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# Output coupling for an atom laser by state change

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We calculate the spectrum of a beam of atoms output from a single-mode atomic cavity. The output coupling uses an internal-state change to an untrapped state. We present an analytical solution for the output energy spectrum from a broadband coupler of this type. An example of such an output coupler, which we discuss in detail, uses a Raman transition to produce a nontrapped state. [S1050-2947(97)50308-8]

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As a result of recent experiments in which a Bose-Einstein condensate (BEC) has been produced in the laboratory [1–4], there has been considerable interest in coupling the atoms in a BEC out of a trap. This could produce a continuous, coherent, directional beam of atoms—an atom laser beam [5–11]. While initial experiments have succeeded in coupling atoms out of a BEC by changing the internal state of the atoms to a nontrapped state [12,13], there is still much to be understood about the output beam. In this paper we present an analytical solution for the output energy spectrum of atoms in a single trapped mode coupled to free space by a change of internal state. Our analysis is based on the atom field input-output theory presented by Hope [14]. We discuss the dependence of the spectrum on output coupling strength, and relate these findings to the atom laser experiment of Mewes *et al.* [12].

In a BEC a large number of bosonic atoms are cooled into a single energy eigenstate of a trap. This is an important step towards producing a monoenergetic beam of atoms. Nevertheless, we still have the problem of how to coherently couple the atoms out of such a trap in a way that preserves their monoenergetic nature. There are many ways in which atoms can be coupled out of a trap. The simplest method is to turn off the trap [8,13]. The result of rapidly turning off the trap is to reproduce the BEC wave function in free space. In particular, the wave function momentum width is conserved. As a result, the atoms have the corresponding range of energies in free space, and the monoenergetic nature of the original BEC is lost. Fortunately, energy-conserving output coupling is possible. One example is quantum-mechanical tunneling of atoms through the trap walls. This is the atomic analog to the use of partially transparent mirrors on an optical laser. Such a process has been considered in a model of an atom laser proposed by Wiseman and Collett [6]. It would be difficult in practice, however, to use tunneling to produce sufficient fluxes of atoms due to the exponential dependence of the tunneling rate on the trap potential barrier.

Another approach to the output coupling problem would be to change the internal state of the trapped atoms to an untrapped state. Experimentally such a method has been used by implementing radio-frequency pulses to induce spin flips on trapped atoms in a BEC [12,13]. The use of Raman transitions as a method of output coupling has also been suggested [11]. A Raman transition can have an extremely narrow linewidth so that lasers can be tuned so as to only couple atoms from a particular trap mode, due to energy conservation.

We model here an output coupler based on change of state, focusing on the specific case of a Raman output coupler. This uses two lasers tuned to a two-photon resonance to couple atoms between an initial atomic state and a final atomic state. We assume that each of the lasers is far detuned from single photon resonance, giving an effective two-level Hamiltonian. In this Hamiltonian we ignore the energies of higher atomic modes of the trap. Initially these other modes are empty, as we assume all the atoms are condensed in the ground mode. Ignoring these higher-energy modes for a later time is valid for very narrow linewidth Raman lasers that are only on resonance with the ground trap mode. This ensures that higher modes do not become populated by atoms in the output state transferring back into the initial state at later times. In addition, population of other modes is suppressed by Bose enhancement of transitions into the ground mode [11]. We also ignore the effects of atom-atom interactions. The resulting effective Hamiltonian is then of the form

$$H_{\mathrm{eff}}=H_{\mathrm{sys}}+H_{\mathrm{ext}}+H_{\mathrm{int}}, \tag{1}$$

$$H_{\mathrm{sys}}=\hbar\tilde{\omega}_0 a^\dagger a, \tag{2}$$

$$H_{\mathrm{ext}}=\int dk\hbar\tilde{\omega}_k b_k^\dagger b_k, \tag{3}$$

$$H_{\mathrm{int}}=-i\hbar\int dk[\kappa(k,t)b_k a^\dagger-\kappa^*(k,t)b_k^\dagger a], \tag{4}$$

with

$$\tilde{\omega}_0=\omega_1+\omega_0-\frac{\Omega_1^2}{\Delta_1}, \tag{5}$$

$$\tilde{\omega}_k=\omega_2+\frac{\hbar k^2}{2m}-\frac{\Omega_2^2}{\Delta_2}, \tag{6}$$

$$\kappa(k,t)=\Gamma^{1/2}[-ie^{-i(\omega_{2L}-\omega_{1L})t}\psi^*(k-k_{1L}-k_{2L})], \tag{7}$$

$$\Gamma^{1/2}=\frac{\Omega_1\Omega_2}{\Delta_1}. \tag{8}$$

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Here, the single trap mode is described by the creation operator $a^\dagger$ and is coupled by the Raman lasers to a continuous spectrum of external modes described by creation operators $b_k^\dagger$. $\hbar\omega_1$ ($\hbar\omega_2$) is the energy of the trap (output) atomic state, $\hbar\omega_0$ is the ground-state trap energy, $m$ is the mass of the trapped atoms, and $\hbar k_{1L}$ and $\hbar k_{2L}$ are the momenta of the two lasers inducing the Raman transition, with frequencies $\omega_{1L}$ and $\omega_{2L}$, respectively. Thus $\hbar(k_{1L}+k_{2L})$ is the total momentum kick received by atoms making the Raman transition. $\Omega_1$ ($\Omega_2$) is the Rabi frequency of the transition between the trapped (output) state and the excited state that mediates the Raman transition. $\Delta_1$ and $\Delta_2$ are the detunings of the two Raman lasers from the excited state. We have assumed that these are large to obtain an effective two-level Hamiltonian by adiabatically eliminating the upper level. If the lasers are tuned close to the two-photon resonance, $\Delta_1 \approx \Delta_2$. $\psi(k)$ is the momentum-space wave function of the ground mode of the trap. $\Gamma$ is a coupling strength, given here in terms of the Rabi frequencies and single-photon detuning.

The Hamiltonian, Eqs. (2)–(4), describes an arbitrary output coupling through state change from a single-mode system to a continuous spectrum of external modes. In the following we discuss the Raman coupling case, given by Eqs. (5)–(8) for definiteness. The results, however, can be extended to a general output coupler with the coupling strength $\Gamma$ and the energies $\hbar\tilde{\omega}_0$ and $\hbar\tilde{\omega}_k$ suitably defined.

We are interested in the output energy spectrum $\langle b_k^\dagger b_k\rangle$, which is the mean population density of the continuum of free-space momentum eigenstate modes, labeled by the momentum $\hbar k$. We obtain this by solving the Heisenberg equations of motion for the operators $a(t)$ and $b_k(t)$, using our Hamiltonian given in Eqs. (1)–(8). The Heisenberg equations for $a(t)$ are linear Volterra equations of the convolution type, and can be solved using Laplace transforms. These equations have been solved by Hope [14], with the resulting output spectrum, in the case where initially the external modes are empty, being given by

$$\langle b_k^\dagger(t) b_k(t)\rangle = |\kappa(k,t)|^2 \langle a^\dagger(0)a(0)\rangle |M_k(t)|^2, \qquad (9)$$

with

$$M_k(t) = \mathcal{L}^{-1}\left\{\frac{1}{[s+\mathcal{L}(f')(s)](s+i\delta_k)}\right\}(t), \qquad (10)$$

$$f'(t) = \int dk |\kappa(k,t)|^2 e^{-i\delta_k t}, \qquad (11)$$

$$\delta_k = \tilde{\omega}_k - \tilde{\omega}_0 - \omega_{1L} + \omega_{2L} = \frac{\hbar k^2}{2m} - \omega_0 . \qquad (12)$$

The final equality holds for the case when the lasers are tuned to the two-photon resonance in free space, which we assume here. $\mathcal{L}$ and $\mathcal{L}^{-1}$ are the Laplace transform and inverse Laplace transform, respectively. The $|M_k(t)|^2$ term in Eq. (9) determines the time dependence of the spectrum evolution and is related to the Fourier transform of the system two-time correlation function $\langle a^\dagger(t')a(t'')\rangle$ by

$$|M_k(t)|^2 = \frac{\int_0^t dt' \int_0^t dt'' \langle a^\dagger(t')a(t'')\rangle e^{-i\omega_k(t'-t'')}}{\langle a^\dagger(0)a(0)\rangle}. \qquad (13)$$

This Fourier transform can be shown to be equivalent to Eq. (10) using the expressions for $a(t)$ given by Hope [14]. Unfortunately, the inverse Laplace transform required to obtain $M_k(t)$, and hence the output spectrum, cannot be obtained analytically for most physical situations. Moreover, numerical solutions are unstable and can only be obtained in the limits of short time or small coupling strength.

Our central result is an analytic solution for the spectrum in the limit of broadband coupling. For simplicity, we consider the case where the total momentum kick from the Raman lasers is very small. That is, we assume $k_{1L} \approx -k_{2L}$. This is analogous to the output coupling experiments in which the atoms receive a negligible momentum kick in changing state [12,13]. We also assume that the coupling function $\kappa(k,t)$ is broad. The shape of $\kappa(k,t)$ is given in terms of the ground-state momentum wave function of the trap, $\psi(k)$ in Eq. (7). We consider here a harmonic trap, with a Gaussian ground state of standard deviation $\sigma_k$ in wave-number space, given by

$$\psi(k) = (2\pi\sigma_k^2)^{-1/4} \exp[-k^2/(4\sigma_k^2)]. \qquad (14)$$

Substituting Eq. (14) and Eq. (7) into Eq. (11) and taking the Laplace transform of $f'(t)$, we obtain

$$\mathcal{L}(f')(s) = \int_{-\infty}^{\infty} dk \frac{|\kappa(k,t)|^2}{s+i\delta_k} \qquad (15)$$

$$= \Gamma c \frac{\sqrt{i}}{\sqrt{s-i\omega_0}} G(r), \qquad (16)$$

where

$$G(r) = \exp[r^2](1 - \mathrm{Erf}[r]), \qquad (17)$$

$$r = \sqrt{-im(s-i\omega_0)/(\hbar\sigma_k^2)}, \qquad (18)$$

$$c = -i\left(\frac{m\pi}{\hbar\sigma_k^2}\right)^{1/2}, \qquad (19)$$

and Erf is the error function. We must simplify this expression for $\mathcal{L}(f')(s)$ in order to evaluate Eq. (10). We first note that we can approximate $G(r) \approx 1$ if $|r| \ll 1$. Noting that the abscissa of convergence [16] for the inverse Laplace transform, Eq. (10), is zero, we can set the real part of $s$ to any small, real positive number in the inverse Laplace transform. We also assume here that $\sigma_k$ is sufficiently broad that $\omega_0 \ll \hbar\sigma_k^2/m$. Typically $\omega_0$ is of the order of hundreds of hertz for atomic traps, with the atomic mass of order $10^{-26}$ kg, thus this inequality will hold for broadband coupling with $\sigma_k \gg 2\times10^5$ m$^{-1}$. The approximation $|r| \ll 1$ then holds in the regime where $\mathrm{Im}(s) \ll \hbar\sigma_k^2/m$. Using the approximation $G(r) \approx 1$ to calculate $M_k(t)$ is, thus, equivalent to smoothing over high ($>\hbar\sigma_k^2/m$) frequency components in the time dependence of $M_k(t)$. As we increase the width of the coupling

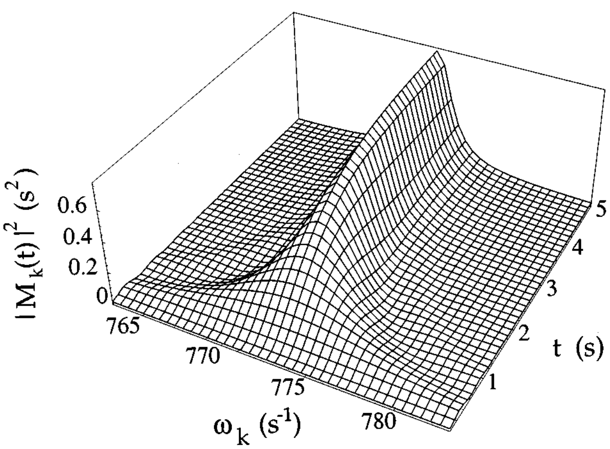


FIG. 1. Plot of $|M_k(t)|^2$ as a function of $\omega_k$ and time for $t=0$ s to $t=5$ s, and $\omega_k$ ranging from 762 $\mathrm{s}^{-1}$ to 783 $\mathrm{s}^{-1}$ about the single-mode trap frequency, $\omega_0 \approx 772$ $\mathrm{s}^{-1}$. $\Gamma = 1.8\times 10^3$ $\mathrm{s}^{-2}$.

in momentum space, given by $\sigma_k$, our solution for $M_k(t)$ becomes valid for increasingly high frequencies, Im$(s)$. For an infinitely broad coupling, the approximation $G(r)=1$ becomes exact and our expression is equivalent to the form of the general broadband coupling discussed by Hope [14]. We substitute Eq. (16) for $\mathcal{L}(f')(s)$ with $G(r)=1$ into Eq. (10). We then use the shift theorem for Laplace transforms [15], to write $M_k(t)$ as

$$M_k(t) = e^{i\omega_0 t}\mathcal{L}^{-1}\{h(s^{1/2})\}(t), \tag{20}$$

where

$$h(p) = \frac{p}{[p^2 + i(\omega_0 + \delta_k)](p^3 + i\omega_0 p + \Gamma c\sqrt{i})}. \tag{21}$$

The inverse Laplace transform $\mathcal{L}^{-1}\{h(s^{1/2})\}$ can be written in terms of an integral involving $\mathcal{L}^{-1}\{h(p)\}$ using a standard theorem [16]. In this integral, the inverse Laplace transform of $h(p)$ is standard, after factorizing the cubic in the denominator as $p^3 + i\omega_0 p + \Gamma c\sqrt{i} = (p-\alpha)(p-\beta)(p-\gamma)$. Finally, we obtain

$$\begin{aligned} M_k(t) = {} & \frac{e^{-i\Delta_k t}}{\omega_k \Delta_k^2 - \Gamma^2 c^2}\left[\frac{-e^{i\omega_k t} i\sqrt{i}\Gamma c}{\sqrt{\pi t}} + i\omega_k \Delta_k \right. \\ & \left. + \frac{1}{2}\sqrt{\frac{\pi}{t}} i\sqrt{i}\Gamma c L_{1/2}^{-1/2}(i\omega_k t)\right] + W(\alpha,\beta,\gamma) \\ & + W(\beta,\alpha,\gamma) + W(\gamma,\beta,\alpha), \end{aligned} \tag{22}$$

where

$$W(x,y,z) = \frac{x^2 \exp[(x^2 + i\omega_0)t]}{(y-x)(z-x)(x^2 + i\omega_k)}[1 + \mathrm{Erf}(x\sqrt{t})],$$

and we have defined $\omega_k = \hbar k^2/(2m)$ and $\Delta_k = \omega_k - \omega_0$. The function $L_{1/2}^{-1/2}(x)$ is a Laguerre polynomial.

Figure 1 shows the behavior of $|M_k(t)|^2$ as a function of $\omega_k$ and time after we turn on the output coupling interaction. Initially $|M_k(t)|^2$ is small, and for short enough times, arbitrarily broad in $k$ space. Initially $|M_k(t)|^2$ agrees with the perturbative solutions presented by Hope [14]. For longer times, we can see that the spectrum reaches a stable shape. For very large values of the coupling strength, the long-time limit becomes very broad in $k$ space. As a result, the shape of the output spectrum, as given by Eq. (9), simply reflects the momentum distribution of the cavity wave function $\psi(k)$. As a result there is no narrowing of linewidth in momentum space. The recent experiment of Mewes *et al.* [12] is an example of an output coupling with an extremely large coupling strength. In these experiments a short, 5 $\mu$s rf pulse was used to couple atoms out of a BEC, making a pulsed atom laser.

We consider here a continuous coupler, turned on at time $t=0$, and examine the resulting long-time spectrum in the external modes described by $b_k^\dagger$. We observe in Fig. 1 that for longer times, $|M_k(t)|^2$ narrows into a $\mathrm{sinc}^2$ function centered about the trap ground-state frequency $\omega_0$. Eventually $|M_k(t)|^2$ reaches a stationary state with a Lorentzian-like profile, as shown in Fig. 1. This long-time behavior is given by

$$\lim_{t\to\infty} M_k(t) = \frac{i\sqrt{\omega_k}e^{-i\Delta_k t}}{\sqrt{\omega_k}\Delta_k - \Gamma c} + W(\gamma,\beta,\alpha), \tag{23}$$

where $\alpha$, $\beta$, and $\gamma$ are the particular solutions to the cubic discussed above, given by the expressions

$$\alpha = e^{-i(5\pi/12)}(2^{1/3}\omega_0/\xi^{1/3} + e^{i(\pi/3)}\xi^{1/3}/54^{1/3}) = i\beta^*, \tag{24}$$

$$\gamma = e^{i(\pi/4)}(2^{1/3}\omega_0/\xi^{1/3} - (4\xi)^{1/3}/6), \tag{25}$$

$$\xi = -27i\Gamma c + [-(27\Gamma c)^2 + 108\omega_0^3]^{1/2}. \tag{26}$$

The long-time expression for $M_k(t)$, Eq. (23), contains two terms. The first of these terms dominates in the case of small $\Gamma$, while the second dominates for very large $\Gamma$. As a result, the long-time spectrum has two distinct behaviors, depending on the strength of the coupling. We consider the case of slow coupling (small $\Gamma$) initially. In this case, the long-time expression for $M_k(t)$ is dominated by the first term in Eq. (23) above, and the resulting long-time spectrum is given by

$$\langle b_k^\dagger b_k\rangle = \Gamma|\psi(k)|^2 \frac{1}{(\Delta_k^2 + |\Gamma c|^2/\omega_k)}. \tag{27}$$

Plots of the long-time spectrum, Eq. (27), as a function of $\omega_k$ are presented in Fig. 2 for various coupling strengths. Figure 2 shows that for increasing coupling strength the linewidth of the long-time spectrum increases. The values for $\Gamma$ chosen lie about $\Gamma = 4\times 10^4$ $\mathrm{s}^{-2}$, which corresponds approximately to values of Raman laser Rabi frequencies, $\Omega_1 \approx 2\pi\times 50$ kHz and $\Omega_2 \approx 2\pi\times 1.6$ MHz and detuning, $\Delta_1 \approx 2\pi\times 2.5$ GHz similar to values presented in [11]. However, much smaller or larger coupling strengths can be achieved by suitably adjusting the intensities of the lasers and their detunings. The figures assume a trap with ground-state frequency $\omega_0 = 2\pi\times 123$ $\mathrm{s}^{-1}$, typical of magnetic traps for ultracold atoms [17]. A ground-state Gaussian with width $\sigma_k \approx 10^6$ $\mathrm{m}^{-1}$ has been assumed, which corresponds to a

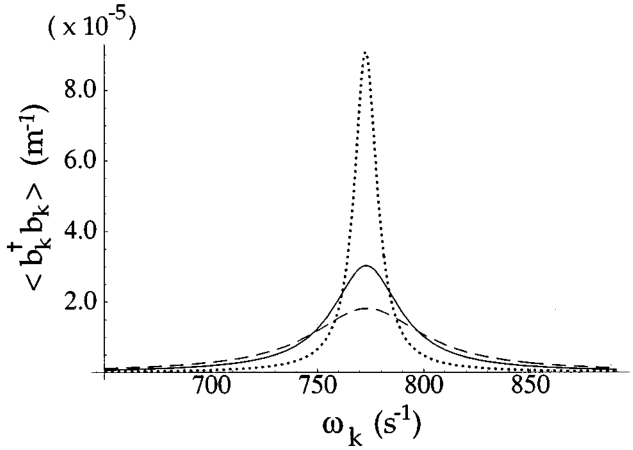


FIG. 2. Plots of the long-time behavior of $\langle b_k^\dagger b_k \rangle$ as a function of $\omega_k$ for various coupling strengths, $\Gamma = 10^4\ \mathrm{s}^{-2}$ (dotted line), $\Gamma = 3\times 10^4\ \mathrm{s}^{-2}$ (solid line), and $\Gamma = 5\times 10^4\ \mathrm{s}^{-2}$ (dashed line).

position-space wave function of size of the order of 2 $\mu$m. This value of $\sigma_k$ corresponds to a width in $\omega_k$ space of $\sigma_{\omega_k} \approx 10^4\ \mathrm{s}^{-1}$.

For each of the graphs shown in Fig. 2, the Lorentzian-like spectrum is centered about $\omega_0$, the ground-state frequency of the single-mode trap, with the width of the spectrum dependent on the strength of the coupling, as mentioned above. In all cases, however, the linewidth is much less than that which would be obtained if the trap were rapidly turned off; that is, $\sigma_{\omega_k} \approx 10^4\ \mathrm{s}^{-1}$. We see from Eq. (27) that the distribution is not exactly Lorentzian, due to the presence of $\omega_k$ in the second part of the denominator. However, for large $\omega_0$ the spectrum is well approximated by a Lorentzian distribution of width $|\Gamma c|/\sqrt{\omega_0}$.

We have already noted that for large coupling rates the width of the long-time limit of $|M_k|^2$, and hence of the long-time spectrum, is increased. When $\Gamma$ is very large, $|\Gamma c|/\sqrt{\omega_0} \gg \sigma_{\omega_k}$, the width of $M_k(t)$ becomes large compared with $\kappa(k,t)$, and the spectrum becomes dominated by the cavity momentum spread $\psi(k)$. As a result, for sufficiently fast coupling (large $\Gamma$) the output spectrum changes significantly from the Lorentzian shape considered above, and instead reflects the momentum spread of the cavity. For very large $\Gamma$ the spectrum is centred about zero, and falls away exponentially in $\omega_k$ space, as required for a Gaussian distribution in momentum space given by $\psi(k)$.

In summary, we have shown how the long-time spectrum from an output coupler based on state change depends on the strength of the output coupling. For very strong coupling, the output spectrum is given by the cavity spectrum, and is very broad in momentum space. As the strength of the coupling is reduced, however, the long-time linewidth is correspondingly reduced. For small coupling strengths the linewidth is effectively Lorentzian, centered about the energy of the cavity with a linewidth proportional to the coupling strength $\Gamma$.

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