Hydrodynamic excitations of Bose condensates in anisotropic traps

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The collective excitations of Bose condensates in anisotropic axially symmetric harmonic traps are investigated in the hydrodynamic and Thomas-Fermi limit. We identify an additional conserved quantity, besides the axial angular momentum and the total energy, and separate the wave equation in elliptic coordinates. The solution is thereby reduced to the algebraic problem of diagonalizing finite-dimensional matrices. The classical quasiparticle dynamics in the local-density approximation for energies of the order of the chemical potential is shown to be chaotic. [S1050-2947(97)51510-1]

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The Bose condensates of alkali-metal atoms in magnetic traps [1–3] offer a unique way to investigate the low-lying collective excitations in Bose condensates [4–16]. Experimentally collective modes with a given symmetry have been excited by time-dependent modulations of the trapping potential, and their evolution has been followed in real time by measurements of the resulting shape oscillations of the condensates. The measurements performed so far have involved turning off the trap after a given time [4–6], but in the future they could even be performed nondestructively by elastic off-resonant light scattering [7]. Theoretically the collective modes have been analyzed by using the Bogoliubov equations or by linearizing the time-dependent Gross-Pitaevskii equation around the time-independent condensate and solving these equations numerically [8,9] or analytically in various approximations [10–16]. Very good agreement between the numerical and the experimental results has been found.

In a seminal paper Stringari [16] has shown how the coupled wave equations for the collective excitations are simplified in the hydrodynamic limit to become a single second-order wave equation for density waves, and he obtained analytical solutions for all its modes in spherically symmetric harmonic traps and, remarkably, also for some of its modes in axially symmetric harmonic traps. The latter are particularly important, because all experiments have been performed with traps of this symmetry [4–6].

In the present paper it is our goal to study in more detail by analytical means the hydrodynamic wave equation in the axially symmetric case. We wish to find an explanation why at least some analytical solutions have been possible in this case and intend to use this insight to construct more solutions in a systematic way.

In principle the collective mode problem looks very different for isotropic and for axially symmetric traps: In the isotropic case the rotational symmetry ensures that angular-momentum conservation gives two good quantum numbers, and therefore the wave equation is separable in spherical coordinates. For axial symmetry, however, only the axial component of angular momentum remains a good quantum number, besides the energy, and one may expect that the system, having three degrees of freedom, is not integrable. In fact, this expectation is borne out for collective excitations whose energies are neither very large nor very small compared to the chemical potential (see below), which spoils all hopes of finding exact analytical solutions for the modes and their spectrum in this energy range for axially symmetric traps. Why then are such solutions possible at energies in the hydrodynamic regime, i.e., for energies much smaller than the chemical potential?

The answer is provided by the existence, in that regime, of an additional conserved quantity, which we exhibit explicitly below. Its existence permits the separation of the hydrodynamic wave equation in elliptical coordinates. Thereby the task of solving the wave equation can be reduced to the purely algebraic problem of diagonalizing finite-dimensional matrices. We use this method to obtain the spectrum of the low-lying hydrodynamic modes as a function of the ratio of the axial and the radial trap frequencies.

A convenient starting point of our analysis is the linearized hydrodynamic equations as derived in [16]. They read

$$\dot{\varphi} = -\frac{4\pi\hbar a}{M}\delta\rho, \quad \delta\dot{\rho} = -\frac{\hbar}{M}\nabla\cdot\rho_0(\boldsymbol{x})\nabla\varphi. \tag{1}$$

These equations are valid on time scales much longer than $\hbar/\mu$. Here φ is the phase of the macroscopic wave function, $\delta\rho$ is the local perturbation of the number density in the collective mode, $\rho_0(\boldsymbol{x})$ is the number density of the time-independent condensate in the Thomas-Fermi approximation [18], $\rho_0(\boldsymbol{x}) = M/4\pi\hbar^2 a[\mu - U(\boldsymbol{x})]\Theta(\mu - U(\boldsymbol{x}))$, where μ is the chemical potential, and M and a are the mass and positive s-wave scattering length of the atoms. For the connection of Eq. (1) with the Bogoliubov equations see [17]. The trapping potential $U(\boldsymbol{x})$ is assumed to have the form $U(\boldsymbol{x}) = (M/2)\omega_0^2(x^2+y^2) + (M/2)\omega_z^2 z^2$. The surface of the condensate, in the Thomas-Fermi limit, is defined by $U=\mu$. Equations (1) are obtained by a gradient expansion and assuming $\delta\rho$ and $\nabla\varphi$ to be small. Eliminating $\nabla\varphi$ from both equations and making the ansatz $\delta\rho(\boldsymbol{x},t) = e^{-i\omega t}\psi(\boldsymbol{x})$, one obtains the time-independent wave equation [16] $M\omega^2\psi(\boldsymbol{x}) = -\nabla\cdot[\mu - U(\boldsymbol{x})]\nabla\psi(\boldsymbol{x})$, which holds inside the condensate. We look for boundary conditions on the surface of the condensate that make the operator $\hat{G} = -M^{-1}\nabla\cdot[\mu$

$-U(\boldsymbol{x})]\boldsymbol{\nabla}$ Hermitian, so that for two eigenfunctions ψ, $\tilde{\psi}$ with eigenvalues ω^2, $\tilde{\omega}^2$ we have orthogonality according to

$$(\omega^2-\tilde{\omega}^2)\int_V d^3x\,\psi^*\tilde{\psi}$$
$$=\int_V d^3x[\psi^*\hat{G}\tilde{\psi}-\tilde{\psi}\hat{G}^*\psi^*]$$
$$=-M^{-1}\int_{\partial V} df(\mu-U)(\psi^*\partial\tilde{\psi}/\partial n-\tilde{\psi}\partial\psi^*/\partial n)=0\,.$$

Here V and ∂V denote the volume of the condensate and its surface, respectively. Because $\mu-U$ vanishes on the boundary, it is enough to require that ψ and its normal derivative $\partial\psi/\partial n$ remain bounded there. In the following it will be useful to measure lengths in units of the radial Thomas-Fermi radius $r_0=(2\mu/M\omega_0^2)^{1/2}$, which brings the wave equation into the dimensionless form $\omega^2\psi=\hat{G}\psi$ with

$$\hat{G}=-(\omega_z^2/2)\boldsymbol{\nabla}\cdot[(1-\epsilon^2)(1-\rho^2)-z^2]\boldsymbol{\nabla}\,, \tag{2}$$

where we use cylindrical coordinates $\rho=\sqrt{x^2+y^2}$,z and the azimuthal angle ϕ, and define $\epsilon^2=1-\omega_0^2/\omega_z^2$, which is positive for $\omega_0<\omega_z$. The operator $\hat{G}$ commutes, of course, with the axial angular-momentum operator $\hat{L}_z=-i(\partial/\partial\phi)$. However, there is an additional nontrivial operator

$$\hat{B}=-\boldsymbol{\nabla}^2+(\boldsymbol{x}\cdot\boldsymbol{\nabla})^2+3\boldsymbol{x}\cdot\boldsymbol{\nabla}+\epsilon^2\partial^2/\partial z^2\,, \tag{3}$$

which commutes with $\hat{L}_z$ and with $\hat{G}$, as one may check by direct calculation of the commutators. In the isotropic case $\epsilon=0$, $\hat{B}$ may be expressed by the square of the angular momentum $\hat{L}^2=\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2$ and $\hat{G}$ via $\hat{B}=2\hat{G}/\omega_0^2+\hat{L}^2$. Because of the existence of the three commuting operators $\hat{G}$, L_z, and $\hat{B}$ in the system with the three degrees of freedom ρ, z, and ϕ, it is now manifest that the system is integrable, i.e., the eigenvalues of $\hat{G}$ can be labeled by the quantum numbers of L_z and $\hat{B}$. To see this explicitly we now look for variables in which the wave equation separates, and introduce cylindrical elliptical coordinates ξ, η, which in their oblate spheroidal form [19] are defined by $\rho=\sigma\sqrt{(\xi^2+1)(1-\eta^2)}$, $z=\sigma\xi\eta$. These coordinates are orthogonal. Surfaces of $\xi=$const are confocal ellipsoids with foci at $z=0$, $\rho=\sigma$. Surfaces with constant η are confocal hyperboloids with the same foci. For $\omega_z\geqslant\omega_0$, i.e., $0\leqslant\epsilon^2\leqslant1$, the foci at $z=0$ $\rho=\sigma$ are made to coincide with the foci of the ellipsoidal Thomas-Fermi surface if we choose $\sigma=\epsilon$. The Thomas-Fermi surface is given by $\xi_{TF}=(1/\epsilon^2-1)^{1/2}$. Then the interior of the condensate is described by ξ in the range $[0,\xi_{TF}]$ and η in the range $[-1,1]$.

For $\omega_z\leqslant\omega_0$ the parameter ϵ^2 is no longer useful as it becomes negative. Instead one can define $\epsilon'^2=1-\omega_z^2/\omega_0^2=-\epsilon^2/(1-\epsilon^2)$, which lies in the range $0\leqslant\epsilon'^2<1$. The foci of the Thomas-Fermi ellipse now lie at $z=\pm\sigma'$ where $\sigma'=\epsilon'/\sqrt{1-\epsilon'^2}$. Therefore we now need the prolate spheroidal form of elliptical coordinates [19] with foci at $z=\pm\sigma'$, $\rho=0$. These coordinates are defined by $\rho\rho=\sigma'\sqrt{(\xi^2-1)(1-\eta^2)}$, $z=\sigma'\xi\eta$, where inside the condensate ξ now has the range $[1,1/\epsilon']$ while η has the same range as before. The treatments in the two cases are equivalent via the transformation connecting ϵ and ϵ'. In the following we shall present the equations for the case $\omega_z>\omega_0$. The final formulas for ω^2 apply for $\omega_z\geqslant\omega_0$ and $\omega_z\leqslant\omega_0$. After the change of coordinates with $\sigma=\epsilon$ the operator $\hat{G}$ takes the form

$$\hat{G}=-\frac{\omega_z^2}{2\epsilon^2}\frac{1}{\xi^2+\eta^2}$$
$$\times\Bigg[[1-\epsilon^2(1-\eta^2)]\frac{\partial}{\partial\xi}[1-\epsilon^2(\xi^2+1)](\xi^2+1)\frac{\partial}{\partial\xi}$$
$$+[1-\epsilon^2(\xi^2+1)]\frac{\partial}{\partial\eta}[1-\epsilon^2(1-\eta^2)](1-\eta^2)\frac{\partial}{\partial\eta}$$
$$+\frac{[1-\epsilon^2(\xi^2+1)][1-\epsilon^2(1-\eta^2)](\xi^2+\eta^2)}{(1-\eta^2)(1+\xi^2)}\left(\frac{\partial}{\partial\phi}\right)^2\Bigg], \tag{4}$$

which is now separable. The ϕ dependence of its eigenfunctions is taken care of by factors $e^{im\phi}$ with the integer azimuthal quantum number m. Separating the operator in ξ and η by making the ansatz $\Psi_\xi(\xi)\Psi_\eta(\eta)e^{im\phi}$ for its eigenfunctions we obtain two equations: one for Ψ_η,

$$\left[\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta}-\frac{m^2}{1-\eta^2}+\frac{2\epsilon^2(1-\eta^2)\eta}{1-\epsilon^2(1-\eta^2)}\frac{d}{d\eta}\right]\Psi_\eta$$
$$-\frac{2\omega^2/\omega_1^2}{1-\epsilon^2(1-\eta^2)}\Psi_\eta=-\beta\Psi_\eta\,; \tag{5}$$

the other for Ψ_ξ. It turns out that both equations are identical if in the equation for Ψ_η we substitute $i\xi$ for η, i.e., $\Psi_\xi(\xi)\equiv\Psi_\eta(i\xi)$. The solution for one coordinate is the analytic continuation of the solution of the other from the real to the imaginary axis. It is easy to check that the separation constant β is just the eigenvalue of the operator $\hat{B}$ for the eigenfunction $\Psi_\xi(\xi)\Psi_\eta(\eta)\exp(im\phi)$. To do this one needs to express $\hat{B}$ also in the elliptic coordinates:

$$\hat{B}=\frac{1}{\epsilon^2(\xi^2+\eta^2)}\Bigg\{-\frac{\partial}{\partial\xi}[1-\epsilon^2(\xi^2+1)](\xi^2+1)\frac{\partial}{\partial\xi}$$
$$-\frac{\partial}{\partial\eta}[1-\epsilon^2(1-\eta^2)](1-\eta^2)\frac{\partial}{\partial\eta}$$
$$-\frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2}{\partial\phi^2}\Bigg\}\,. \tag{6}$$

Equations (5) contain m only quadratically. Therefore the energy levels are the same for $\pm m$. Expanding Ψ_η for fixed $|m|$ in terms of associated Legendre functions $P_\ell^{|m|}(\eta)$ with coefficients a_ℓ, where $|m|\leqslant\ell\leqslant\infty$, we obtain from Eq. (5) a second-order recursion relation for the coefficients a_ℓ relating only even or only odd indices ℓ. The eigenstates therefore have even and odd parity. The recursion relation itself is straightforward to obtain but lengthy and

TABLE I. The 43 lowest levels labeled by the quantum numbers $|m| \leq 2, n, j$ for $(\omega_z/\omega_0)^2 = 8$.

n	j	$m=0$	$m=1$	$m=2$
0	0	0.000 000 00	1.000 000 00	1.414 213 56
2	0	1.797 128 37	2.317 294 58	2.723 416 01
1	0	2.828 427 12	3.000 000 00	3.162 277 66
4	0	2.911 930 10	3.365 376 41	3.731 310 58
3	0	3.273 025 89	3.518 321 07	3.741 657 39
5	0	3.826 573 18	4.098 941 74	4.343 248 21
6	0	3.840 996 26	4.215 323 29	4.522 793 02
7	0	4.407 483 68	4.679 210 86	4.922 217 57
8	0	4.590 810 95	4.898 147 04	5.160 102 07
9	0	4.971 054 60	5.229 358 96	5.461 449 95
2	1	4.976 979 97	5.160 440 47	5.346 307 63
10	0	5.206 481 19	5.470 813 54	5.704 176 81
4	1	5.450 307 61	5.744 562 65	(6.041 424 31)
11	0	5.499 251 14	5.740 535 18	5.959 256 95
12	0	5.738 689 80	5.976 163 20	(6.190 777 58)

will not be written out here. The condition that the expansion terminate at $\ell_{max} = (|m| + n)$ quantizes the eigenvalue β

$$\beta = (n + |m|)(n + |m| + 3), \tag{7}$$

which means that Ψ_η becomes $(1-\eta^2)^{|m/2|}$ times a polynomial of order n.

In the *isotropic* case $\epsilon = 0$, the operator $\hat{B}$ can be diagonalized in spherical coordinates and its spectrum then found as $\beta = (2n_r + \ell)(2n_r + \ell + 3)$ with radial quantum number n_r and angular quantum number ℓ. Together with the connection of $\hat{G}$ and $\hat{B}$ for isotropic traps this gives the result of [16] for the spectrum in the isotropic case.

The eigenvalue condition for ω^2 takes the form of the characteristic equation of a tridiagonal matrix of dimension $N = 1 + \text{int}[n/2]$, which can be symmetrized by a suitable similarity transformation with a given diagonal matrix. For fixed numbers n, $|m|$ we have N different solutions for ω^2, which we label by our third quantum number $j = 0, \ldots, \text{int}[n/2]$. In the *isotropic* case $\epsilon = 0$ the quantum number j can be expressed as $j = (n - \ell + |m|)/2 = n_r$, as one finds by expressing the isotropic spectrum of [16] in terms of the new quantum numbers n, $j \leq \text{int}[n/2]$.

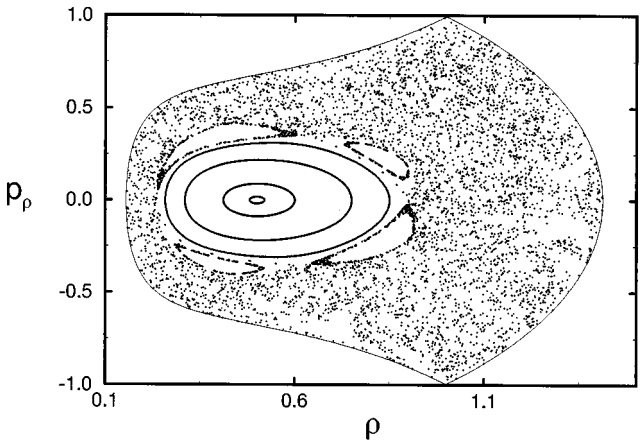

FIG. 1. Poincaré sections of the effective classical dynamics in cylindrical coordinates ρ, z, ϕ, after the elimination of the conserved axial angular momentum L_z and the azimuthal angle ϕ, for energy $E/\mu = 1$. The cut through the three-dimensional energy surface in the four-dimensional phase space (ρ, z, p_ρ, p_z) is taken at $z = 0$ and displayed in the scaled variables ρ, p_ρ measured in units of $\sqrt{2\mu/m\omega_0^2}$ and $\sqrt{2m\mu}$, respectively. The anisotropy is chosen as $\omega_z/\omega_0 = \sqrt{8}$; the angular momentum was fixed as $\omega_0 L_z/E = 0.2$.

Calculating the first levels $\omega(n,j,m)$ we get with $\lambda = (\omega_0/\omega_z)^2$

$$\omega^2(0,0,m) = \omega_0^2|m|, \quad \omega^2(1,0,m) = \omega_z^2 + \omega_0^2|m|$$

$$\omega^2(2,j(=0,1),m) = \omega_z^2\Big(\frac{3}{2} + 2(|m|+1)\lambda - \frac{(-1)^j}{2}[9 - 4(|m|+4)\lambda + 4(|m|+2)^2\lambda^2]^{1/2}\Big)$$

$$\omega^2(3,j(=0,1),m) = \omega_z^2\Big(\frac{7}{2} + 2(|m|+1)\lambda - \frac{(-1)^j}{2}[25 + 4(|m|-4)\lambda + 4(|m|+2)^2\lambda^2]^{1/2}\Big). \tag{8}$$

In the limit $\lambda^{-1} \to 0$, which is relevant for the experiments reported in [5], the mode frequencies for *arbitrary* integer $n \geq 2j$ not too large can be expanded in the small parameter λ^{-1} and are found as

$$\omega^2(n,j,m) = \omega_0^2(|m| + 2j(j + |m| + 1) + O(\lambda^{-1})),$$

$$\omega^2(n,0,0) = \omega_z^2(n(n+3)/4 + O(\lambda^{-1})). \tag{9}$$

Remarkably, the leading-order term for $|m|, j$ both not vanishing is independent of n, i.e., the levels consist in this case of bands of closely spaced levels split only by small frequencies of order ω_z^2/ω_0.

Equation (8) contains, as special cases, the particular solutions previously obtained by Stringari [16] (the $n = 0,1$; $j = 0,1$ modes for all $|m|$ and the two $n = 2$; $j = 0,1$; $m = 0$ modes).

After our calculation had been completed and while this paper was being prepared unpublished results became available [20], in which the mode frequencies (8) were also found by solving the wave equation directly via a polynomial ansatz, which can be shown to be equivalent to ours. This method works because, as we have shown above, no special boundary conditions except regularity need to be imposed on the wave function at the surface of the condensate. However, the deeper *reason* for the solvability of the equation, i.e., the additional conservation law, has not been identified in [20].

Because of the necessity to diagonalize N-dimensional matrices the analytical determination of the ω^2 by the present method for λ neither very small nor very large is possible up to $N = 4$, even though the formulas for $N = 3$ and $N = 4$ are too cumbersome to be of much practical value.

However, it is straightforward to diagonalize the matrices numerically and to prepare a list of the numerical values of the eigenvalues for a given ratio λ; e.g., for the experimental value $\lambda=1/8$. In Table I we give such a list for the 43 lowest-lying eigenvalues with $|m|\leqslant 2$ in units of ω_0 for $\lambda=1/8$, together with their quantum numbers m, n, j. A numerical evaluation of the energy levels of the collective modes as a function of the scaling variable [16] Na/d_0 for $\lambda=1/8$ was recently reported in [9]. Here N is the number of atoms and $d_0=\sqrt{\hbar/2M\omega_0}$ is a measure of the size of the ground state in the trap. The present results apply for $Na/d_0\rightarrow\infty$. Comparing our results with those in [9] we find that the rate of convergence to the asymptotic case $Na/d_0\rightarrow\infty$ depends on the quantum number n and is much slower, e.g., for $n=6$ than for $n=2$. For $m=0$ we find four additional levels (with quantum numbers $n=5,7,8,9$ and $j=0$) below the highest $m=0$ level considered in [9], where the motion of the levels was tracked from the free-particle case to the strongly interacting case. The explanation can be that levels having rather high frequency in the free trap can move down into the considered low-energy regime as the parameter Na/d_0 is increased.

Following the levels one can expect a number of avoided-level crossings due to the nonintegrability of the system at intermediate energies. We demonstrate the chaoticity (see also [22]) of the corresponding classical system whose Hamiltonian

$$H(\boldsymbol{p},\boldsymbol{x})=\sqrt{\epsilon_{HF}^2(\boldsymbol{p},\boldsymbol{x})-K^2(\boldsymbol{x})}\,,$$

$$\epsilon_{HF}(\boldsymbol{p},\boldsymbol{x})=\frac{\boldsymbol{p}^2}{2M}+U(\boldsymbol{x})-\mu+2K(\boldsymbol{x}),$$

$$K(\boldsymbol{x})=[\mu-U(\boldsymbol{x})]\Theta(\mu-U(\boldsymbol{x})) \tag{10}$$

is the Bogoliubov quasiparticle energy in local density approximation [21] by examining a typical Poincaré cross section (Fig. 1). Indeed, we find the typical result for a mixed phase space, a regular island of tori [corresponding to the perturbations around a stable periodic orbit of the effective (ρ,z) dynamics], surrounded by higher-order elliptic islands and a chaotic sea.

In conclusion, we have demonstrated that the quasiparticle dynamics in axially symmetric traps is chaotic for energies comparable to the chemical potential, but for energies much smaller than μ we found a third conserved variable $\hat{B}$ commuting with the wave operator $\hat{G}$ and axial angular momentum $\hat{L}_z$. As a consequence, the wave equation in this regime is proven to be integrable, which explains why some solutions have already been found in the literature. We have separated the wave equations in elliptical coordinates, and reduced the eigenvalue problem to the diagonalization of finite-dimensional matrices, which is solved in terms of three integer quantum numbers m, n, j. As a final remark we mention that we have found even the fully anisotropic case to be integrable [22]. *Two* conserved quantities can be identified in that case that reduce to $\hat{L}_z$ and $\hat{B}$ in the axially symmetric limit.

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