Motion of a two-level atom in an optical cavity

A. C. Doherty, A. S. Parkins, S. M. Tan, and D. F. Walls
Department of Physics, University of Auckland, Private Bag 92019, Auckland, New Zealand
(Received 18 December 1996)

A semiclassical model of the force and momentum diffusion on a point particle is used to describe the motion of a two-level atom strongly coupled to a single Gaussian cavity mode. The effects of the momentum diffusion on the motion of an atom in a cavity are investigated in a regime similar to that of the experiment performed by Mabuchi *et al.* [Opt. Lett. **21**, 1393 (1996)]. It is found that a slow atom quickly develops significant velocities along the cavity axis. The limited bandwidth in the experiment of Mabuchi *et al.* means that the full intensity signal due to atomic motion in the standing wave is filtered leading to the apparently smaller velocities observed. It is shown that a negative detuning of the laser and cavity from the atomic resonance would lead to nonzero dipole forces and significantly reduced velocities along the standing wave. An analysis of the intensity signal with a larger bandwidth is proposed, which would track the velocity of the atom along the cavity axis. These results are compared with a Monte Carlo wave-function simulation similar to that used to treat Doppler cooling. [S1050-2947(97)00407-1]

PACS number(s): 42.50.Ct, 42.50.Lc, 42.50.Vk

I. INTRODUCTION

Experimental research in the field of cavity quantum electrodynamics (cavity QED) is now able to operate routinely in the so-called ''strong-coupling regime'' for which the maximal atom-field dipole coupling strength g_0 exceeds the cavity field decay rate κ and the atomic spontaneous emission rate, Γ (see, for example, [1]). In this regime excitations can be exchanged coherently between the atom and the field several times before decay processes are likely to occur, the properties of the field can be significantly altered by the presence of even a single atom and, similarly, the prescence of a single photon in the field will saturate the response of the atom. Recently, Mabuchi *et al.* [2] have achieved something of a breakthrough in cavity QED research by replacing the commonly used (fast) atomic beam source with very cold atoms ($T \simeq 100\ \mu$K) dropped into the cavity (a few at a time) from a magneto-optical trap (MOT). This allows for very long interaction times and observation of the effects of single atoms on the cavity field in real time. In particular, the presence of an atom in the cavity, which is tuned to the atomic transition and resonantly driven through one of its mirrors with a laser field, can lead to a dramatic drop in the transmitted cavity intensity [3]. This intensity is measured in the experiment by a balanced heterodyne detector.

The intensity drop that is observed in the experiment is strongly dependent on the position of the atom in the cavity as a result of the spatial variation of the electromagnetic mode function $\psi(\mathbf{r})$, which describes the Gaussian standing wave structure of the mode. Given the dependence of the intensity on the atomic position, it should in principle be possible to follow individual atomic trajectories. So, for example, oscillations in the transmitted intensity, which were observed by Mabuchi *et al.*, may reflect atomic motion along the cavity standing wave, with each value of the intensity corresponding to a particular position of the atom (modulo one wavelength). The possibility of such atomic position measurements via measurements of the transmitted field has been theoretically investigated by a number of authors [4–7] for similar cavity QED systems, although typically for far off-resonant excitation or for weak resonant excitation, such that atomic spontaneous emission does not figure significantly. In particular Quadt *et al.* [8] showed that in such a regime it should be possible to continuously monitor the atomic position. In contrast to these earlier works, the present analysis, and indeed the experiment of Mabuchi *et al.*, focuses on a (resonant) regime in which repeated absorption and spontaneous and stimulated emission all play a major role and strongly influence the atomic motion within the cavity mode. This must in turn have a strong influence on the intensity signal from which one might attempt to extract information about the atomic position or momentum. It is therefore essential to have effective models for the atomic motion in the cavity field.

This paper presents two approaches to modeling the motion of an atom in a single mode cavity within the rotating-wave and two-level-atom approximations. The first employs a semiclassical theory in which the atom is treated as a point particle subject to a position-dependent force and momentum diffusion rate. This is a standard method used to model laser cooling and other mechanical effects of light in classical fields [9–13]. The second approach is to quantize the external atomic coordinates as well as the internal atomic coordinates and the cavity mode. This is very challenging since the resulting Hilbert space is so large. A quantum Monte Carlo wave-function simulation of the system is presented that discretizes the atomic momentum in units of the photon momentum. This is an extension of the treatment of Mølmer *et al.* [14], who do not quantize the cavity field. In Sec. II we discuss the master equation to be used in the rest of the paper; the semiclassical approximations to this master equation are treated in Sec. III. Section IV develops an approximation that treats the cavity field as a classical standing wave for the purposes of calculating the effect of the field on the atomic motion, but varies the intensity of the standing wave depending on the atomic position. This allows single atomic trajectories to be simulated with greater efficiency; the results of these simulations are given in Sec. V, while the

1050-2947/97/56(1)/833(12)/$10.00

effect of achievable detection bandwidths is considered in Sec. VI. Section VII considers the effect of cavity field detunings on the atomic motion within the semiclassical approximation and the possibility of obtaining reduced atomic velocities. Section VIII proposes a simple scheme to recover velocity information from the intensity signal given a sufficiently broad detection bandwidth. Finally, Sec. IX presents the results of fully quantum-mechanical Monte Carlo wave-function simulations of the system and compares these results with those obtained in the semiclassical approximation.

II. THE MASTER EQUATION

The Hamiltonian for a two-level atom interacting with a single mode of the electromagnetic field in an optical cavity using the electric dipole and rotating-wave approximations (in the interaction picture with respect to the laser frequency) is

$$H=\frac{\mathbf{p}^2}{2m}+\hbar(\omega_0-\omega_L)\sigma_+\sigma_-+\hbar(\omega_c-\omega_L)a^\dagger a+i\hbar g_0\psi(\mathbf{r})\times(a^\dagger\sigma_- -\sigma_+ a)+\hbar E(a^\dagger+a). \tag{2.1}$$

The atomic transition frequency is ω_0, the cavity has a resonance at the frequency ω_c, and the system is driven by a coherent (laser) driving field of intensity E and frequency ω_L. In this paper we will always consider the case where the cavity is resonantly driven ($\omega_c=\omega_L$) with a possible detuning of the light field from the atomic resonance ($\Delta=\omega_L-\omega_0$). The cavity mode function is $\psi(\mathbf{r})=\cos(k_Lx)\exp[-(y^2+z^2)/w_0^2]$, describing the Gaussian standing-wave structure of the field in the Fabry-Pérot cavity; for the experimental cavity the mode waist is $w_0\simeq 45$ μm and the optical wavelength is $\lambda_L=852.359$ nm for the cesium transition employed ($k_L=2\pi/\lambda_L$).

Dissipation in the system is due to cavity losses and spontaneous emission. By treating the modes external to the cavity as heat reservoirs at zero temperature it is possible to derive the standard master equation for the density operator of the system [15] ρ,

$$\frac{\partial\rho}{\partial t}=\frac{-i}{\hbar}(H\rho-\rho H)+\kappa(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a)+\frac{3\Gamma}{8\pi}\int d^2\hat{\mathbf{k}}\sum_{\boldsymbol{\varepsilon}}\langle\mathbf{u}_d\cdot\boldsymbol{\varepsilon}\rangle^2\exp(-ik_L\hat{\mathbf{k}}\cdot\mathbf{r})\sigma_-\rho\sigma_+\times\exp(ik_L\hat{\mathbf{k}}\cdot\mathbf{r})-\frac{\Gamma}{2}(\sigma_+\sigma_-\rho-\rho\sigma_+\sigma_-). \tag{2.2}$$

The decay of the upper atomic level to the lower is given by $\Gamma/2\pi=5$ MHz, while the cavity field decay constant is $\kappa/2\pi=3.5$ MHz. These are both less than the coupling strength $g_0/2\pi=11$ MHz, corresponding to the strong-coupling regime. The third term describes the effect of spontaneous emission on the atomic motion, $\mathbf{u}_d$ is a unit vector in the direction of the atomic dipole moment, $\hat{\mathbf{k}}$ is a unit vector in the direction of an emitted photon. In this paper the polarization of the light in the cavity field, ε, will be taken to be circular, the atomic transition will be taken to be $\Delta J=1$, and the atoms will be optically pumped into a two-level system before the atom enters the cavity.

The intensity of the light transmitted through the cavity when the laser is on resonance with the atomic transition and the cavity mode is much reduced if the factor $g_0\psi(\mathbf{r})$ is large, that is, when the atom is close to an antinode of the field. Thus it is possible to some extent to track the motion of the atom in the cavity mode using information about the transmitted intensity. The intensity of light in the cavity for a particular value of the coupling constant was calculated by finding steady-state solutions to the master equation where the atom was assumed to be stationary [16]. In this paper the cooling of the atoms prior to entering the cavity effects a separation of time scales of the dynamics of the external degrees of freedom of the atom and the other degrees of freedom in the problem. The frequency with which the atom passes through wavelengths of the standing wave in the cavity is much less than the other frequencies involved,

$$\frac{1}{g}\nabla g(\mathbf{r})\frac{d\mathbf{r}}{dt}\ll g,\kappa,\gamma. \tag{2.3}$$

A truncated basis of Fock states can be used to model this system, since the cavity is typically driven such that only a few photons on average are present in the cavity. For the experimental parameters the transmitted intensity is reduced to around a tenth of its empty cavity value when the atom is at an antinode of the standing wave along the cavity axis. On the other hand, when the atom is introduced at a node the transmitted intensity is unchanged from its empty cavity value. Thus the oscillations in transmitted intensity observed by Mabuchi *et al.* [2] are taken to correspond to motion of the atom along the cavity axis.

III. SEMICLASSICAL EQUATIONS OF MOTION

This system differs from that usually considered in the literature on the semiclassical equations of motion for a two-level atom. The atom is normally assumed to be in an electromagnetic field, usually either a standing or a traveling wave of laser light, which is unaffected by the presence of the atom. In the system under consideration the intensity of light in the mode is strongly dependent on the atomic position and the electromagnetic field is a driven mode in a high finesse cavity excited typically with only a few photons; in principle, a fully quantum-mechanical treatment of the cavity mode field is necessary. Moreover, the atomic velocity inside the cavity will initially be very small; the mirrors of the cavity are about 3 mm in diameter and the cavity is only about 100 μm long, thus atoms reaching the center of the cavity mode must have very small velocities along the cavity axis, around 1 cm/s or less. However, the momentum kick associated with the absorption or emission of a single photon results in a change of velocity for a cesium atom of around 0.3 cm/s. Since the atom is on resonance with the atomic transition, the effects of repeated spontaneous and stimulated emission and of absorption should quickly lead to much increased velocities of the atom. This will allow the use of the semiclassical approximations, which assume velocities significantly greater than the recoil velocity, at least once the atom has been in the cavity mode for some time.

The system is in the range in which the semiclassical approximation is good. The dimensionless quantity $\hbar^2 k_L^2/2m\hbar\Gamma$, which must be small in the semiclassical approximation [17,13] is of the order of 10^{-3} for this cesium transition. Thus, the recoil energy for the atom is much less than the natural width and so after the emission of a single photon the atom will still be on resonance with the transition. This ensures that a constant atomic velocity approximation can be made when calculating the force and momentum diffusion coefficient and that it is possible to define the atomic position within a wavelength while defining the velocity sufficiently well that the Doppler shift of the transition frequency is well defined on the scale of the atomic linewidth. It is also expected for our system that the atom will very quickly acquire momentum several times greater than $\hbar k_L$ so that individual momentum kicks as a result of emission or absorption have little effect on the total momentum. Mølmer *et al.* found very good agreement between the results of the semiclassical treatment and a fully quantum calculation in the range of $\mathbf{p}/\hbar k_L$ that is relevant to this experiment [14].

Heisenberg's equation of motion gives the force operator for the Hamiltonian (2.1),

$$\mathbf{f}=\frac{d\mathbf{p}}{dt}=\frac{1}{i\hbar}[\mathbf{p},H]=-i\hbar g_0 \nabla\psi(\mathbf{r})(a^\dagger\sigma_- - \sigma_+ a), \qquad (3.1)$$

omitting fluctuations in the force due to spontaneous emmission. In the semiclassical approximation the atom is treated as a point particle located at $\langle\mathbf{r}\rangle$ with momentum $\langle\mathbf{p}\rangle$ moving subject to a force $\langle\mathbf{f}\rangle$. The fluctuations in this force due to spontaneous emission and atomic dipole fluctuations are modeled by calculating a momentum diffusion coefficient for a Fokker-Planck equation describing the distribution of atoms in phase space, where

$$D=\frac{1}{2}\lim_{t\to\infty}\frac{\partial}{\partial t}[\langle\mathbf{p}^2(t)\rangle-\langle\mathbf{p}(t)\rangle^2]. \qquad (3.2)$$

This can be expressed in terms of the time integral of the force covariance [11],

$$D=\lim_{t\to\infty}\mathrm{Re}\int_0^\infty\langle\mathbf{f}(t),\mathbf{f}(t-\tau)\rangle d\tau. \qquad (3.3)$$

These quantities are calculated by assuming that the atomic velocity $\mathbf{v}$ is constant; that is, the mass is treated as infinite. The resulting force and diffusion are used to describe the motion of any particular collection of atoms. In the regime of validity of the semiclassical theory the atomic velocity will change slowly enough, due to the resulting forces, that this will provide a good approximation for the atomic motion. The largest contribution to $\nabla\psi$ will be from the rapidly varying cosine factor due to the standing wave along the cavity axis, so the greatest forces and momentum diffusion coefficient will be along this axis. It will be sufficient to assume that the atom is roughly stationary in the other two dimensions. The coupling can be redefined to accommodate different positions of the atom in the Gaussian profile of the mode, thus $g=g_0\exp[-(y^2+z^2)/w_0^2]$ and

$$H_{sc}=-\hbar\Delta\sigma_+\sigma_- + \hbar E(a^\dagger+a)+i\hbar g\cos(v_x t)\times(a^\dagger\sigma_- - \sigma_+ a). \qquad (3.4)$$

There are several ways to calculate $\langle\mathbf{f}\rangle$ and D for this Hamiltonian. Fourier components of the steady-state density matrix of the resulting master equation can be calculated by a matrix continued fraction technique since the Liouvillian has only sinusoidal time dependence (see, for example, [18]). From the steady-state density matrix it is straightforward to find the force and transmitted intensity as a function of time and therefore position. A further matrix continued fraction allows the momentum diffusion coefficient to be calculated [13]. However, this calculation is prohibitively long when it is necessary to calculate these quantities for a range of velocities over the entire Gaussian profile of the mode, and uses very large amounts of computer memory if more than a few Fourier terms are needed. The matrix continued fraction technique does have the advantage of giving the cycle averages of these quantities as the first Fourier component, these can be difficult to find since values of both the force and diffusion vary widely over a wavelength and accurate averages can be difficult to obtain. A practical alternative is to integrate the master equation numerically for the truncated basis from some initial density matrix that is close to the steady state until the steady state is reached and evaluate the force and transmitted intensity as a function of time. To calculate the momentum diffusion coefficient it is necessary to use the quantum regression theorem and integrate the master equation a second time using $\mathbf{f}\rho$ as an initial condition in order to find the force correlation function. The integral of the force correlation gives the momentum diffusion coefficient according to Eq. (3.3). For a stationary atom the force is not time dependent and the calculation of the momentum diffusion coefficient is relatively simple. The momentum diffusion coefficient for zero velocity and with other parameters close to those realized in the experiment is plotted in Fig. 1; note that the dipole force is zero since the atom is on resonance. Figure 2 plots the force and momentum diffusion coefficient for the same parameters where the cavity and driving fields are slightly detuned from the atomic resonance.

IV. CLASSICAL STANDING-WAVE-FIELD APPROXIMATION

In order to develop a realistic computer simulation for the system it is necessary to find an approximation to the full calculation given above that is capable of quickly calculating the force and momentum diffusion coefficient for a wide range of velocities and values of g. The cavity is not very different from a Gaussian standing wave in free space created by two counterpropagating lasers, however, the atomic position has a very strong effect on the intracavity intensity, while the laser beams can be assumed to be undepleted. The approach taken here is to treat the standing wave as a laser but to modify the intensity of that standing wave according to the intracavity intensity appropriate to the atomic position. In this way the cavity mode is treated as a c-number constant; for any given position of the atom in the cavity a is replaced by $\langle a\rangle$, reducing the size of the Hilbert space roughly tenfold in this case. Standard treatments of the force

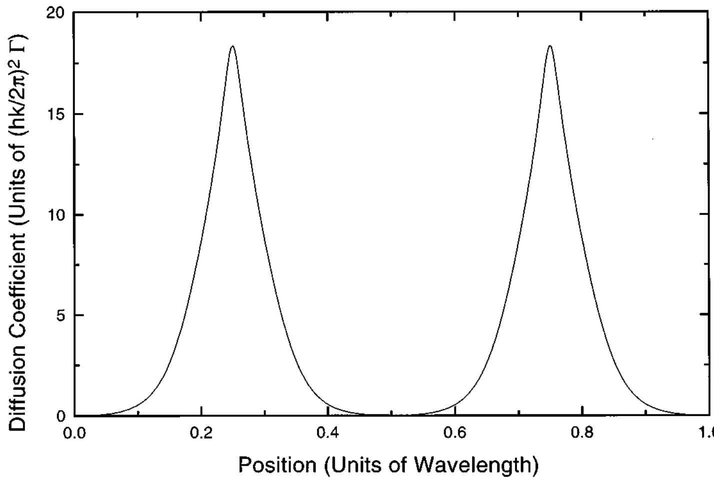

FIG. 1. The semiclassical momentum diffusion coefficient, along the cavity axis, of an atom strongly coupled to a few photon optical cavity mode. The atom is on resonance with the optical field with parameters $(g_0,\kappa,\Gamma)=(11,3,5)/2\pi$ MHz and the cavity is driven such that the mean photon number in the empty cavity is 2.

and momentum diffusion coefficient assume that the field is in a particular coherent state [11], or is a classical driving field [9]; that is, they also effectively treat the operator a as a c-number constant and so these works are immediately relevant to the present situation.

As a first approximation it is possible to find, for zero atomic velocity, the intracavity intensity appropriate to a given atomic position, which is easy to calculate since the resulting master equation has no time dependence, and then find the force and momentum diffusion coefficient of a two-level atom in a classical standing wave of this intensity; this is valid as a result of the separation of time scales mentioned above, the motion of the atom is not sufficiently fast to strongly perturb the cavity field. The force (setting $\Omega=g\langle a\rangle$ where $\langle a\rangle$ is the field expectation value calculated from the full master equation)

$$\langle\mathbf{f}\rangle=-i\hbar\nabla\Omega(\langle\mathbf{r}\rangle)\langle\sigma_- - \sigma_+\rangle \tag{4.1}$$

can be found by solving the optical Bloch equations for a particle of velocity $\mathbf{v}$; thus cavity decay must presumably be slow enough to not significantly affect the atomic response except insofar as it serves to establish the steady-state field. Moreover the atom and field operators effectively decorrelate, $\langle a\sigma_+\rangle\simeq\langle a\rangle\langle\sigma_+\rangle$, which must require that g is not too large. The effect of these approximations can be found by comparing the force and momentum diffusion coefficient calculated in this way with those calculated from the original semiclassical approximation in Sec. III. This comparison is made in Fig. 3 for the force and momentum diffusion coefficient of a detuned cavity. The agreement of the diffusion in the resonant case is similarly good, which justifies their use in simulations for the parameters of the experiment of Mabuchi *et al.*

If significant velocities are to be modeled this new problem requires a continued fraction technique for the Fourier components of the force [9]. The time correlations of the force obey similar equations to atomic operators [19], and this allows Fourier components of the diffusion to be found by using matrix continued fractions as before [13]. The force and momentum diffusion coefficient of an atom in a laser field have been solved analytically and written in a closed form by Gordon and Ashkin [11], to first order in the $k_L v/\Gamma$. This means that the force and momentum diffusion coefficient can be computed very fast for a wide range of values of g and of the detuning $\Delta=\omega_L-\omega_0$ over the range of velocities of interest in the experiment. For zero detuning when the atom is in a standing wave the dipole force is zero and the diffusion has a particularly simple form:

$$D_{xx}=2\hbar^2k_L^2\Omega^2\sin^2(k_Lx)/\Gamma+\hbar^2\Gamma\langle\mathbf{k}_L\cdot\hat{\mathbf{x}}\rangle^2\langle\sigma_+\sigma_-\rangle/2. \tag{4.2}$$

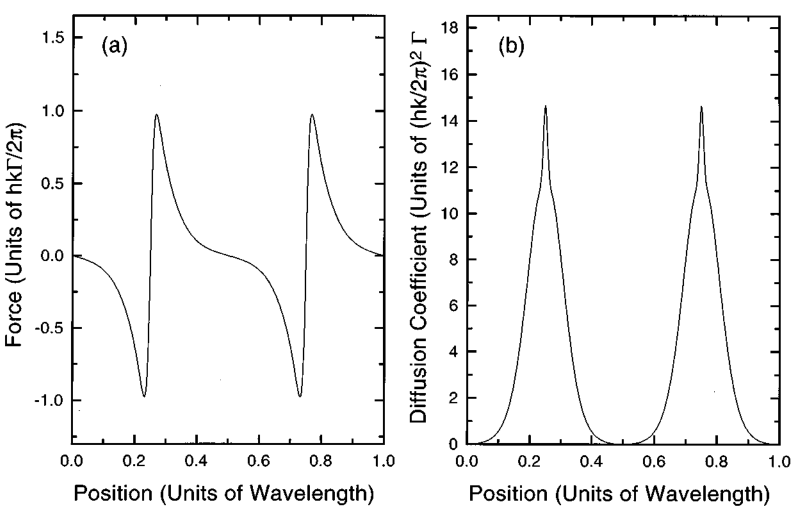

FIG. 2. (a) The semiclassical force and (b) momentum diffusion coefficient, along the cavity axis, of an atom strongly coupled to a few photon optical cavity mode. The optical field is detuned from the atom with parameters $(g_0,\kappa,\Gamma)=(11,3,5)/2\pi$ MHz, the detuning is $\Delta=-0.3\Gamma$, and the cavity is driven such that the mean photon number in the empty cavity is 2.

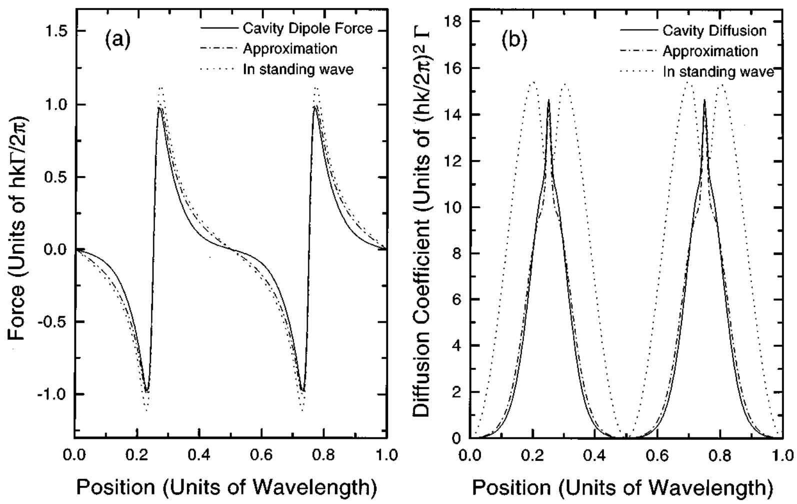

FIG. 3. Comparison between (a) the force and (b) the momentum diffusion coefficient of an atom strongly coupled to a few photon optical cavity mode calculated from the full master equation and in the approximation used in the simulations. The optical field is detuned from the atom with parameters $(g_0,\kappa,\Gamma)=(11,3,5)/2\pi$ MHz, the detuning is $\Delta=-0.3\Gamma$, and the cavity is driven such that the mean photon number in the empty cavity is 2.

The factor $\langle \mathbf{k}_L \cdot \hat{\mathbf{x}} \rangle^2$reflects the distribution of spontaneous emission along the cavity axis [19]. For the simulation it was assumed that the atom was optically pumped into a two-level system by circularly polarized light on a $\Delta J=1$ transition and the values of $\langle \mathbf{k}_L \cdot \hat{\mathbf{x}} \rangle^2$ were chosen accordingly. The first term is the most important contribution to the momentum diffusion coefficient along the cavity axis. In the other directions the second term (spontaneous emission) is the more significant as the first is reduced by a factor $(1/w_0 k_L)^2$ [9] due to the slower Gaussian variation of the field in those directions.

In Fig. 4 the momentum diffusion coefficient for zero detuning with parameters very similar to those achieved in [2], when the atom is near the center of the Gaussian profile of the mode, is plotted for the atom in the cavity, calculated as in Sec. III and in the classical standing-wave approximation. For comparison, the momentum diffusion coefficient for a two-level atom in a laser standing wave of the same intensity is also shown. Note that these quantities have very different behaviors when the atom is in a cavity as opposed to a laser beam and the good agreement of the full calculation with the approximation of the cavity as a standing wave of position-dependent intensity.

V. SIMULATION OF ATOMIC MOTION

A Langevin-equation-like simulation was run to plot the progress of a single atom through the cavity. At each time step the field intensity in the cavity $\langle a \rangle$ is calculated as for a stationary atom and this is used to calculate the force and momentum diffusion coefficient on the particle at that point in time, as in Sec. IV. A vector of Gaussian random variables, **W**, of mean zero and standard deviation 1 is then generated and each component v_α of the atomic velocity altered at the ith time step according to

$$v_\alpha^{i\prime}=v_\alpha^i+\sqrt{2D_{\alpha\alpha}^i\Delta t/m^2}W_\alpha^i, \tag{5.1}$$

where Δt is chosen such that the cavity intensity changes little in a single time step. For the plane perpendicular to the cavity axis spontaneous emmission was the dominant contribution to the momentum diffusion coefficient and this was evaluated as in the second term of Eq. (4.2), which refers to the diffusion for the cavity axis itself. The position and velocity of the atom are then moved forward under the standard kinematic equations

$$\mathbf{x}^{i+1}=\mathbf{x}^i+\mathbf{v}_\alpha^{i\prime}\Delta t+\tfrac{1}{2}\mathbf{f}^i(\Delta t)^2/m,$$
$$\mathbf{v}^{i+1}=\mathbf{v}_\alpha^{i\prime}+\mathbf{f}^i\Delta t/m. \tag{5.2}$$

The atoms in each simulation began just above the cavity mode. They were evenly distributed over a wavelength of the standing wave along the cavity axis and were normally dis-

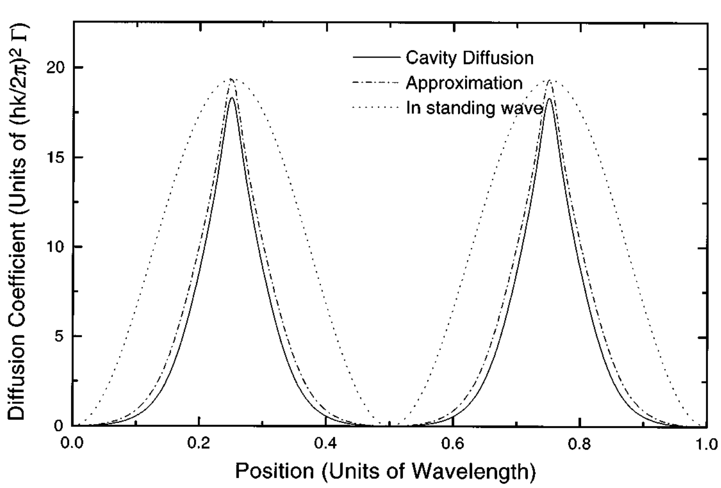

FIG. 4. Comparison of the semiclassical momentum diffusion coefficient in a standing wave in free space and strongly coupled to a few photon optical cavity mode. The atom is on resonance with the optical field with parameters $(g_0,\kappa,\Gamma)=(11,3,5)/2\pi$ MHz and the driving such that the mean photon number in the empty cavity is 2.

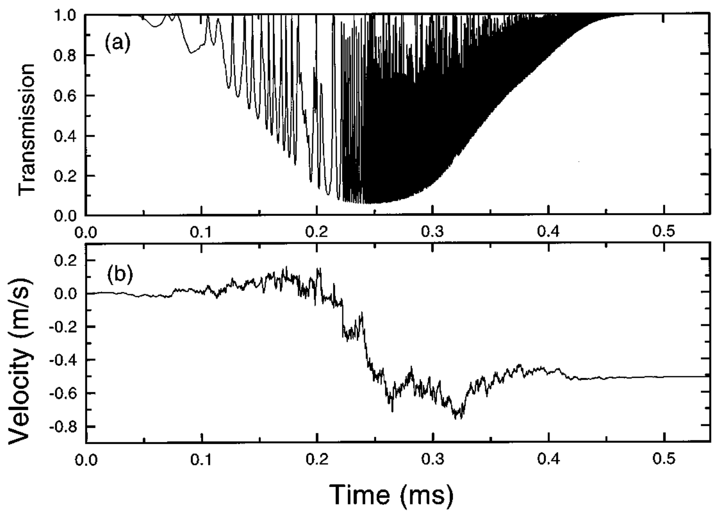

FIG. 5. Plot of (a) cavity transmission and (b) atomic velocity along the cavity axis for a simulation of atomic motion in the cavity. The atom is on resonance with the cavity and the driving field, $(g_0,\kappa,\Gamma)=(11,3,5)/2\pi$ MHz, and the cavity is driven such that there are on average two photons in the cavity when the atom is at a node of the standing wave.

tributed across the Gaussian mode profile along the y axis. The vertical velocity component was appropriate to that gained from the 7-mm drop from the MOT to the cavity mode and the horizontal velocity components were taken to be normally distributed about zero velocity with a standard deviation of 1 cm/s. This is the order of velocities that would make it from the MOT to the cavity mode.

The on-resonance case has zero dipole force but the largest momentum diffusion coefficient as a result of fluctuations in the atomic dipole and of spontaneous emission. Thus, it was found that the velocity of the atoms could increase very quickly, with atoms attaining velocities of the order of 1 m/s. The fluctuations of the velocity were very large and the atom would frequently pass through regions of small velocity. The transmitted intensity signal shows very fast oscillations as a result of high velocities, which are at first sight not in agreement with the results given in [2]. Figure 5 gives a typical example of the simulations along with a plot of the velocity of the atom along the cavity axis. Note that the atomic velocity remains small until the atom approaches the center of the mode, where the increased field intensity leads to a very large momentum diffusion coefficient and in this case a dramatic change in the atom's velocity. As the atom leaves the mode the momentum diffusion coefficient again decreases and the velocity fluctuations are reduced such that the atom leaves the mode with a much larger velocity along the cavity axis than initially. This is the origin of the very asymmetric transmitted intensity signal in Fig. 5 and in many other runs of the simulation. Moreover the Gaussian envelope of the transmitted intensity signal, which reflects the Gaussian profile of the mode function, is slightly modified by the effects of spontaneous emission on the atomic velocity perpendicular to the cavity axis; more pronounced examples of this will be seen in Fig. 6 below.

VI. THE EFFECT OF FINITE DETECTION BANDWIDTH

The bandwidth of the heterodyne apparatus, which measures the intensity of the light transmitted through the cavity, is 100 kHz in the experiment of Mabuchi *et al.*, which is equivalent to a period of the intensity oscillations of 10^{-5} s or to atomic velocities of only 4 cm/s. Higher frequencies in the intensity signal will not be detected and the time-intensity information obtained by the measurement will be some average of an infinite bandwidth signal over short time spans. Where the atomic motion happens to be slow the intensity measurement follows the changes in the transmitted intensity with atomic position and the signal has structures like those found in the experiment. Where the atomic motion is faster the oscillations in intensity due to the motion of the atom along the standing wave are averaged out in the observed signal. The variation in intensity is then due to the slowly varying Gaussian profile of the cavity mode.

To model this effect the intensity data were passed through a digital low-pass Butterworth filter with a cutoff frequency of 100 kHz and then plotted against time. The order of the filter used is limited by the accuracy with which the filtering could be computed given the sampling rate of the simulations. The resulting curves, which are plotted in Fig. 6 along with the original infinite bandwidth intensity signal, are very like those observed experimentally and plotted in [2]. These plots show structure where the atom has chanced to slow down as a result of its random walk in momentum space. Increasing the bandwidth should result in signals with more and more structure even without a change to any experimental parameters. Note, however, that the structures in the signals in [2] could also be the result of effects outside the two-level approximation used in these calculations. There was no optical pumping of the atoms prior to entering the cavity, which would trap the atoms in a two-level system. This will mean that transitions between the different sublevels of the ground and excited states that have different couplings to the field will have an effect on the atomic motion. Moreover asymmetry of the signals could be due to optical pumping of the atom into another hyperfine ground state that is not coupled to the cavity field [2].

VII. COOLING IN THE CAVITY

A possible extension of the experiment is to detune the electromagnetic field from the atomic transition. This will allow a nonzero mean dipole force, which, if the detuning is in the right direction, will keep the atomic velocity small. In the on-resonance case the absence of a dipole force means

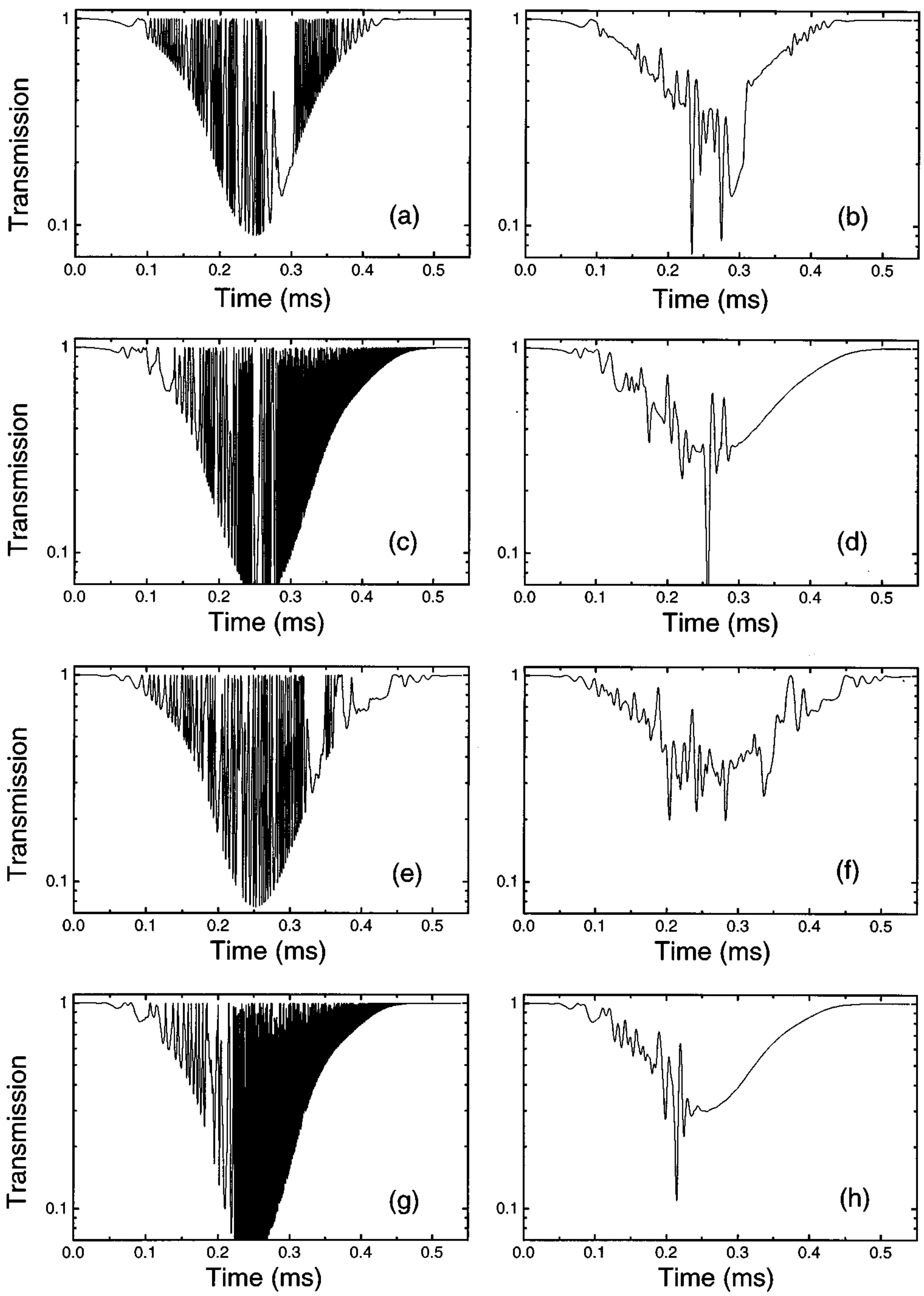

FIG. 6. (a), (c), (e), and (g) Transmitted intensity signal for several simulations. (b), (d), (f), and (h) The transmission signal passed through a digital low-pass Butterworth filter of fourth order with cutoff frequency 100 kHz. This simulates the current experimental bandwidth of Mabuchi *et al.* [2]. The atom is on resonance with the cavity and the driving field, $(g_0,\kappa,\Gamma)=(11,3,5)/2\pi$ MHz, and the cavity is driven such that there are on average two photons in the cavity when the atom is at a node of the standing wave.

that the atomic velocity varies more and more wildly with time and the simulations show that atoms can emerge from the cavity with velocities as high as 1 m/s along the cavity axis. Where there is a cooling force, its interplay with the momentum diffusion coefficient, that is fluctuations in the force, would lead to some steady-state velocity distribution for the atom in the cavity if it were confined to a given point in the Gaussian mode profile, just as the force and diffusion determine the eventual temperature of a system of atoms in the case of Doppler cooling.

The simplest adjustment is to negatively detune the laser from the atomic transition and then tune the cavity to the laser beam, just as in the current experiment. The force and momentum diffusion coefficient were calculated in Sec. III and plotted for the parameters used in the previous simulations and $\Delta=-0.3\Gamma$; in Fig. 3 they are plotted in comparison to those relevant to a standing wave in free space and in the approximation used in the simulations. Once again the picture of an atom moving in a classical standing wave of position-dependent intensity gives a very good characterization of the momentum diffusion coefficient of an atom strongly coupled to a cavity mode. The value of the detuning was chosen to give the largest cycle-averaged cooling force. By tuning the cavity to the driving laser field this scheme retains a very significant drop in transmitted intensity when the atom is in the cavity.

Simulations were run for this detuned case using both the matrix continued fraction technique and the small velocity expressions given by Gordon and Ashkin for the force and momentum diffusion coefficient of a two-level atom in a

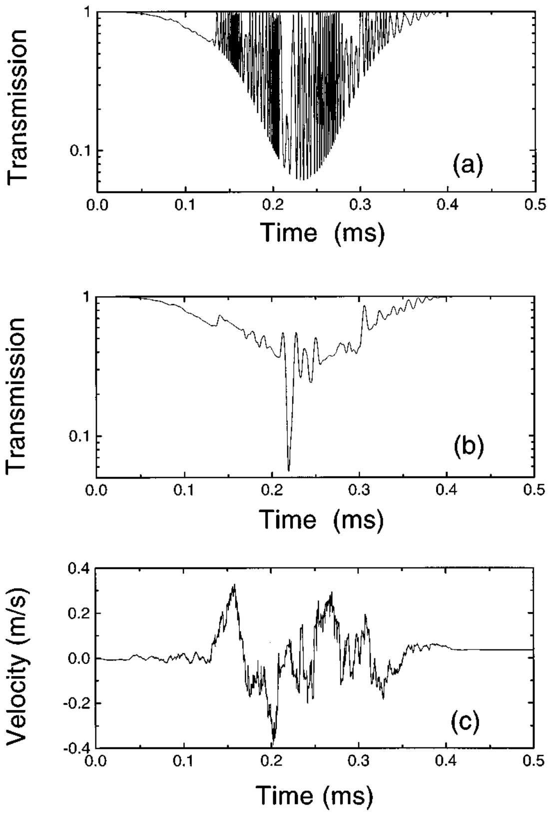

FIG. 7. Results of a typical simulation with laser detuning $\Delta = -0.3\Gamma$ and other parameters as in Fig. 6. (a) Cavity transmission. (b) The transmission signal passed through a digital low-pass Butterworth filter of fourth order with cutoff frequency 100 kHz. This simulates the current experimental bandwidth of Mabuchi *et al.* [2]. (c) The atomic velocity along the cavity axis.

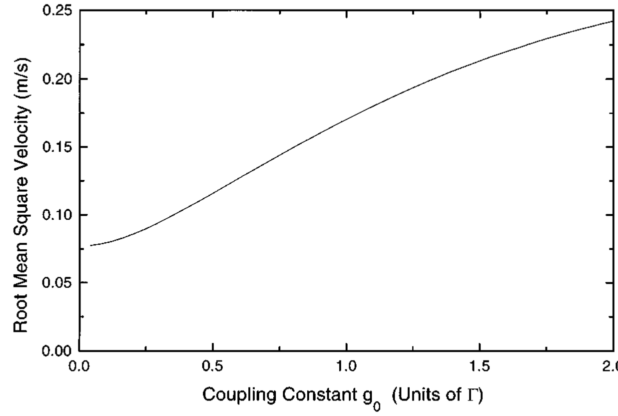

FIG. 8. Steady state root-mean-square velocities of the atom confined to move only along the cavity axis.

classical standing wave or coherent state. A problem with the matrix continued fraction calculation was the large number of Fourier terms necessary to get an accurate description of the diffusion around a node, which made these simulations significantly longer. However, the two methods give similar results. Atomic velocities in the center of the mode reached little more than 0.4 m/s, which is significantly slower than those observed where there is no cooling force, although not by an order of magnitude. However, the atoms were on average much slower as they left the cavity mode. The results for a single run of the simulation are plotted in Fig. 7, which, by comparison with Fig. 6, demonstrates this feature of small final velocities.

There is a simple way to understand why the atoms are likely to have smaller velocities and smaller velocity fluctuations as they leave the cavity mode. In the standard theory of Doppler cooling the final velocity distribution of atoms in the laser field is the result of the competing effects of momentum diffusion coefficient acting to spread the atomic velocity distribution and the cooling force acting to slow the atoms. At low velocities the cycle averaged force F is linear with velocity, $F = -\alpha v$, and this results in a Gaussian velocity distribution with temperature

$$k_B T = m\langle v^2\rangle = D/\alpha, \tag{7.1}$$

where D is the zero-velocity cycle-averaged momentum diffusion coefficient (see, for example, [13]). Although in this system there is only one atom, if it were confined to move only in one dimension it would reach a steady state in which its velocity had this Gaussian distribution in time. Such a single atom cannot be said to have a temperature, but the mean-squared velocity predicted by Eq. (7.1) does have a physical meaning. Although the real atom moves across the Gaussian profile of the cavity mode it does so slowly and the steady-state root-mean-square (rms) velocities appropriate to the value of g at each point will be a good indication of the velocities to be expected in different parts of the mode. To find these steady-state rms velocities as a function of g, the force on the atom in the cavity for a range of small velocities was calculated and cycle averaged in order to calculate α. The zero-velocity cycle-averaged momentum diffusion coefficient was evaluated and Eq. (7.1) used to calculate the steady-state rms velocities for the values of $g_0,\Gamma,\Delta,\kappa,E$, used in the simulations. Note that this was done in the approximation of Sec. IV, which was used in the simulations. The results are plotted in Fig. 8 as a function of g. In agreement with the simulations, this predicts large rms velocities for larger values of g, that is, in the most intense parts of the Gaussian mode. As the atom falls out of the mode and down the intensity profile the atom is likely to have much lower velocity and lower velocity fluctuations. The largest mean-squared velocity is equivalent to a temperature of a system of atoms of around 1 mK and at low intensity the atomic velocity distribution tends towards the Doppler limit for cesium of 127 μK or rms velocity 8.9 cm/s. The value on the graph is slightly lower than this, but this is most likely the result of the approximations made in calculating the force and momentum diffusion coefficient. The Doppler limit, which depends only on the atomic mass and linewidth, would not be improved upon by cooling in the cavity unless more atomic levels and cavity mode polarizations were considered.

In the experiment of Mabuchi *et al.* the laser driving of the system must be sufficiently intense to ensure a good signal-to-noise ratio in heterodyne detection of the transmitted intensity. It appears that the resonant and intense laser driving quickly results in atomic velocities that are much larger than when the cold atoms reach the cavity mode even where the field is detuned from atomic resonance. If this is the case the velocities that are attained should leave that information in the intensity signal that is measured, as discussed below. However, this calculation suggests that if the driving field could be altered quickly enough after an atom is detected in the cavity its velocity would be reduced by ramping down the driving intensity in this detuned situation. It is possible to change the driving laser intensity and detuning on a time scale of the order of 100 ns [20], which is much less than the tens of microseconds that the atom remains in the very center of the mode so it is not unrealistic to investigate this possibility; this is also somewhat shorter than the time scale of the atomic motion along the standing wave. Interesting possibilities would then include trapping this single atom near an antinode of the standing wave by detuning the laser field and the cavity much farther from resonance (the laser intensity would have to be increased in this process). This would allow a transition from the present regime that is sensitive to the presence of a single atom to the more frequently considered case of a single atom trapped at an antinode of a far detuned optical standing wave, in which as noted above it has been shown that it would be possible to track the position of the atom through measurement of the quadrature phase of the transmitted light [8]. Such a system has been investigated by Wong *et al.*, who have shown that it may be possible to ''juggle'' the atom, keeping it at an antinode and thus overcoming the effects of gravity [21]; in some circumstances it may even be possible to map out the atomic Wigner function [22].

VIII. VELOCITY INFORMATION FROM THE TRANSMITTED LIGHT INTENSITY

Given an experimental measurement of the transmitted intensity output from the cavity, we would like to get as much information as possible about the system. For a stationary atom ideas similar to the classical theory of system identification have been applied to find the coupling g, which also relates to the atomic position, from the timing of cavity and atomic decays [23]. In this section we take rather a different approach, applying some signal processing techniques to get an estimate of the atomic velocity from the transmitted intensity signal.

It is not necessary to have an infinite bandwidth in the experiment to get good information about the atomic velocity along the cavity axis. With a bandwidth of 1 MHz it would be possible to measure atomic velocities of up to 0.4 m/s, which would allow the atom's velocity to be known for much of its time in the cavity. There is a simple scheme to recover this information from just the transmitted intensity signal. The frequency of the intensity oscillations can be calculated as a function of time and since one period of the signal is equal to the time it takes the atom to travel from node to node in the standing wave this frequency is proportional to the velocity along the cavity axis. To do this the signal is fast Fourier transformed and all the negative frequency components are set to zero. Then the inverse transform is taken, giving the complex analytic signal that rotates in the complex plane around the time axis at an angular velocity, which is the frequency of the transmitted intensity signal, just as the frequency of the function $\exp(i\omega t)$ is simply the angular velocity of its argument.

To achieve a good correspondence between the frequency of the complex analytic signal and the velocity that is recorded by the simulation it is necessary to remove the average value from the signal. Due to the Gaussian profile of the mode this average value is a slowly varying function of time and so this is most easily achieved by filtering out the smallest frequencies prior to taking the Fourier transform. The magnitude of the complex signal now reflects the magnitude of the oscillations in intensity, large in the center of the mode and small at the edges. It is also necessary to filter out high-frequency noise in the simulated signal. In the simulated signals, this noise could be the result of the discontinuous changes in velocity at each time step. However, in a real application it would still be necessary to low-pass filter in order to remove shot noise from the signal. This also serves the purpose of simulating a realistic experimental bandwidth, since measuring the transmitted intensity of the cavity with bandwidths larger than 1 MHz will not be achievable in the near future. The number of data points affects the rate at which particular frequencies cause the complex signal to rotate around the time axis; there must be enough data points that the frequencies that correspond to atomic motion cause rotation around the axis that is only a very small part of a full cycle for each time step. This means that the filters that can be used are low order and thus have large phase distortion, which is the information in the signal that we wish to preserve. Thus the signal is forward and reverse filtered to remove the phase distortion from the first pass; this process is noncausal and so was not considered in modeling the effects of real measurement apparatus but could easily be employed in the post-analysis of actual data. With these precautions it is possible to reproduce very closely, over a wide velocity range, the velocity curve generated by the simulation just from the transmitted intensity information.

Figure 9 gives an example of this procedure for the results plotted in Fig. 7. The infinite bandwidth intensity signal is first band-pass filtered. In this case a Butterworth filter of order 2 with pass band 4–800 kHz is used, although this can be optimized for a particular signal depending on the atomic velocity. After removing the negative frequencies the rate of change of the argument of this complex signal is compared to the actual velocity information obtained from the simulation. Note that very small velocities are not well modeled; this is due to the low-pass filtering, which is necessary to remove the average value of the signal, but which also removes information about very small atomic velocities. Where the atom is leaving the cavity the variations of intensity, and hence the magnitude of the complex signal, become much smaller and the model does not give an accurate value of the velocity. The argument of the signal is in this case very poorly defined. However, the velocity of the atom is reproduced remarkably faithfully for a wide range of velocities while the atom is in the most intense part of the mode.

IX. A QUANTUM MODEL OF THE MOTION: MONTE CARLO WAVE-FUNCTION SIMULATION

An alternative to the above semiclassical theory is to attempt to retain the quantization of the atomic external vari-

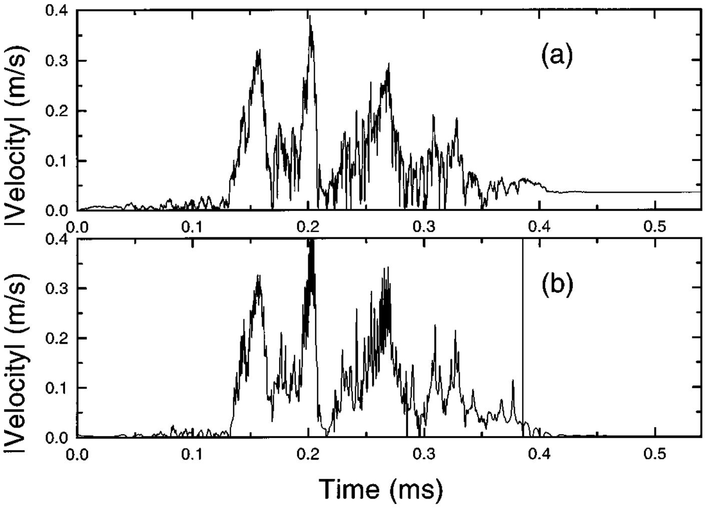

FIG. 9. Comparison of velocity information obtained from (a) the transmitted intensity signal with (b) the actual (absolute value of) velocity along the cavity axis in a single run of the simulation at detuning $\Delta = -0.3\Gamma$.

ables. This will be especially important in modeling the initial atomic motion since the atom is expected to be moving at only a few times the recoil velocity. We performed a momentum space Monte Carlo wave-function simulation for the system along the lines of Mølmer *et al.* [14]. Only one dimension was considered and the momentum was quantized in units of the photon recoil momentum and truncated at 200 possible momentum states; this allows atomic velocities of only around 0.3 m/s. The system obeys the master equation (2.2) where the position and momentum are no longer vector quantities. An unraveling of this master equation was chosen that corresponds to perfect photon counting of light emitted from the cavity mirror and out the sides of the cavity. This does not correspond directly to the experiment of Mabuchi *et al.*, which employs heterodyne detection; a stochastic Schrödinger equation would be necessary to correctly model the measurement backaction in this case [24]. Each spontaneous emission results in a momentum kick to the atom but due to the discretization of momentum the only possible kicks are $-\hbar k_L, 0, \hbar k_L$ along the cavity axis. The probability of these three collapses was assigned in the ratio $\frac{1}{5}:\frac{3}{5}:\frac{1}{5}$ as this is the best representation of the directional distribution of spontaneous emission (this is consistent with that chosen above for the distribution of spontaneous emission and assumes a $\Delta J = 1$ transition in a σ_+-polarized beam). Thus there are four possible collapses for the system:

$$C_1 = \sqrt{2\kappa}a, \tag{9.1a}$$

$$C_2 = \sqrt{\Gamma/5}\exp(-ik_L x)\sigma_-, \tag{9.1b}$$

$$C_3 = \sqrt{3\Gamma/5}\sigma_-, \tag{9.1c}$$

$$C_4 = \sqrt{\Gamma/5}\exp(ik_L x)\sigma_-, \tag{9.1d}$$

and this results in the non-Hermitian Hamiltonian for the Monte Carlo wave-function simulation

$$H_{\text{eff}} = H - \frac{i\hbar}{2}\sum_m C_m^\dagger C_m, \tag{9.2}$$

$$H = \frac{p^2}{2m} - \hbar\Delta\sigma_+\sigma_- + i\hbar g_0\cos(k_L x)(a^\dagger\sigma_- - \sigma_+ a) + \hbar E(a^\dagger + a).$$

The wave function for the system is evolved according to this non-Hermitian Hamiltonian until its norm is less than some randomly generated number between zero and one, at which point one of the four possible collapses takes place and the wave function is renormalized. These collapses are recorded as a classical record of the evolution and at regular intervals in the evolution expectation values of the transmitted intensity and atomic momentum are calculated from the wave function. The Fock state basis for the cavity mode was truncated as before at the tenth number state. Note that Mølmer *et al.* assume a classical laser field in their simulations of Doppler cooling, so once again the necessity of treating the cavity mode quantum mechanically differentiates this simulation from conventional Doppler cooling calculations. The atom was initially in the ground state; its momentum distribution was a Gaussian wave packet with initial mean momentum $3\hbar k_L$ (equivalent to a velocity of around 1 cm/s) and with a standard deviation of $\hbar k_L$. This is roughly equivalent to localization in around $\lambda/7$. Each individual trajectory was taken to approximate a possible path of a particular atom in the cavity given the idealized measurement scheme of perfect photodetection.

These simulations gave oscillations in the transmitted intensity very similar to those predicted earlier and the momentum information was also in good agreement with the predictions of the semiclassical theory. They were performed with the cavity field both on-resonance and detuned from the atomic transition, although in the on-resonance case long simulations were not possible due to the likelihood of the atom developing momenta too large for the basis of momentum states that was employed. In Fig. 10 the expectation value of the transmitted intensity and the atomic momentum are plotted for a 30 μs simulation for the on-resonance situation corresponding to the experiment of Mabuchi *et al.*, the atom quickly develops a significantly increased velocity and

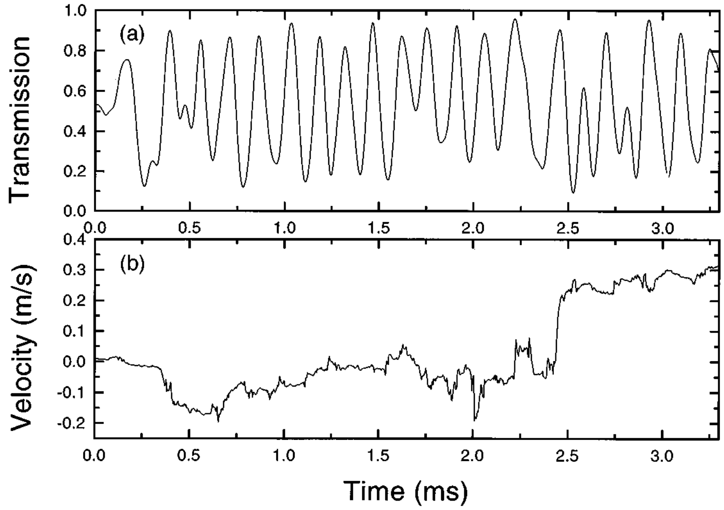

FIG. 10. Results of the Monte Carlo simulation for atom, cavity, and driving field on resonance. The (a) cavity transmission and (b) velocity of the atom are calculated from the expectation value of the transmitted intensity and momentum of the system at each time step. The parameters for the system are $(g_0,\kappa,\Gamma)=(11,3,5)/2\pi$ MHz and the cavity is driven such that there are on average two photons in the cavity when the atom is at a node of the standing wave.

the transmitted intensity shows oscillations very similar to those that resulted from the semiclassical simulations. Only 30 μs is shown since in that time the atom develops velocities that make the truncation of the momentum basis states invalid. Figure 11 shows the results of a simulation lasting 100 μs with the field detuned from the atomic resonance. This also has the expected range of velocities and oscillations in the transmitted intensity. The results of the simulation are very similar to the detuned case, in order to check the predictions of the semiclassical cooling theory it would be necessary to vary the coupling between the atom and the cavity during the simulation so as to model the motion of the atom through the cavity mode and observe the distinctive ''cooling'' of the atom, as in Fig. 7. Using a larger momentum basis and running the simulations for longer times would also help to distinguish the two cases, as that would allow larger velocities to develop in the case of resonant excitation. Finally Fig. 12 shows the result of an attempt to reconstruct the velocity of the atom from the transmitted intensity information as in Sec. VIII. This is obviously less successful than in the semiclassical case; each of the oversized peaks in the estimated velocity of the atom correspond to a decrease in the magnitude of the complex analytic signal at that point in time that leads, as noted above, to less accurate evaluation of the velocity.

While this model could be developed further (for instance, to include atomic motion in more than one dimension as suggested above) we have shown that the results of a fully quantum model of the motion are consistent with the earlier semiclassical analysis. It appears that a fully quantum model is not essential to give the basic features of the experiment of Mabuchi *et al.*

X. CONCLUSION

In this paper we have investigated two simple models for the motion of a single two-level atom in a single mode cavity with which it is nearly resonant. A study of the motion of the atom in the semiclassical approximation, common in the theory of laser cooling, shows that such an atom, even if initially very slow, will quickly develop large velocities and large velocity fluctuations. By filtering the transmitted intensity signals to model the actual bandwidth with which intensity is measured in the experiment of Mabuchi *et al.* we have produced signals that closely correspond to those reported in [2]. We have suggested that one way of limiting the effect of momentum diffusion coefficient in the resonant standing wave is to detune the cavity and the laser driving field so as to introduce a cooling force that will limit atomic velocities. In such a system the atom would be ''heated'' and then

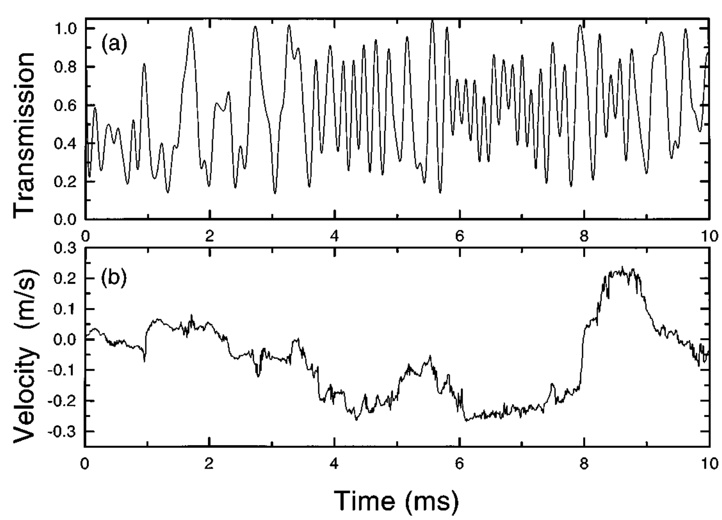

FIG. 11. Results of the Monte Carlo wavefunction simulation with a detuning of $\Delta=-0.3\Gamma$ and the other parameters as in Fig. 10. Note the different time scale from Fig. 10. (a) Cavity transmission. (b) Atomic velocity.

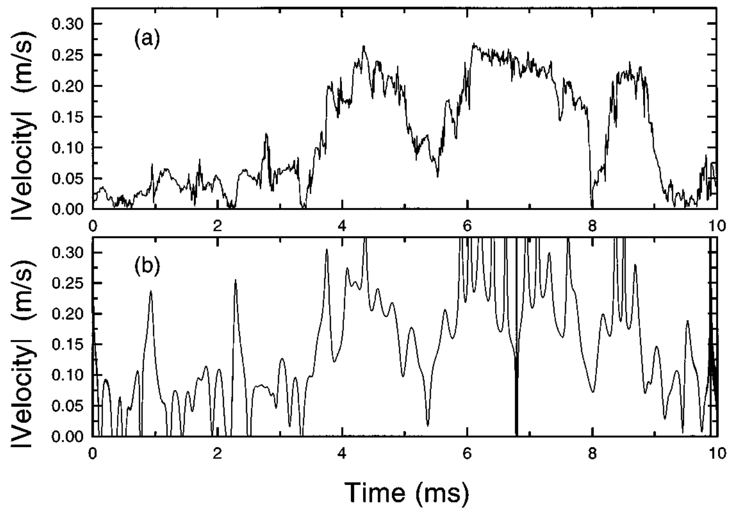

FIG. 12. Comparison of (a) the absolute value of the velocity recorded at each time step of the simulation of Fig. 11 with (b) the velocity information to be gained from the transmitted intensity signal. A low-pass filter of cutoff 700 kHz was applied to the signal and the average value subtracted before removing the negative frequency and finding the angular velocity of the argument of the resulting complex signal. The peaks in the signal each relate to a small value for the magnitude of the complex analytic signal, which makes the calculation of the angular velocity inaccurate.

''cooled'' as it passed through the cavity mode, and this kind of control of the atomic velocity could lead to further interesting experiments. A simple means of obtaining the velocity from the transmitted intensity signal is proposed, which would track the atomic velocities if the experimental bandwidth could be broadened. Finally a simulation of the atom cavity system that quantizes all the degrees of freedom for a one-dimensional model has been investigated and found to give very similar results.

ACKNOWLEDGMENTS

This project was supported by the Marsden Fund of the Royal Society of New Zealand, the University of Auckland Research Committee and the New Zealand Lotteries Commission. A.D. would like to thank the members of the Quantum Optics group at the California Institute of Technology for hospitality and many valuable discussions during his stay.

[1] *Cavity Quantum Electrodynamics, Advances in Atomic, Molecular and Optical Physics, Supplement 2*, edited by P. Berman (Academic, New York, 1994).
[2] H. Mabuchi, Q. A. Turchette, M. S. Chapman, and H. J. Kimble, Opt. Lett. **21**, 1393 (1996).
[3] Q. A. Turchette, R. J. Thompson, and H. J. Kimble, Appl. Phys. Lett. **60**, S1 (1995), observe drops in transmission in the so-called bad-cavity system with $\kappa \gg g^2/\kappa \gg \gamma$.
[4] P. Storey, M. J. Collett, and D. F. Walls, Phys. Rev. Lett. **68**, 472 (1992).
[5] M. Marte and P. Zoller, Appl. Phys. B **54**, 477 (1992).
[6] P. Storey, M. J. Collett, and D. F. Walls, Phys. Rev. A **47**, 405 (1993).
[7] G. Rempe, Appl. Phys. B **60**, 233 (1995).
[8] R. Quadt, M. Collett, and D. F. Walls, Phys. Rev. Lett. **74**, 351 (1995).
[9] V. G. Minogin and O. T. Serimaa, Opt. Commun. **30**, 373 (1979).
[10] R. J. Cook, Phys. Rev. Lett. **44**, 976 (1980).
[11] J. P. Gordon and A. Ashkin, Phys. Rev. A **21**, 1606 (1980).
[12] J. Dalibard and C. Cohen-Tannoudji, J. Opt. Soc. Am. B **2**, 1707 (1985).
[13] A. G. Agarwal and K. Mølmer, Phys. Rev. A **47**, 5158 (1993).
[14] K. Mølmer, Y. Castin, and J. Dalibard, J. Opt. Soc. Am. B **10**, 524 (1993).
[15] H. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer-Verlag, Berlin, 1993).
[16] The calculations in this paper were performed with the aid of S. M. Tan (unpublished).
[17] J. Dalibard and C. Cohen-Tannoudji, J. Phys. B **18**, 1661 (1985).
[18] H. Risken, *The Fokker-Planck Equation, Methods of Solution and Applications* (Springer-Verlag, Berlin, 1989).
[19] V. G. Minogin and V. Letokhov, *Laser Light Pressure on Atoms* (Gordon and Breach, New York, 1987).
[20] H. Mabuchi (private communication).
[21] K. S. Wong, M. J. Collett, and D. F. Walls, Opt. Commun. **137**, 269 (1997).
[22] K. S. Wong, M. J. Collett, and D. F. Walls (unpublished).
[23] H. Mabuchi, Quantum Semiclass. Opt. **8**, 1103 (1996).
[24] H. M. Wiseman and G. J. Milburn, Phys. Rev. A **47**, 1652 (1993).