* (Oxford, New York, 1997), p. 481.

[18] For $\xi^2=-\omega^2$ and a vacuum between the plates, that is $\varepsilon(\omega^2)=1$, we have $p=\cos\theta$, where θ is the angle of incidence of the mode under consideration on one of the walls.

[19] Consider, for instance, the simple case for which the region between the plates is the physical vacuum [not the ideal vacuum with $\varepsilon(\omega)=1$ everywhere]. For any value of l of physical interest one can set $\varepsilon(\omega)=1$, but for l of the order of a Compton wavelength, vacuum polarization is affected by the presence of the walls and ε for all but very high frequencies is independent of ω but dependent upon l. See G. Barton and K. Scharnhorst, J. Phys. A **26**, 2037 (1993).

[20] See, for example, E. Merzbacher, *Quantum Mechanics*, 2nd ed. (Wiley, New York, 1970), p. 458.

[21] The mass in Eq. (3.28) is the electron's observed *physical* mass m, since this relation is derived for (renormalized) physical quantities, analogous to the fact that the similarly derived relation between plasma frequency and electron density in Eq. (3.24) also involves only the observed electron mass.

[22] R. P. Feynman, in *The Present Status of Quantum Electrodynamics*, edited by R. Stoops (Wiley Interscience, New York, 1961).

[23] See also E. A. Power, Am. J. Phys. **34**, 516 (1966), and Ref. [10], p. 92.

[24] G. Barton, Proc. R. Soc. London, Ser. A **320**, 251 (1970).

[25] L. Spruch and Y. Tikochinsky, Phys. Rev. A **48**, 4213 (1993).

[26] R. J. Glauber and M. Lewenstein, Phys. Rev. A **43**, 467 (1991).

[27] M. Matloob, R. Loudon, S. M. Barnett, and J. Jeffers, Phys. Rev. A **52**, 4823 (1995); M. Matloob and R. Loudon, *ibid.* **53**, 4567 (1996).

[28] D. Kupiszewska and J. Mostowski, Phys. Rev. A **41**, 4636 (1990).

[29] G. Barton, Proc. R. Soc. London, Ser. A (to be published).

[30] P. W. Milonni, J. Mod. Opt. **42**, 1191 (1995).