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Relative phase for a quantum field interacting with a two-level system

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Several quantum descriptions of the relative phase between an atomic dipole and a one-mode electromagnetic field are examined. Positive-operator measures for this variable are derived from dipole and field-phase formalisms. They are analyzed and compared with an operator defined by a polar decomposition. Some examples of time evolution are discussed. [S1050-2947(97)09407-9]

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I. INTRODUCTION

Phase plays an outstanding role in solving and understanding a great variety of classical problems, especially those concerning optical processes. Perhaps one of the most relevant examples is that of the atom-field interaction, in which the atom is coupled to a single electromagnetic field through the atomic dipole. Many features of their coupled dynamics can be fruitfully accounted for by resorting to phase relations.

In quantum terms, this role has been played by the field quadratures [1,2] which, in many respects, properly account for phase relations: These are the variables which effectively couple the field to the atom; they are free from the difficulties that phase encounters in the quantum domain.

However, in recent years much effort has been devoted to the problem of a suitable description of phase in quantum terms (for recent reviews see Refs. [3–7]) and many difficulties in this context have been overcome. Most of the effort has focused on the absolute phase of a one-mode field. Nevertheless, the phase difference is more operationally meaningful than the absolute phase. In our context, it is the relative phase between the atomic dipole and the field which is of importance. The quantum description of this variable, and its properties, is the subject of this work.

In principle, it seems that, concerning the dipole phase, the whole atomic space (i.e., all the energy levels) should be involved. In the semiclassical theory of radiation it is common to work with a two-part assumption: The field is quasimonochromatic, and its frequency coincides almost exactly with one of the transition frequencies of the atom. The two-level atom is the natural consequence of these hypotheses and allows us to describe the interaction between matter and radiation in a simple and analytical way, with the hope that this knowledge can be generalized to more realistic situations [1]. Therefore, in what refers to the coupling of the atom to a single-mode field, we can replace the whole atomic system by an effective two-level system which should account for the relevant details of the interaction. It is the atomic-dipole phase within this two-level approximation which concerns us in this work.

To study the relative phase two different procedures can be followed. We can focus directly on the phase difference and derive a suitable operator for this variable without making any previous assumption about how absolute phases should be described. It may be thought that this reasoning cannot lead to any new conclusion with the phase difference inheriting the same difficulties and the same solutions as the absolute phase. However, the actual situation appears to be different and it has been shown that the quantum translation of relative phases encounters fewer difficulties [8,9]. This procedure has been carried out previously for the dipole-field relative phase [10].

Besides, we have the natural possibility of describing the relative phase in terms of the absolute phases. This procedure leads to results other than a direct definition. Nevertheless, some points of coincidence can be found. In this work we follow this procedure for the atom-field relative phase.

For such a derivation suitable formalisms for the field phase are at hand and some of them are recalled in Sec. II. The atomic-dipole phase has received less attention. Several possibilities for describing it are examined and compared in Sec. III. They are applied to the study of the relative phase in Sec. IV, where the results are compared with the operator provided for by a direct definition. The time evolution of all these approaches within the Jaynes-Cummings model for the atom-field interaction is studied in Sec. V.

II. PHASE FOR A ONE-MODE ELECTROMAGNETIC FIELD

The first approach to the problem of the phase of a one-mode quantum field is due to Dirac [11], who proposed that the exponential of the phase E_θ should emerge from the polar decomposition [12]

$$a = E_\theta \sqrt{a^\dagger a}, \tag{2.1}$$

where a is the complex amplitude for the field. It is known that there are no unitary solutions for E_θ, and so Eq. (2.1) does not define a proper phase operator. Instead, it has the nonunitary solution introduced by Susskind and Glogower [13],

$$E_{\theta,\mathrm{SG}} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|, \tag{2.2}$$

where $|n\rangle$ are the number states. The eigenstates of $E_{\theta,\mathrm{SG}}$ with unit-modulus eigenvalue $E_{\theta,\mathrm{SG}}|\theta\rangle = e^{i\theta}|\theta\rangle$ are known as Susskind-Glogower phase states,

$$|\theta\rangle=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}e^{in\theta}|n\rangle, \tag{2.3}$$

which prove to be of general interest.

The failure of the polar decomposition (2.1) has allowed for the introduction of alternative approaches to this problem. Virtually all of them can be formulated in terms of positive-operator measures $\Delta_j(\theta)$ that define a probability distribution $P_j(\theta)$ as

$$P_j(\theta)=\mathrm{tr}[\rho_{\text{field}}\Delta_j(\theta)], \tag{2.4}$$

for any field state ρ_{field}. The reality, positiveness, and normalization of $P_j(\theta)$ impose

$$\Delta_j(\theta)^{\dagger}=\Delta_j(\theta),\quad \Delta_j(\theta)\geqslant 0,\quad \int d\theta\Delta_j(\theta)=I, \tag{2.5}$$

but, in general, $\Delta_j(\theta)$ are not orthogonal projectors. The mean value of any function $f(\theta)$ is

$$\langle f(\theta)\rangle_j=\int d\theta f(\theta)P_j(\theta). \tag{2.6}$$

If the positive-operator measure $\Delta_j(\theta)$ is canonically conjugate to the number operator (in the sense of a weak Weyl relation), it should verify the shifting property [14]

$$e^{i\theta' a^{\dagger}a}\Delta_j(\theta)e^{-i\theta' a^{\dagger}a}=\Delta_j(\theta+\theta'). \tag{2.7}$$

This condition leads to the following general form for $\Delta_j(\theta)$ in the number basis:

$$\Delta_j(\theta)=\frac{1}{2\pi}\sum_{n,n'=0}^{\infty}G_j(n,n')e^{i(n-n')\theta}|n\rangle\langle n'|, \tag{2.8}$$

where the coefficients $G_j(n,n')$ must ensure the statistical conditions (2.5).

Although these positive-operator measures avoid the definition of a phase operator, the evaluation of mean values, for instance, of phase exponentials $\langle e^{ik\theta}\rangle_j$, becomes the mean values of the sequence of operators [15]

$$\int d\theta e^{ik\theta}P_j(\theta)=\mathrm{tr}[\rho_{\text{field}}E_{\theta,j}^{(k)}], \tag{2.9}$$

where

$$E_{\theta,j}^{(k)}=E_{\theta,j}^{(-k)\dagger}=\int d\theta e^{ik\theta}\Delta_j(\theta). \tag{2.10}$$

These operators are not unitary,

$$E_{\theta,j}^{(k)}E_{\theta,j}^{(k')}\neq E_{\theta,j}^{(k+k')},\quad E_{\theta,j}^{(k)}\neq E_{\theta,j}^{k}, \tag{2.11}$$

where $E_{\theta,j}=E_{\theta,j}^{(1)}$.

Most of the quantum descriptions of phase fall within this kind of positive-operator-measures formalism. Some of them have focused on the concept of phase as an observable canonically conjugate to the photon number. This is the case of the Pegg-Barnett formalism which has been extensively studied in recent years [16]. This approach and some others embodying this concept have been shown to be equivalent, leading to the positive-operator measure $\Delta_{\text{SG}}(\theta)$ induced by the Susskind-Glogower phase states [17,18]

$$\Delta_{\text{SG}}(\theta)=|\theta\rangle\langle\theta|, \tag{2.12}$$

which is in the form of Eq. (2.8) with

$$G_{\text{SG}}(n,n')=1. \tag{2.13}$$

Another widely used conception of phase is based on examining quasiprobability distributions in phase space. Among them, one of the most interesting and studied comes from the Q function [19–22] $Q(\alpha)=\langle\alpha|\rho_{\text{field}}|\alpha\rangle/\pi$, where $|\alpha\rangle$ is a coherent state. A positive-operator measure for the field phase $\Delta_Q(\theta)$ can be defined in terms of the radial integration of $|\alpha\rangle\langle\alpha|$,

$$\Delta_Q(\theta)=\frac{1}{\pi}\int_0^{\infty}dr\quad r|\alpha=re^{i\theta}\rangle\langle\alpha=re^{i\theta}|, \tag{2.14}$$

which is also in the form of Eq. (2.8) with

$$G_Q(n,n')=\frac{\Gamma((n+n')/2+1)}{\sqrt{n!n'!}}. \tag{2.15}$$

This less abstract and more pictorial method has the advantage that it corresponds to a realistic scheme for phase measurement, though a generalized or noisy one. This noisy character is reflected by the fact that the integration of $Q(\alpha)$ over the real or imaginary parts of α does not give the probability for the corresponding quadratures of the field [23], nor does the integration over θ give the probability distribution for the action $a^{\dagger}a$ [24]. On the other hand, definitions based on generalized measurements give different results for different quantum implementations of measurement schemes which are classically equivalent, and so they do not provide a unique quantum description of phase [25,26].

In addition to these two formalisms there are other approaches to the problem. We have singled out these two descriptions because we think they are representative enough of how phase can be described in the quantum domain.

III. PHASE FOR A TWO-LEVEL ATOM

In this section we focus on the quantum translation of the atomic-dipole phase for a two-level atom.

Denoting by $|g\rangle$ and $|e\rangle$ the ground and excited energy levels of the isolated atom, the effective component of the atomic dipole which couples the atom to a one-mode field can be written as

$$d=d_{eg}(|e\rangle\langle g|+|g\rangle\langle e|)=d_{eg}(S_{+}+S_{-})=2d_{eg}S_x, \tag{3.1}$$

where $S_{\pm}$ are the Pauli spin-flip operators of a 1/2 angular momentum. It has been assumed that $\langle g|d|g\rangle=\langle e|d|e\rangle=0$ and d_{eg} can be taken as real.

Considering the atom in the most general pure state, $|\psi\rangle_{\text{atom}}=c_g|g\rangle+c_e e^{i\varphi}|e\rangle$, where c_g and c_e are real, the evolution of the mean value of d under the Hamiltonian of the isolated atom (in units $\hbar=1$),

$$H_{\rm atom}=\frac{\omega}{2}(|e\rangle\langle e|-|g\rangle\langle g|)=\omega S_z, \tag{3.2}$$

is

$$\langle d\rangle=2c_g c_e d_{eg}\cos(\omega t-\varphi). \tag{3.3}$$

This leads us to identify φ as the phase of the atomic dipole in the state $|\psi\rangle_{\rm atom}$.

The subject that concerns us in this section is the proper description of this dipole phase φ as a quantum variable instead of as a state parameter. At this stage two alternative routes can be followed. On the one hand and according to the usual procedure of ascribing an operator to each variable, it can be asked which operator could serve as a quantum translation of φ. On the other hand and motivated by the field phase problem, we can consider that the optimum description of the dipole phase should be provided by a positive-operator measure. These two procedures are briefly examined in the following.

The mean value (3.3) of the atomic dipole is proportional to the real part of

$$\langle S_-\rangle=c_g c_e e^{-i\omega t}e^{i\varphi}. \tag{3.4}$$

Thus, it seems appropriate to define the exponential of the atomic phase E_φ as the unitary solution of the polar decomposition

$$S_-=\sqrt{S_-S_+}E_\varphi=E_\varphi\sqrt{S_+S_-}, \tag{3.5}$$

which is the operator counterpart of Eq. (3.4). After these equations are solved for E_φ, an operator Φ_φ can be defined by $E_\varphi=e^{i\Phi_\varphi}$. This is a particular case of a general definition of the SU(2) phase [27–29]. Here we will just briefly recall the main properties of this definition, particularizing it to a two-level system.

Contrary to Eq. (2.1), in this case there are unitary solutions of the form

$$E_\varphi=|g\rangle\langle e|+e^{i2\varphi_0}|e\rangle\langle g|, \tag{3.6}$$

where φ_0 is arbitrary. The last term corresponds to a matrix element undefined by Eqs. (3.5) and appears due to the unitarity requirement. Although the main features of this operator are independent of φ_0, its eigenvectors and eigenvalues depend on φ_0. For the sake of concreteness we can make a definite choice by imposing further conditions. For instance, according to Eq. (3.3), the complex conjugation of the wave function in the energy basis should reverse the sign of Φ_φ [10]. This leads to $e^{i2\varphi_0}=-1$, and the exponential of the dipole phase becomes

$$E_\varphi=|g\rangle\langle e|-|e\rangle\langle g|, \tag{3.7}$$

with eigenvectors

$$|\varphi_\pm\rangle=\frac{1}{\sqrt{2}}(|g\rangle\pm i|e\rangle) \tag{3.8}$$

and $E_\varphi|\varphi_\pm\rangle=\pm i|\varphi_\pm\rangle$. To any function $f(\varphi)$ we can associate the operator

$$f(\Phi_\varphi)=\sum_\pm |\varphi_\pm\rangle f(\varphi_\pm)\langle\varphi_\pm| \tag{3.9}$$

and the mean value

$$\langle f(\Phi_\varphi)\rangle=\sum_\pm f(\varphi_\pm)P(\varphi_\pm), \tag{3.10}$$

where $P(\varphi_\pm)$ is the probability distribution,

$$P(\varphi_\pm)={\rm tr}(\rho_{\rm atom}|\varphi_\pm\rangle\langle\varphi_\pm|). \tag{3.11}$$

With this choice for φ_0, E_φ is proportional to S_y. Also, because $E_\varphi^\dagger=-E_\varphi$, we have $\cos\Phi_\varphi=0$. For other values of φ_0 we get similar equations. This is a rather pathological behavior caused by the small dimension of the system, as such strong relations no longer hold for dimensions other than two. In other words, a two-dimensional Hilbert space is not large enough to distinctly accommodate all different variables. This means that this behavior is not exclusive to this approach, and we will see later that equivalent unexpected features also appear when other approaches are used.

Another striking consequence of an operator description is that the dipole phase can take only two values $\pm\pi/2$, due as well to the dimension of the atomic-state space.

One may think it preferable to describe the dipole phase by a positive-operator measure $\Delta_j(\varphi)$, taking continuous values in a 2π interval, even though this cannot lead to an operator description. This is the possibility we examine in the following.

In this case the shifting property associated with the dipole phase is

$$e^{i\varphi' S_z}\Delta_j(\varphi)e^{-i\varphi' S_z}=\Delta_j(\varphi+\varphi'). \tag{3.12}$$

The most general $\Delta_j(\varphi)$ fulfilling this property and the statistical conditions (2.5) is of the form

$$\Delta_j(\varphi)=\frac{1}{2\pi}(I+\gamma_j e^{i\varphi}|e\rangle\langle g|+\gamma_j^* e^{-i\varphi}|g\rangle\langle e|), \tag{3.13}$$

with $|\gamma_j|\leq 1$ and we exclude the trivial case $\gamma_j=0$.

Before considering particular examples, let us examine some properties that can be derived from the general form (3.13). Here again we have that mean values of phase functions, in particular $\langle e^{ik\varphi}\rangle_j$, become the mean value of operators $\langle e^{ik\varphi}\rangle_j=\langle E_{\varphi,j}^{(k)}\rangle$ with

$$E_{\varphi,j}=E_{\varphi,j}^{(1)}=\int d\varphi e^{i\varphi}\Delta_j(\varphi)=\gamma_j^*|g\rangle\langle e|=\gamma_j^* S_-, \tag{3.14}$$

and $\langle e^{ik\varphi}\rangle_j=0$ if $|k|>1$. These relations show that the information $P_j(\varphi)$ conveys goes beyond what would strictly be the atomic phase. For example, for every $\rho_{\rm atom}$ we have that $\langle S_x\rangle$ and $\langle S_y\rangle$ can be expressed in the form

$$\langle S_x\rangle=\frac{1}{|\gamma_j|}\int d\varphi\ \cos(\varphi+\delta_j)P_j(\varphi), \tag{3.15}$$

$$\langle S_y \rangle = -\frac{1}{|\gamma_j|}\int d\varphi \, \sin(\varphi + \delta_j) P_j(\varphi),$$

where $\delta_j = \arg \gamma_j$. Since $S_x^2 = S_y^2 = 1/4$, $P_j(\varphi)$ contains the complete statistics of S_x and S_y. In particular, it contains the whole statistics of the dipole operator and not only of its phase.

The most general $P_j(\varphi)$ is of the form

$$P_j(\varphi) = \frac{1}{2\pi}(1 + c_j e^{i\varphi} + c_j^* e^{-i\varphi}), \tag{3.16}$$

with $c_j = \langle g|\rho_{\text{atom}}|e\rangle \gamma_j$. This means that $P_j(\varphi)$ is always broader than $P_k(\varphi)$ if $|\gamma_j| < |\gamma_k|$. This can be expressed quantitatively by using the dispersion [30]

$$D_j^2 = 1 - \left| \int d\varphi e^{i\varphi} P_j(\varphi) \right|^2 = 1 - |\langle g|\rho_{\text{atom}}|e\rangle|^2 \, |\gamma_j|^2, \tag{3.17}$$

as a measure of the phase uncertainty. Clearly, $D_j \geqslant D_k$ when $|\gamma_j| \leqslant |\gamma_k|$.

From Eq. (3.13) or (3.16) we see that any $\Delta_j(\varphi)$ can be expressed as a linear combination of any other $\Delta_k(\varphi)$ in the form

$$\Delta_j(\varphi) = \frac{1}{2\pi}\int d\varphi' \left[1 + \frac{\gamma_j}{\gamma_k} e^{i(\varphi - \varphi')} + \frac{\gamma_j^*}{\gamma_k^*} e^{-i(\varphi - \varphi')} \right] \Delta_k(\varphi'), \tag{3.18}$$

for any γ_j, γ_k. The same relation holds between $P_j(\varphi)$ and $P_k(\varphi)$, and so if one of them is known, the other one can be obtained. This means that all of them contain the same information about the atom state ρ_{atom}.

A relevant feature of this approach, based on positive-operator measures, is that it provides an atomic-phase description where any value for φ is allowed. Some remarks can be made concerning this dependence. This continuous range of variation is not completely effective in the sense that the values $P_j(\varphi)$ at every point φ cannot be independent, and we can find relations between them irrespective of the atomic state. In other words, all $\Delta_j(\varphi)$ cannot be linearly independent because the atomic Hilbert space $\mathcal{H}_{\text{atom}}$ is two dimensional and the algebra of operators acting on $\mathcal{H}_{\text{atom}}$ is four dimensional. The most general $P_j(\varphi)$ depends only on the complex parameter c_j. This c_j can be determined by the value of $P_j(\varphi)$ at two φ points not differing by π. Nevertheless, more manageable expressions emerge if we use three points instead of two, such as $\varphi_r = 2\pi r/3$, $r = -1,0,1$. We have

$$c_j = \frac{2\pi}{3}\sum_{r=-1}^{1} P_j(\varphi_r) e^{-i\varphi_r}, \tag{3.19}$$

which allows us to express $P_j(\varphi)$ for any ρ_{atom} as

$$P_j(\varphi) = \frac{1}{3}\sum_{k,r=-1}^{1} P_j(\varphi_r) e^{ik(\varphi - \varphi_r)}, \tag{3.20}$$

and so the knowledge of the three values $P_j(\varphi_r)$ gives $P_j(\varphi)$ at any other point φ. Similarly,

$$\Delta_j(\varphi) = \frac{1}{3}\sum_{k,r=-1}^{1} \Delta_j(\varphi_r) e^{ik(\varphi - \varphi_r)}. \tag{3.21}$$

This effective discreteness allows us to compute the mean values of any function $f(\varphi)$ in a way very similar to Eq. (3.10),

$$\langle f(\varphi) \rangle_j = \frac{2\pi}{3}\sum_{r=-1}^{1} \tilde{f}(\varphi_r) P_j(\varphi_r), \tag{3.22}$$

where $\tilde{f}$ is related to f by

$$\int d\varphi e^{ik\varphi} \tilde{f}(\varphi) = \int d\varphi e^{ik\varphi} f(\varphi), \quad k = 0, \pm 1,$$

$$\int d\varphi e^{ik\varphi} \tilde{f}(\varphi) = 0, \quad |k| > 1, \tag{3.23}$$

and so

$$\int d\varphi P_j(\varphi) \tilde{f}(\varphi) = \int d\varphi P_j(\varphi) f(\varphi), \tag{3.24}$$

for any $P_j(\varphi)$. Discreteness then is also at the heart of these formalisms.

Finally, we will consider two particular examples of a dipole-phase description which are motivated by the field-phase problem. We can begin with a finite-dimensional translation of Eqs. (2.3) and (2.12),

$$\Delta_{\text{SG}}(\varphi) = |\varphi\rangle\langle\varphi|, \quad |\varphi\rangle = \frac{1}{\sqrt{2\pi}}(|g\rangle + e^{i\varphi}|e\rangle), \tag{3.25}$$

which is in the form of Eq. (3.13) with

$$\gamma_{\text{SG}} = 1. \tag{3.26}$$

The definition of this positive-operator measure seems reasonable in the sense that the operator E_φ corresponds to a selection of an orthogonal basis from the set $|\varphi\rangle$. This positive-operator measure does not privilege any $|\varphi\rangle$ and all of them play the same role.

As another example which parallels Eq. (2.14), we can consider the SU(2) coherent states for a 1/2 angular momentum [31],

$$|\vartheta, \varphi\rangle = \sin\left(\frac{\vartheta}{2}\right)|g\rangle + \cos\left(\frac{\vartheta}{2}\right) e^{i\varphi}|e\rangle, \tag{3.27}$$

and the SU(2) Q function they define,

$$Q(\vartheta, \varphi) = \frac{1}{2\pi}\text{tr}(\rho_{\text{atom}}|\vartheta, \varphi\rangle\langle\vartheta, \varphi|), \tag{3.28}$$

which can be regarded as a probability distribution in the atomic-phase space. This allows us to define a positive-operator measure for the dipole phase φ as

$$\Delta_Q(\varphi) = \frac{1}{2\pi}\int_0^\pi d\vartheta \, \sin\vartheta |\vartheta, \varphi\rangle\langle\vartheta, \varphi|, \tag{3.29}$$

which corresponds to

$$\gamma_Q = \frac{\pi}{4}. \tag{3.30}$$

In this section we focused on phase descriptions for a 1/2 angular momentum because of our interest in its application to a two-level atom. Nevertheless, the generalization of all these definitions to an arbitrary angular momentum is straightforward.

IV. RELATIVE PHASE BETWEEN A TWO-LEVEL ATOM AND A SINGLE-MODE FIELD

In order to describe in quantum terms the relative phase between the atomic dipole and the field two main routes can be followed. We can start from previous descriptions of the field and dipole phases and manage them until we get the probability distribution for their difference $\phi = \varphi - \theta$. Alternatively, we can focus directly on the relative-phase variable, trying to define the corresponding operator without any previous assumption about either the field- or dipole-phase descriptions.

These two routes will lead to different results. Mainly, the second one gives an operator, while the first one leads naturally to a positive-operator measure. However, we will see that, although they look very different, their structure is quite similar. This similarity can be ascribed to those particular features which appear when dealing with the difference of two periodic variables.

For an easier comparison we briefly recall the definition of an operator for the relative phase [10]. The procedure is almost the same as that followed for the dipole phase. A unitary exponential of the relative phase E_ϕ should emerge from the equations

$$S_- a^\dagger = \sqrt{S_- S_+ a^\dagger a}\, E_\phi = E_\phi \sqrt{S_+ S_- a a^\dagger}, \tag{4.1}$$

which come directly from a classical factorization. Let us note that the following equation holds,

$$S_- a^\dagger = \sqrt{S_- S_+}\, E_\varphi \sqrt{a^\dagger a}\, E_\theta^\dagger = \sqrt{S_- S_+ a^\dagger a}\, E_\varphi E_\theta^\dagger, \tag{4.2}$$

which seems to lead to a nonunitary solution $E_\phi = E_\varphi E_\theta^\dagger$. However, it has been shown that Eq. (4.1) has true unitary solutions, so that $E_\phi \neq E_\varphi E_\theta^\dagger$ is allowed. To some extent, this leads us to expect that the relative phase has features not straightforwardly related to the properties of absolute phases.

As occurs in other polar decompositions such as Eqs. (2.1) and (3.5), Eqs. (4.1) do not fix all the matrix elements of E_ϕ and further conditions must be imposed [10]. The most adequate are the translation into quantum mechanics of the classical Poisson brackets verified by the corresponding variables. The only one compatible with unitarity and the polar decomposition is

$$[S_z + a^\dagger a, E_\phi] = 0. \tag{4.3}$$

By imposing this relation, the problem can be reduced to the study of its restriction to each subspace $\mathcal{H}_n$ with fixed $S_z + a^\dagger a$.

The allowed values for $S_z + a^\dagger a$ are $n - 1/2$ with $n = 0,1\ldots,\infty$. The corresponding subspaces $\mathcal{H}_n$ are spanned by the common eigenvectors of S_z and $a^\dagger a$: $\{|g,n\rangle, |e,n-1\rangle\}$ for $n>0$ and $|g,0\rangle$ for $n=0$. The total Hilbert space $\mathcal{H}$ can be expressed as the direct sum of all these subspaces as

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \tag{4.4}$$

Once the polar decomposition (4.1) has been solved in each of these subspaces, obtaining the family of operators $E_\phi^{(n)}$, the solution for the whole space is

$$E_\phi = \sum_{n=0}^{\infty} E_\phi^{(n)}, \tag{4.5}$$

from which a Hermitian relative-phase operator Φ_ϕ can be defined as $E_\phi = e^{i\Phi_\phi}$.

The solutions are

$$\begin{aligned} E_\phi^{(0)} &= |g,0\rangle\langle g,0|, \\ E_\phi^{(n)} &= |g,n\rangle\langle e,n-1| - |e,n-1\rangle\langle g,n|, \end{aligned} \tag{4.6}$$

with eigenvectors

$$\begin{aligned} |\phi_0^{(0)}\rangle &= |g,0\rangle, \\ |\phi_\pm^{(n)}\rangle &= \frac{1}{\sqrt{2}}(|g,n\rangle \pm i|e,n-1\rangle), \quad \text{for } n>0, \end{aligned} \tag{4.7}$$

and eigenvalues

$$\begin{aligned} E_\phi|\phi_0^{(0)}\rangle &= |\phi_0^{(0)}\rangle, \\ E_\phi|\phi_\pm^{(n)}\rangle &= \pm i|\phi_\pm^{(n)}\rangle, \quad \text{for} \quad n>0. \end{aligned} \tag{4.8}$$

As occurs with the dipole-phase operator, we have that $E_\phi^{\dagger(n)} = -E_\phi^{(n)}$ for $n>0$, $E_\phi^{\dagger(0)} = E_\phi^{(0)}$, and therefore $\cos\Phi_\phi = 0$ outside $\mathcal{H}_0$, and $\sin\Phi_\phi = 0$ for $\mathcal{H}_0$. Another striking feature of this result is that the relative phase can take only three values. This may be surprising since any value for the field phase seems allowed. The reasons for these behaviors are the same as those discussed for the dipole phase. Due to the commutation relation (4.3), the operator splits into components acting on two-dimensional subspaces $\mathcal{H}_n$ (one dimensional for $\mathcal{H}_0$), and so the previous features can be ascribed to the particular dimension of the atomic space. This is supported by the fact that this operator behaves properly when considering classical limits for either the atom or the field [10].

Another relevant point is that E_ϕ cannot be written as a product of phase exponentials for each system. This relative phase is not the difference of absolute phases, and it does not have the usual mathematical properties of a difference. It is worth noting that this is not exclusive of this formalism, and it also arises in other relative-phase approaches, as has been shown for two field modes [32].

The other possibility of describing the relative phase is by deriving a positive-operator measure from the corresponding ones for dipole and field absolute phases. This is what we next examine.

The joint-probability distribution for the atomic-dipole and field phases is

$$\begin{aligned} P_j(\theta,\varphi) &= \mathrm{tr}[\rho\Delta_j(\theta,\varphi)], \\ \Delta_j(\theta,\varphi) &= \Delta_j(\theta)\otimes\Delta_j(\varphi), \end{aligned} \tag{4.9}$$

where $\Delta_j(\theta)$ and $\Delta_j(\varphi)$ are the positive-operator measures defined in the preceding sections. From these expressions we have to consistently derive a positive-operator measure $\Delta_j(\phi)$ for the relative phase $\phi=\varphi-\theta$. This can be achieved in many ways.

We could try a change of variables to express $\Delta_j(\theta,\varphi)$ in terms of the phase sum and phase difference, then removing the phase-sum dependence by the corresponding integration. Although this change of variables is nonbijective, it can be carried out by very general methods [24].

Another possibility is to directly define the probability distribution for the relative phase as

$$P_j(\phi)=\int d\theta P_j(\theta,\theta+\phi)=\mathrm{tr}[\rho\Delta_j(\phi)], \tag{4.10}$$

where

$$\Delta_j(\phi)=\int d\theta\Delta_j(\theta,\theta+\phi). \tag{4.11}$$

In the Appendix it is shown that this definition is consistent with the procedure based on the change of variables mentioned above.

The result for $\Delta_j(\phi)$ can be written as

$$\Delta_j(\phi)=\sum_{n=0}^{\infty}\Delta_j(n,\phi), \tag{4.12}$$

with

$$\Delta_j(n,\phi)=\begin{cases} \dfrac{1}{2\pi}[I^{(n)}+\mu_j(n)e^{i\phi}|e,n-1\rangle\langle g,n|+\mu_j^*(n)e^{-i\phi}|g,n\rangle\langle e,n-1|], & n>0, \\ \dfrac{1}{2\pi}I^{(0)}, & n=0, \end{cases} \tag{4.13}$$

where

$$\mu_j(n)=\gamma_j G_j(n-1,n), \tag{4.14}$$

and $I^{(n)}$ is the identity in $\mathcal{H}_n$.

For the two examples considered in the preceding sections we have

$$\mu_{\mathrm{SG}}(n)=1, \quad \mu_Q(n)=\frac{\pi}{4}\frac{\Gamma(n+1/2)}{(n-1)!\sqrt{n}}. \tag{4.15}$$

The exponentials of the relative phase $e^{ik\phi}$ become the mean values of the nonunitary operators

$$\langle e^{ik\phi}\rangle_j=\langle E_{\phi,j}^{(k)}\rangle=\left\langle\int d\phi e^{ik\phi}\Delta_j(\phi)\right\rangle. \tag{4.16}$$

This gives $E_{\phi,j}^{(k)}=0$ unless $k=0,\pm1$. The case $k=0$ gives the identity, while $k=1$ gives

$$E_{\phi,j}=E_{\phi,j}^{(1)}=\sum_{n=1}^{\infty}\mu_j^*(n)|g,n\rangle\langle e,n-1|=E_{\varphi,j}E_{\theta,j}^{\dagger}, \tag{4.17}$$

$k=-1$ being the Hermitian conjugate of this last one.

Perhaps the most important feature of both approaches is expressed in the previous decomposition of the probability distribution over the independent contributions for each subspace $\mathcal{H}_n$. This is a consequence of the shifting property of $\Delta_j(\theta)$ and $\Delta_j(\varphi)$ which leads to

$$e^{i\phi'(S_z+a^\dagger a)}\Delta_j(\phi)e^{-i\phi'(S_z+a^\dagger a)}=\Delta_j(\phi). \tag{4.18}$$

This implies the commutation relation $[\Delta_j(\phi),S_z+a^\dagger a]=0$ and then the previous splitting of $\Delta_j(\phi)$ follows. In other words, this expresses the invariance of $P_j(\phi)$ under any phase-sum shift. The system state ρ and

$$e^{i\phi'(S_z+a^\dagger a)}\rho e^{-i\phi'(S_z+a^\dagger a)} \tag{4.19}$$

have the same $P_j(\phi)$. Thus shifting the phase of the field by ϕ' is equivalent to shifting the dipole phase by $-\phi'$. If this formalism were to emerge from a relative-phase operator, this property would imply its commutation with $S_z+a^\dagger a$ as in Eq. (4.3).

On the other hand, from Eqs. (2.7), (3.12), and (4.11), we have the relative-phase shifting property

$$e^{i\phi'(S_z-a^\dagger a)/2}\Delta_j(\phi)e^{-i\phi'(S_z-a^\dagger a)/2}=\Delta_j(\phi+\phi'). \tag{4.20}$$

The compatibility with $S_z+a^\dagger a$ allows us to introduce the joint-probability distribution $P_j(n,\phi)$ for n and ϕ, in the form

$$P_j(n,\phi)=\mathrm{tr}[\rho\Delta_j(n,\phi)], \tag{4.21}$$

satisfying

$$\sum_{n=0}^{\infty} P_j(n,\phi)=P_j(\phi),$$
$$\int d\phi P_j(n,\phi)=P(n), \tag{4.22}$$

where $P(n)$ is the probability of getting the value $n-1/2$ for $S_z+a^\dagger a$.

We have here a point of comparison with the relative-phase operator obtained from the polar decomposition. From different starting points, we arrive at the same compatibility with $S_z+a^\dagger a$, which is expressed by the relative-phase operator in Eq. (4.3).

Despite this common property, a strong difference still remains. The polar decomposition leads to only three allowed values for the relative phase, while any value seems possible for the positive-operator measures. Due to the previous splitting into independent contributions over each $\mathcal{H}_n$, here we are in a position similar to that found for the dipole phase. Accordingly, we could translate the same analysis here.

Due to the dimension of $\mathcal{H}_n$, any operator has just two eigenvalues for $n>1$ and one for $n=0$. On the other hand, the most general form of $P_j(n,\phi)$ is

$$P_j(n,\phi)=\frac{1}{2\pi}[\alpha^{(n)}+\beta_j^{(n)}e^{i\phi}+\beta_j^{(n)*}e^{-i\phi}], \tag{4.23}$$

where $\alpha^{(n)}=P(n)$ is real. Since this function depends just on three parameters, it can be completely fixed by its value on three properly chosen ϕ points for each n. This corresponds to the analysis in each subspace $\mathcal{H}_n$. In addition to this, we also have that the whole $P_j(\phi)$ can be written in the form

$$P_j(\phi)=\frac{1}{2\pi}(1+c_je^{i\phi}+c_j^*e^{-i\phi}). \tag{4.24}$$

The complete $P_j(\phi)$ function depends just on two parameters and thus can be completely fixed by its value in two points. This means that formulas (3.20), (3.21), and (3.22) could be translated here exactly in the same terms and, therefore, these formalisms show an effective discreteness despite their apparent continuity. Moreover, this effective discreteness applies to any relative-phase description with the properties (4.18) and (4.20).

V. EVOLUTION OF THE ATOM-FIELD RELATIVE PHASE IN THE JAYNES-CUMMINGS MODEL

In this section we study the evolution of the relative phase between the atomic dipole and the field for the Jaynes-Cummings model [33], comparing the approaches developed in the preceding section. The evolution of the absolute phases of both systems is briefly examined as well.

The Jaynes-Cummings Hamiltonian for the atom-field interaction reads (in the rotating-wave approximation, at exact resonance, and $\hbar=1$)

$$H=\omega S_z+\omega a^\dagger a+\lambda(a^\dagger S_-+S_+a). \tag{5.1}$$

Its apparent simplicity belies the fact that the evolution of physical quantities displays an extraordinary complexity, and so it serves as a testing ground for many theoretical concepts. For our purposes, its sensitivity to the relative phase between the atomic dipole and the field has been tested [34,35].

The Hamiltonian commutes with the total excitation number $S_z+a^\dagger a$ and can be diagonalized in the subspaces $\mathcal{H}_n$, giving the dressed states [36]

$$|\Psi_0^{(0)}\rangle=|g,0\rangle,$$
$$|\Psi_\pm^{(n)}\rangle=\frac{1}{\sqrt{2}}(|g,n\rangle\pm|e,n-1\rangle), \tag{5.2}$$

with eigenvalues

$$H|\Psi_0^{(0)}\rangle=-\frac{\omega}{2}|\Psi_0^{(0)}\rangle,$$
$$H|\Psi_\pm^{(n)}\rangle=[(n-1/2)\omega\pm\lambda\sqrt{n}]|\Psi_\pm^{(n)}\rangle. \tag{5.3}$$

A first interesting and simple example of evolution is the case where the initial state of the system is the product of the atom in its ground state and the field in a number state $|\Psi\rangle=|g,n\rangle$ with $n>0$. At later times t, the state is given by

$$|\Psi(t)\rangle=\cos(\lambda\sqrt{n}t)|g,n\rangle-i\sin(\lambda\sqrt{n}t)|e,n-1\rangle, \tag{5.4}$$

and the probabilities of finding the atom in the upper and lower states are

$$P_g(t)=\cos^2(\lambda\sqrt{n}t), \quad P_e(t)=\sin^2(\lambda\sqrt{n}t). \tag{5.5}$$

The population inversion $W=|e\rangle\langle e|-|g\rangle\langle g|=2S_z$ is then

$$\langle W\rangle=-\cos(2\lambda\sqrt{n}t). \tag{5.6}$$

We can begin by considering the operator description of the relative phase. If we ask for the probability of finding the system with phase differences $0,-\pi/2$ or $\pi/2$, we have

$$P_{-\pi/2}(t)=\cos^2(\lambda\sqrt{n}t-\pi/4),$$
$$P_0(t)=0, \tag{5.7}$$
$$P_{\pi/2}(t)=\sin^2(\lambda\sqrt{n}t-\pi/4).$$

This gives the mean value

$$\langle E_\phi\rangle=-i\ \sin(2\lambda\sqrt{n}t), \tag{5.8}$$

and so $\langle\cos\Phi_\phi\rangle=0$.

Concerning the evolution of the relative phase given by the two positive-operator measures of the preceding section, we have the mean values

$$\langle e^{i\phi}\rangle_j=-\frac{i}{2}\mu_j^*(n)\sin(2\lambda\sqrt{n}t). \tag{5.9}$$

For the two positive-operator measures studied here $\mu_j(n)$ is real, and so we have $\langle\cos\phi\rangle_j=0$, as occurs for the operator description. For all approaches we have the same harmonic evolution but with different amplitudes.

If we ask for the evolution of the field and dipole absolute phases, we get $P_j(\theta)=P_j(\varphi)=1/(2\pi)$, and so they are always uniform.

When initially the field is in a coherent state $|\alpha\rangle$, the close relation (simply a time translation) between the evolution of the relative phase and the population inversion of the previous example allows us to expect similar collapses and revivals for the relative phase as those experienced by the population inversion. When the initial state is $|g\rangle|\psi\rangle$, with $|\psi\rangle$ an arbitrary field state, we have the mean value associated with the operator description

$$\langle E_\phi\rangle = P_\psi(0) - i\sum_{n=1}^{\infty} \sin(2\lambda\sqrt{n}t)P_\psi(n), \tag{5.10}$$

where $P_\psi(n)$ is the field photon-number distribution $|\langle n|\psi\rangle|^2$. Equivalently we have

$$\begin{aligned}\langle \sin\Phi_\phi\rangle &= -\sum_{n=0}^{\infty} \sin(2\lambda\sqrt{n}t)P_\psi(n)\\ &= \mathrm{Im}\left[\sum_{n=0}^{\infty} e^{-i2\lambda\sqrt{n}t}P_\psi(n)\right],\end{aligned} \tag{5.11}$$

and $\langle\cos\Phi_\phi\rangle = P_\psi(0)$.

On the other hand, the positive-operator measures give a similar evolution

$$\langle e^{i\phi}\rangle_j = -\frac{i}{2}\sum_{n=1}^{\infty} \mu_j^*(n)\sin(2\lambda\sqrt{n}t)P_\psi(n). \tag{5.12}$$

Taking into account the reality of $\mu_{\mathrm{SG}}(n)$ and $\mu_Q(n)$ this is equivalent to

$$\begin{aligned}\langle \sin\phi\rangle_j &= -\frac{1}{2}\sum_{n=0}^{\infty} \mu_j(n)\sin(2\lambda\sqrt{n}t)P_\psi(n)\\ &= \frac{1}{2}\mathrm{Im}\left[\sum_{n=0}^{\infty} \mu_j(n)e^{-i2\lambda\sqrt{n}t}P_\psi(n)\right],\end{aligned} \tag{5.13}$$

and here again we have $\langle\cos\phi\rangle_j=0$.

We should point out that the equality $\langle\cos\phi\rangle_j=0$ is due to the fact that the atom is initially in its ground state (the excited state will also provide the same result). If at $t=0$ the atom is in a superposition, we will have in general $\langle\cos\phi\rangle_j\neq 0$, whereas the operator will always give $\langle\cos\Phi_\phi\rangle = |c_g|^2P_\psi(0)$.

This relative-phase evolution can be compared with the time dependence of the population inversion

$$\begin{aligned}\langle W\rangle &= -\sum_{n=0}^{\infty} \cos(2\lambda\sqrt{n}t)P_\psi(n)\\ &= -\mathrm{Re}\left[\sum_{n=0}^{\infty} e^{-i2\lambda\sqrt{n}t}P_\psi(n)\right].\end{aligned} \tag{5.14}$$

In Fig. 1 we show $\langle W\rangle$, $\langle\sin\phi\rangle_Q$, and $\langle\sin\phi\rangle_{\mathrm{SG}}$ when the field is initially in a coherent state $|\alpha\rangle$ with mean number of photons $|\alpha|^2=5$. The mean value $\langle\sin\Phi_\phi\rangle$ is not represented since $\langle\sin\Phi_\phi\rangle = 2\langle\sin\phi\rangle_{\mathrm{SG}}$. One can appreciate the similarity between $\langle\sin\phi\rangle_Q$ and $\langle\sin\phi\rangle_{\mathrm{SG}}$, and their close relation with $\langle W\rangle$. In particular $\langle\sin\phi\rangle_j$ and $\langle\sin\Phi_\phi\rangle$ experience the same collapse and revival dynamics of the population inversion.

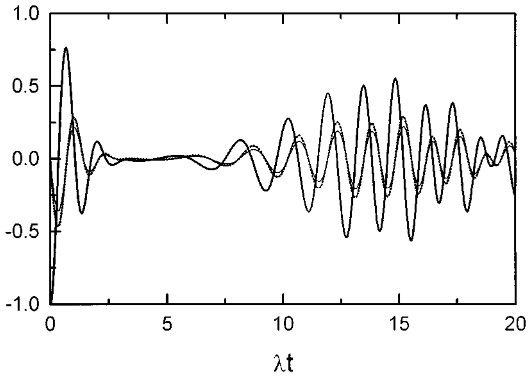

FIG. 1. Evolution of the mean values of the population inversion $\langle W\rangle$ (thick solid line), the sine of the relative phase $\langle\sin\phi\rangle_Q$ (thin solid line), and $\langle\sin\phi\rangle_{\mathrm{SG}}$ (dotted line) when the field is initially in a coherent state with mean-photon number $|\alpha|^2=5$ and the atom is in the ground state.

Next we outline a plausible physical interpretation of the similarity between relative phase and population inversion. The interaction Hamiltonian in the rotating-wave approximation is proportional to the components of the atomic dipole and field quadratures and also proportional to cos ϕ. In classical terms, the dipole energy is maximum or minimum either when $\sin\phi=0$ or when the field quadratures or the atomic dipole components vanish. In the quantum case, for the initial state $|g,n\rangle$, the population inversion has maximum or minimum values precisely when $\langle\sin\Phi_\phi\rangle = \langle\sin\phi\rangle_j = 0$. This relation holds very approximately when the initial state is $|g\rangle|\alpha\rangle$, as can be seen in Fig. 1.

If the atom is initially in its ground state, the mean value of the dipole operator vanishes and therefore so does the interaction Hamiltonian. Since in the resonant case the interaction Hamiltonian is a constant of the motion, we would expect $\cos\phi=0$ and $\sin\phi=\pm 1$ at all later times. Whichever formalism is used, the relative phase is effectively uniform at $t=0$ due to the randomness of the dipole phase. Due to the quantum fluctuations, the condition $\sin\phi=\pm 1$ cannot be established instantaneously, whereas this is possible classically. Nevertheless, the trend to satisfy this phase relation can be recognized in Fig. 1 in the initial stages of the evolution, before the quantum evolution displays its complexity.

On the other hand, we always have $\langle\cos\phi\rangle_j=0$, and so at any time $\arg\langle e^{i\phi}\rangle_j = \pm\pi/2$, in agreement with the preceding discussion. For the operator description $\langle\cos\Phi_\phi\rangle = P_\psi(0)$, and so $\arg\langle E_\phi\rangle \simeq \pm\pi/2$ will occur only provided that $P_\psi(0) \ll \langle\sin\Phi_\phi\rangle$.

Therefore expressions (5.11), (5.13), and (5.14) as well as Fig. 1 show that the previously discussed relationship between relative phase and population inversion extends to the quantum case.

In addition to this we can show that Fig. 1 also conveys relevant information about the evolution of the relative-phase uncertainty. Since we are dealing with a 2π-periodic vari-

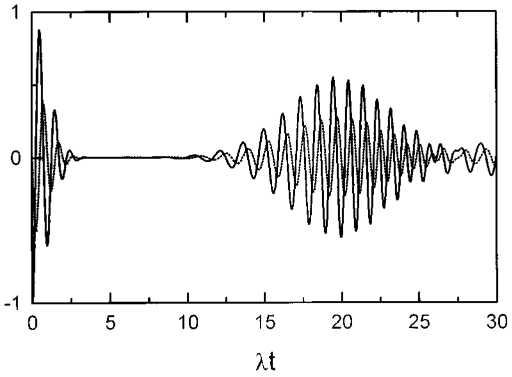

FIG. 2. Evolution of the mean values of the population inversion $\langle W\rangle$ (solid line) and the sine of the relative phase $\langle \sin\phi\rangle_{\mathrm{SG}}$ (dotted line) when the field is initially in a coherent state with mean photon number $|\alpha|^2=10$ and the atom is in the ground state.

able, a suitable measure of this relative-phase uncertainty is provided by the dispersion [30,37]. In the case of the operator description, the dispersion is defined as

$$D^2=1-|\langle E_\phi\rangle|^2=1-\langle \sin\Phi_\phi\rangle^2-\langle \cos\Phi_\phi\rangle^2, \quad (5.15)$$

whereas for the positive-operator measures we have

$$D_j^2=1-|\langle E_{\phi,j}^{(1)}\rangle|^2=1-\langle \sin\phi\rangle_j^2-\langle \cos\phi\rangle_j^2. \quad (5.16)$$

For the cases considered in this work we have always $\langle \cos\phi\rangle_j=0$ and $\langle \cos\Phi_\phi\rangle=P_\psi(0)$, and so the evolution of $\langle \sin\phi\rangle_j$ and $\langle \sin\Phi_\phi\rangle$ represented in Fig. 1 directly gives the evolution of the relative-phase uncertainty. The dispersion is minimum for those interaction times for which $\langle \sin\phi\rangle_j$ is maximum and vice versa.

It is interesting to examine how the evolution of the relative phase depends on the initial state of the field. In the first place, we can consider the dependence of the relative-phase dynamics on the mean-photon number when the field state is initially coherent. In Fig. 2 we show $\langle W\rangle$ and $\langle \sin\phi\rangle_{\mathrm{SG}}$ when $\langle n\rangle=|\alpha|^2=10$ and the atom is in the ground state. Here again the relative phase closely follows the population inversion, the oscillations being always $\pi/2$ out of phase. The first revival occurs later than in Fig. 1, it is slightly broader, and the maximum amplitude of the oscillations is almost the same. This dependence on $\langle n\rangle$ is known for $\langle W\rangle$ and here we show that it is also experienced by the relative phase. Later we will show these features more clearly when considering the limit of high photon numbers.

There are other interesting initial field states like ideally squeezed states,

$$|\alpha,\xi\rangle=e^{\alpha a^\dagger-\alpha^* a}e^{\xi^* a^2-\xi a^{\dagger 2}}|0\rangle. \quad (5.17)$$

These states offer the possibility of examining how the dipole-phase dynamics is influenced by the initial field-phase uncertainty. This is because the field-phase distribution depends on α and ξ. For fixed values of $r=2|\xi|$ and $|\alpha|$, it depends strongly on the squeezing direction $\delta=\arg(\alpha)-\arg(\xi)/2$ (the angle between the direction of the coherent component α and the minor axis of the uncertainty ellipse in the quadrature phase space).

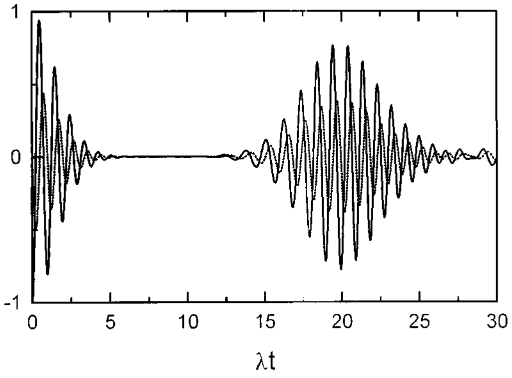

FIG. 3. Evolution of the mean values of the population inversion $\langle W\rangle$ (solid line) and the sine of the relative phase $\langle \sin\phi\rangle_{\mathrm{SG}}$ (dotted line) when the field is initially in a squeezed state with $\langle n\rangle=10$, $r=0.5$, $\delta=0$ and the atom is in the ground state.

If $\delta=0$, the initial squeezed state has phase uncertainty higher than that of a coherent state of the same mean photon number and a narrower photon-number distribution. If $\delta=\pi/2$, it has lesser phase uncertainty and broader photon-number distribution.

In Fig. 3 we have represented $\langle W\rangle$ and $\langle \sin\phi\rangle_{\mathrm{SG}}$ for an initial squeezed state with $\langle n\rangle=10$, $r=0.5$, and $\delta=0$. In comparison with the unsqueezed coherent state in Fig. 2 we can see that the first revival time coincides (the mean photon number is the same). The revival is narrower and the maximum amplitude of the oscillations is higher.

In Fig. 4 we have represented $\langle W\rangle$ and $\langle \sin\phi\rangle_{\mathrm{SG}}$ for an initial squeezed state with $\langle n\rangle=10$, $r=0.5$, and $\delta=\pi/2$. In comparison with Fig. 2 the revival time is the same but now the envelope is broader and the maximum value of the oscillations is smaller.

Throughout we have pointed out the close relation between population inversion and relative phase. In Eqs. (5.11), (5.13), and (5.14) they are the real and imaginary parts of the same expression. We can take advantage of this fact to translate to the relative phase the previous analysis of the dynamics of the population inversion. In particular, some approximate analytical expressions for $\langle W\rangle$ in Eq. (5.14) are available [38,39] which can be directly translated to the rela-

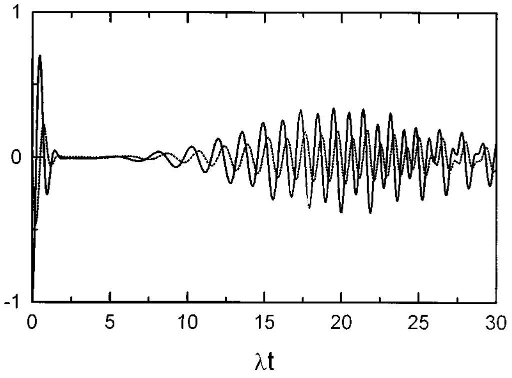

FIG. 4. Evolution of the mean values of the population inversion $\langle W\rangle$ (solid line) and the sine of the relative phase $\langle \sin\phi\rangle_{\mathrm{SG}}$ (dotted line) when the field is initially in a squeezed state with $\langle n\rangle=10$, $r=0.5$, $\delta=\pi/2$ and the atom is in the ground state.

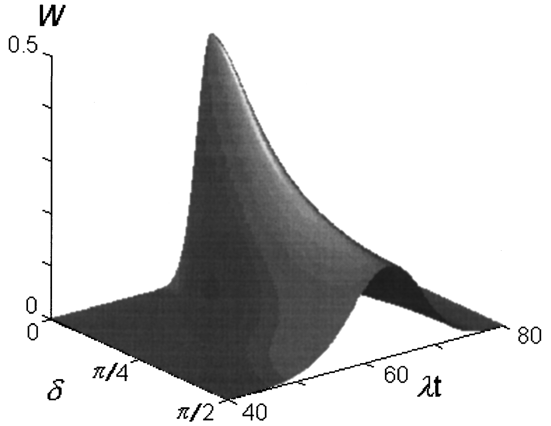

FIG. 5. Behavior of the envelope for the first revival as a function of the squeezing direction δ when the field is initially in a squeezed state with $\langle n\rangle = 100$, $r=0.5$ and the atom is in the ground state.

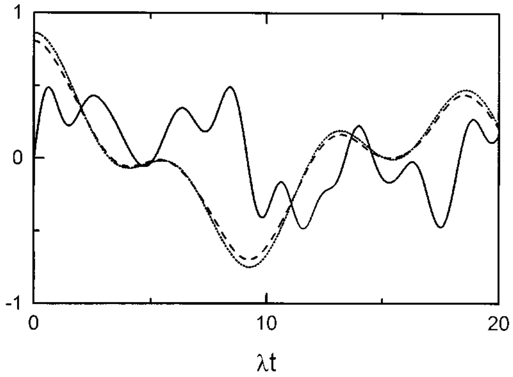

FIG. 6. Evolution of the $R_{\theta,Q}$ (dashed line), $R_{\theta,\mathrm{SG}}$ (dotted line), and R_φ (solid line) factors modulating the harmonic evolution of the field and atomic-dipole phases, respectively, when the field is initially in a coherent state with mean photon number $|\alpha|^2=1.6$ and the atom is in the ground state.

tive phase in Eqs. (5.11) or (5.13). Such an approximation can be very useful to prove the previous features of the phase evolution when the mean photon number is very large.

Under appropriate conditions it has been shown [39] that it is essentially the shape of the photon-number distribution which governs the first revivals envelope. For instance, the first revival of the population inversion becomes

$$\langle W\rangle \simeq -P_\psi\left(n=\frac{\lambda^2 t^2}{4\pi^2}\right)\frac{\lambda t}{\pi\sqrt{2}}\cos\left(\frac{\lambda^2 t^2}{2\pi}-\frac{\pi}{4}\right). \tag{5.18}$$

The same approximation can be applied to Eq. (5.13), leading to a first revival of $\langle \sin\phi\rangle_{\mathrm{SG}}$ of the form

$$\langle \sin\phi\rangle_{\mathrm{SG}} \simeq -\frac{1}{2}P_\psi\left(n=\frac{\lambda^2 t^2}{4\pi^2}\right)\frac{\lambda t}{\pi\sqrt{2}}\sin\left(\frac{\lambda^2 t^2}{2\pi}-\frac{\pi}{4}\right). \tag{5.19}$$

We can note that the envelope is the same and the relation between the oscillations agrees with the previous discussions. Replacing $P_\psi(n)$ by the corresponding expression for a coherent or squeezed state the dependence of the relative phase with $\langle n\rangle$ and δ in Figs. 2, 3, and 4 can be easily derived. In the case of large photon number and moderate squeezing $P_\psi(n)$ can be replaced by a Gaussian approximation

$$P_\psi(n) \simeq \frac{1}{\sqrt{2\pi}\Delta n}\exp\left[-\frac{(n-\langle n\rangle)^2}{2(\Delta n)^2}\right], \tag{5.20}$$

with

$$(\Delta n)^2 = (\langle n\rangle - \sinh^2 r)(e^{-2r}\cos^2\delta + e^{2r}\sin^2\delta) + 2\sinh^2 r\,\cosh^2 r. \tag{5.21}$$

Equations (5.19), (5.20), and (5.21) show how the relative phase dynamics (revival time, width, and peak of the envelope modulating the oscillations) depends on the field initial conditions.

For example, in Fig. 5 we have represented the variation of the first revival envelope with δ for fixed mean photon number $\langle n\rangle = 100$. This dependence agrees with that shown in Figs. 3 and 4 for smaller values of the mean photon number.

Although throughout this work we have focused on the relative phase, it could be also interesting to consider briefly the time variation of the dipole and field absolute phases. When the system is initially in the state $|g\rangle|\alpha\rangle$ with $\alpha = |\alpha| e^{i\delta}$ we have, for the dipole and field phases,

$$\begin{aligned}\langle e^{i\varphi}\rangle_j &= -i\gamma_j^* R_\varphi(|\alpha|,t)e^{-i(\omega t-\delta)},\\ \langle e^{i\theta}\rangle_j &= R_{\theta,j}(|\alpha|,t)e^{-i(\omega t-\delta)},\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}R_\varphi(|\alpha|,t) &= |\alpha|\sum_{n=0}^{\infty}\sin(\lambda\sqrt{n+1}t)\cos(\lambda\sqrt{n}t)\\ &\quad\times\frac{1}{\sqrt{n+1}}P_\alpha(n),\\ R_{\theta,j}(|\alpha|,t) &= |\alpha|\sum_{n=0}^{\infty}[G_j(n,n+1)\cos(\lambda\sqrt{n+1}t)\cos(\lambda\sqrt{n}t)\\ &\quad + G_j(n-1,n)\sin(\lambda\sqrt{n+1}t)\sin(\lambda\sqrt{n}t)]\\ &\quad\times\frac{1}{\sqrt{n+1}}P_\alpha(n).\end{aligned} \tag{5.23}$$

These functions $R_\varphi(|\alpha|,t)$ and $R_{\theta,j}(|\alpha|,t)$ are always real, and they are plotted in Fig. 6 for $|\alpha|^2=1.6$. We can note that $R_{\theta,Q}$ is smaller than $R_{\theta,\mathrm{SG}}$, leading to slightly greater values of the phase dispersion, in accordance with the general relation between these two approaches [14]

The evolutions (5.22) show the natural harmonic variation of both phases with the $\pm\pi/2$ phase difference discussed above. The harmonic evolution is modulated by the terms $R_\varphi(|\alpha|,t)$ and $R_{\theta,j}(|\alpha|,t)$, which contain the quantum details of the evolution, since in a classical theory they are constant.

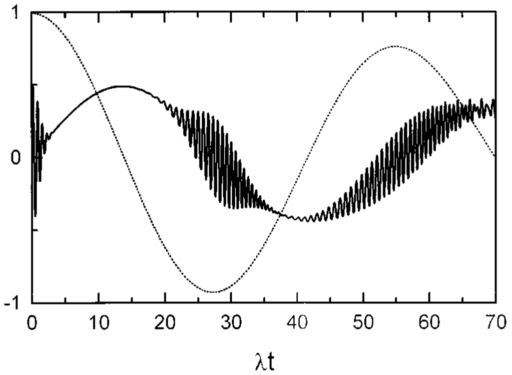

FIG. 7. Evolution of the $R_{\theta,\mathrm{SG}}$ (dotted line) and R_φ (solid line) factors when the field is initially in a coherent state with mean photon number $|\alpha|^2=20$ and the atom is in the ground state.

In Eqs. (5.23) it can be seen that these terms contain two different kinds of frequencies. The product of harmonic terms gives rise to the frequencies $\lambda(\sqrt{n+1}\pm\sqrt{n})$ which, for high enough n, are of the form $\lambda\sqrt{n}$ and $\lambda/(2\sqrt{n})$. The $\lambda\sqrt{n}$ terms give rise to the Rabi oscillations and the collapse and revival dynamics. On the other hand, the $\lambda/(2\sqrt{n})$ frequencies are much smaller with a period close to the first revival time. The effect of both kinds of frequencies is clearly shown by $R_\varphi(|\alpha|,t)$ in Fig. 7, where $|\alpha|^2=20$.

Concerning the field evolution, the SG case contains only the low frequencies. The Q case also contains the higher frequencies. However, they are modulated by $G_Q(n,n+1)-G_Q(n-1,n)$ which is close to zero for high enough n, and so only the slowly varying part is effective. In the case of mean photon number $|\alpha|^2=20$ in Fig. 7, $R_{\theta,Q}$ and $R_{\theta,\mathrm{SG}}$ coincide and cannot be distinguished.

These modulating factors also provide a measure of the phase uncertainty through the dispersion. For instance, we can see that the dispersion of the field phase becomes maximum $R_{\theta,j}(|\alpha|,t)=0$ near the collapse time. A detailed study of the field phase evolution within this framework can be found in Refs. [40,41]. Also and according to the conclusions of Sec. III, the atomic dipole moment displays this same kind of evolution [42].

VI. CONCLUSIONS

In this work we have studied and compared several possible descriptions of the atom-field relative phase. These are an operator defined in terms of a polar decomposition and two positive-operator measures derived from the corresponding measures for field and dipole phases. This led us to examine quantum descriptions of the dipole phase for a two-level atom.

The evaluation of these formalisms shows that, although strictly speaking they give different results, they share a lot of properties. In all the cases, we have that this variable is compatible with the total excitation number. This inevitably leads to an effective discreteness even if in principle a continuous range of variation is assumed. These two facts are naturally reflected by the operator description.

We have also found similarities when studying the evolution of the relative phase within the Jaynes-Cummings model. Although numerical values are different, all these formalisms provide the same relevant features. Such an evolution is in agreement with some classical and semiclassical relations and also with the notable quantum character of this kind of dynamics.

The relative phase studied here involves a system describable by a two-dimensional Hilbert space and a very dissimilar one described by an infinite-dimensional space. This case can be generalized along the same lines to situations involving other spinlike systems describable by Hilbert spaces of arbitrary finite dimension.

For all the approaches considered, some unexpected relations arise which are entirely due to the particular dimension of the atomic state space. These pathological behaviors disappear as soon as the dimension is increased, and so they do not imply any flaw for the phase formalisms studied here.

ACKNOWLEDGMENTS

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APPENDIX: POSITIVE-OPERATOR MEASURE FOR THE RELATIVE PHASE

Let us consider a joint-probability distribution function $P(\phi_1,\phi_2)$ for two 2π-periodic variables ϕ_1,ϕ_2 that in general will arise from a positive-operator measure $\Delta(\phi_1,\phi_2)$,

$$P(\phi_1,\phi_2)=\mathrm{tr}[\rho\Delta(\phi_1,\phi_2)], \tag{A1}$$

where ρ is any system state. We will refer to ϕ_1 and ϕ_2 as phases, although the procedure to be followed can be applied to any pair of 2π-periodic variables.

As a first step, we want to find the probability distribution function $\mathcal{P}(\phi_+,\phi_-)$ for the phase sum and difference,

$$\phi_+=\phi_1+\phi_2, \quad \phi_-=\phi_2-\phi_1, \tag{A2}$$

where we assume that ϕ_+ and ϕ_- are also 2π periodic. Then we integrate $\mathcal{P}(\phi_+,\phi_-)$ over the phase sum in order to obtain the probability-distribution function for the phase difference

$$\mathcal{P}(\phi_-)=\int d\phi_+\mathcal{P}(\phi_+,\phi_-). \tag{A3}$$

Our final objective is to get a positive-operator measure $\Delta(\phi_-)$ such that

$$\mathcal{P}(\phi_-)=\mathrm{tr}[\rho\Delta(\phi_-)]. \tag{A4}$$

As far as we consider all these variables as 2π periodic, the change (A2) is nonbijective since the pairs (ϕ_1,ϕ_2) and $(\phi_1+\pi,\phi_2+\pi)$ give the same (ϕ_+,ϕ_-) mod(2π). Therefore, in order to obtain $\mathcal{P}(\phi_+,\phi_-)$ from $P(\phi_1,\phi_2)$, the change (A2) must be followed by the addition of these two contributions, so that, taking into account the Jacobean of transformation (A2), we get $\mathcal{P}(\phi_+,\phi_-)$ as

$$\mathcal{P}(\phi_+,\phi_-)=\frac{1}{2}\left[P\left(\frac{\phi_+-\phi_-}{2},\frac{\phi_++\phi_-}{2}\right)+P\left(\frac{\phi_+-\phi_-}{2}+\pi,\frac{\phi_++\phi_-}{2}+\pi\right)\right], \tag{A5}$$

which is a 2π-periodic function of ϕ_+ and ϕ_-. This relation is equivalent to imposing the following equalities for the mean values of the ϕ_+,ϕ_- exponentials:

$$\int d\phi_+ d\phi_- e^{ik\phi_+}e^{i\ell\phi_-}\mathcal{P}(\phi_+,\phi_-) = \int d\phi_1 d\phi_2 e^{ik(\phi_1+\phi_2)}e^{i\ell(\phi_2-\phi_1)}P(\phi_1,\phi_2). \tag{A6}$$

Since $\mathcal{P}(\phi_+,\phi_-)$ and $P(\phi_1,\phi_2)$ are both 2π periodic, we have the Fourier decomposition

$$P(\phi_1,\phi_2)=\frac{1}{(2\pi)^2}\sum_{k,\ell} P_{k,\ell}e^{-ik\phi_1}e^{-i\ell\phi_2},$$

$$\mathcal{P}(\phi_+,\phi_-)=\frac{1}{(2\pi)^2}\sum_{k,\ell} \mathcal{P}_{k,\ell}e^{-ik\phi_+}e^{-i\ell\phi_-}. \tag{A7}$$

Both Eqs. (A5) and (A6) lead to the equation for the Fourier amplitudes,

$$\mathcal{P}_{k,\ell}=P_{k-\ell,k+\ell}. \tag{A8}$$

Concerning the probability distribution $\mathcal{P}(\phi_-)$ for the phase difference it has the Fourier expression

$$\mathcal{P}(\phi_-)=\frac{1}{2\pi}\sum_{\ell} \mathcal{P}_{\ell}e^{-i\ell\phi_-}. \tag{A9}$$

Using Eqs. (A3) and (A6) we have the relations

$$\mathcal{P}_{\ell}=\mathcal{P}_{0,\ell}=P_{-\ell,\ell}. \tag{A10}$$

On the other hand, the following equation holds:

$$P_{-\ell,\ell}=\int d\phi_- e^{i\ell\phi_-}\int d\theta P(\theta,\theta+\phi_-), \tag{A11}$$

and so Eqs. (A9), (A10), and (A1) give

$$\mathcal{P}(\phi_-)=\int d\theta P(\theta,\theta+\phi_-)=\mathrm{tr}\left[\rho\int d\theta\Delta(\theta,\theta+\phi_-)\right]. \tag{A12}$$

Then, Eq. (A5) provides the desired positive-operator measure as

$$\Delta(\phi_-)=\int d\theta\Delta(\theta,\theta+\phi_-), \tag{A13}$$

which is Eq. (4.11).

This shows that the transformation law (A5) is consistent with the natural definition (4.10).

[1] L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Dover, New York, 1987).
[2] P. Meystre and M. Sargent III, *Elements of Quantum Optics* (Springer-Verlag, Berlin, 1991).
[3] J. Bergou and B. G. Englert, Ann. Phys. (NY) **209**, 479 (1991).
[4] Phys. Scr. **T48**, 1 (1993), special issue on quantum phase.
[5] A. Lukš, and V. Peřinová, Quantum Opt. **6**, 125 (1994).
[6] R. Lynch, Phys. Rep. **256**, 367 (1995).
[7] T. Tanaś, A. Miranowicz, and T. Gantsog, Prog. Opt. **36**, 161 (1996).
[8] A. Luis and L. L. Sánchez-Soto, Phys. Rev. A **48**, 4702 (1993).
[9] L. L. Sánchez-Soto and A. Luis, Opt. Commun. **105**, 84 (1994).
[10] A. Luis and L. L. Sánchez-Soto, Opt. Commun. **133**, 159 (1997).
[11] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **114**, 243 (1927).
[12] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 441 (1968).
[13] L. Susskind and J. Glogower, Physics (Long Island City, NY) **1**, 49 (1964).
[14] U. Leonhardt, J. A. Vaccaro, B. Böhmer, and H. Paul, Phys. Rev. A **51**, 84 (1995).
[15] B.-G. Englert and K. Wódkiewicz, Phys. Rev. A **51**, R2661 (1995); B.-G. Englert, K. Wódkiewicz, and P. Riegler, *ibid.* **52**, 1704 (1995).
[16] S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989).
[17] J. H. Shapiro, S. R. Shepard, and N. C. Wong, Phys. Rev. Lett. **62**, 2377 (1989); J. H. Shapiro and S. R. Shepard, Phys. Rev. A **43**, 3795 (1991).
[18] M. J. W. Hall, Quantum Opt. **3**, 7 (1991).
[19] U. Leonhardt and H. Paul, Phys. Rev. A **45**, R2460 (1993).
[20] M. Freyberger, K. Vogel, and W. Schleich, Phys. Lett. A **176**, 41 (1993).
[21] Y. Lai and H. A. Haus, Quantum Opt. **1**, 99 (1989).
[22] U. Leonhardt and H. Paul, Phys. Rev. A **48**, 4598 (1993).
[23] A. Ortowski, and A. Wünsche, Phys. Rev. A **48**, 4697 (1993).
[24] A. Luis and L. L. Sánchez-Soto, Phys. Rev. A **53**, 495 (1996).
[25] J. W. Noh, A. Fougères, and L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991); Phys. Rev. A **45**, 424 (1992); **46**, 2840 (1992).
[26] U. Leonhardt, Phys. Rev. A **48**, 3265 (1991).
[27] J. M. Lévy-Leblond, Rev. Mex. Fis. **22**, 15 (1973).
[28] D. Ellinas, J. Math. Phys. (N.Y.) **32**, 135 (1990).
[29] A. Vourdas, Phys. Rev. A **41**, 1653 (1990).
[30] A. Bandilla, H. Paul, and H. H. Ritze, Quantum Opt. **3**, 267 (1991).
[31] A. Perelomov, *Generalized Coherent States and their Applications* (Springer-Verlag, Berlin, 1986).
[32] J. R. Torgerson and L. Mandel, Phys. Rev. Lett. **76**, 3939 (1996).
[33] B. W. Shore and P. L. Knight, J. Mod. Opt. **40**, 1195 (1993).
[34] K. Zaheer and M. S. Zubairy, Phys. Rev. A **39**, 2000 (1989).

[35] J. I. Cirac and L. L. Sánchez-Soto, Phys. Rev. A **42**, 2851 (1990).

[36] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms: Introduction to Quantum Electrodynamics* (John Wiley, New York, 1989).

[37] T. Opatrný, J. Phys. A **27**, 7201 (1994).

[38] J. I. Cirac and L. L. Sánchez-Soto, Phys. Rev. A **40**, 3743 (1989).

[39] M. Fleischhauer and W. Schleich, Phys. Rev. A **45**, 4258 (1993).

[40] H. T. Dung, R. Tanaś, and A. S. Shumovsky, Opt. Commun. **79**, 462 (1990); J. Mod. Opt. **38**, 2069 (1991).

[41] B. M. Garraway and P. L. Knight, Phys. Rev. A **46**, 5346 (1992).

[42] N. B. Narozhny, J. J. Sánchez-Mondragon, and J. H. Eberly, Phys. Rev. A **23**, 236 (1981).