Gaussian-Wigner distributions and hierarchies of nonclassical states in quantum optics: The single-mode case

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A recently introduced hierarchy of states of a single-mode quantized radiation field is examined for the case of centered Gaussian-Wigner distributions. It is found that the onset of squeezing among such states signals the transition to the strongly nonclassical regime. Interesting consequences for the photon-number distribution, and explicit representations for them, are presented. The effects of nonideal detection are also carefully analyzed. [S1050-2947(97)00512-X]

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I. INTRODUCTION

Squeezed states of light, and other states exhibiting either antibunching or sub-Poissonian photon statistics or both, are well known examples of so-called ''nonclassical'' states of radiation [1–4]. In fact these are the simplest and most familiar ones out of an infinite hierarchy of independent signatures of nonclassical states in quantum optics; many other signatures have been presented in the literature [5].

The precise definition of a nonclassical state of radiation is based upon the diagonal coherent state expansion of the density matrix $\hat{\rho}$ of the state in the quantum theory. Limiting ourselves to the single-mode radiation field this expansion is [6]

$$\hat{\rho}=\int \frac{d^2z}{\pi}\,\phi(z)|z\rangle\langle z|, \tag{1.1}$$

where the coherent states $|z\rangle$ are the familiar normalized eigenstates of the photon annihilation operator $\hat{a}$ with complex eigenvalue z and $\phi(z)$ is a real normalized weight function which is in general a distribution. The state $\hat{\rho}$ is said to be ''classical '' if $\phi(z)$ is pointwise nonnegative, and nowhere more singular than a δ function, so that it can be interpreted as a classical probability density over the complex plane. Otherwise $\hat{\rho}$ is a ''nonclassical '' state. This classification is clearly invariant under rotations and translations in phase space.

It has been shown elsewhere that there is a dual operator based approach to this distinction between classical and nonclassical states, which is physically quite instructive [7]. The representation (1.1), as is well known, is closely related to the normal ordering rule of correspondence between classical dynamical variables and quantum operators. Given any real classical function $f(z^\star,z)$ of a complex variable z and its conjugate, one defines a Hermitian operator $\hat{F}$ in quantum theory by the replacement $z\rightarrow\hat{a}, z^\star\rightarrow\hat{a}^\dagger$ and then brings all factors $\hat{a}^\dagger$ ''by hand'' to the left of all factors $\hat{a}$:

$$f(z^\star,z)\rightarrow\hat{F}=f(\hat{a}^\dagger,\hat{a})|_{\hat{a}^\dagger \text{ to left, } \hat{a} \text{ to right}},$$

$$\langle z|\hat{F}|z\rangle=f(z^\star,z). \tag{1.2}$$

Then the quantum mechanical expectation value of $\hat{F}$ in the state $\hat{\rho}$ is

$$\langle\hat{F}\rangle=\mathrm{Tr}(\hat{\rho}\hat{F})=\int\frac{d^2z}{\pi}\,\phi(z)f(z^\star,z). \tag{1.3}$$

The key observation now is that while the correspondence $f\leftrightarrow\hat{F}$ is linear and takes real functions to Hermitian operators and vice versa, a real non-negative $f(z^\star,z)$ may well lead to a Hermitian indefinite $\hat{F}$. A state $\hat{\rho}$ is then said to be classical if this permitted ''quantum negativity'' in operators never shows up in expectation values, nonclassical otherwise:

$$\hat{\rho}\ \text{ classical}\Leftrightarrow\mathrm{Tr}(\hat{\rho}\hat{F})\geqslant 0\quad\text{for every}\quad f(z^\star,z)\geqslant 0,$$

$$\hat{\rho}\ \text{ nonclassical}\Leftrightarrow\mathrm{Tr}(\hat{\rho}\hat{F})<0\quad\text{for some}\quad f(z^\star,z)\geqslant 0. \tag{1.4}$$

With this alternative characterization (completely equivalent to the usual one), one has the possibility of defining several degrees or levels of nonclassicality, if one restricts in various ways the collection of operators $\hat{F}$ for which one tests the conditions given in Eq. (1.4) [7]. Specifically, for a single-mode system, it has been shown by considering the

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subset of phase invariant (number conserving) operators $\hat{F}$ which arise from $f(z^\star,z)$ obeying

$$f(z^\star e^{-i\alpha},ze^{i\alpha})=f(z^\star,z) \tag{1.5}$$

that an exhaustive and mutually exclusive threefold classification of states is possible. If $f(z^\star,z)$ obeys Eq. (1.5), then for the expectation value of the corresponding $\hat{F}$ it suffices to use an angle average of $\phi(z)$:

$$[\hat{F},\hat{a}^\dagger\hat{a}]=0 \Rightarrow \mathrm{Tr}(\hat{\rho}\hat{F})=\int_0^\infty dI\mathcal{P}(I)f(I^{1/2},I^{1/2}),$$

$$\mathcal{P}(I)=\int_0^{2\pi}\frac{d\theta}{2\pi}\phi(I^{1/2}e^{i\theta}). \tag{1.6}$$

One can then obtain the following finer classification of all states:

$$\hat{\rho}\ \text{ classical} \Leftrightarrow \phi(z)\geqslant 0,\quad \text{so}\quad \mathcal{P}(I)\geqslant 0,$$

$$\hat{\rho}\ \text{ weakly nonclassical} \Leftrightarrow \mathcal{P}(I)\geqslant 0,\quad \text{but}\quad \phi(z)\not\geqslant 0,$$

$$\hat{\rho}\ \text{ strongly nonclassical} \Leftrightarrow \mathcal{P}(I)\not\geqslant 0,\quad \text{so}\quad \phi(z)\not\geqslant 0. \tag{1.7}$$

Thus the previous ''nonclassical'' has been subdivided now into ''weakly nonclassical'' and ''strongly nonclassical'' states. Up to and including the weakly nonclassical level, $\mathcal{P}(I)$ can be treated as a classical probability density for intensity, whether or not $\phi(z)$ can be regarded as a probability distribution over the complex plane; in the third strongly nonclassical regime, even $\mathcal{P}(I)$ ceases to be a probability density.

The aim of this paper is to illustrate these ideas in the concrete case of states described by Gaussian-Wigner distributions on phase space. It is well known that in a wide variety of physical processes the states of radiation that are produced are indeed of this type [8]. Their description also lends itself to direct analytical treatment. The photon-number distribution for Gaussian states has been studied by several authors [9]. What we shall demonstrate is that within this set of states, the onset of squeezing signals an abrupt change from classical to the strongly nonclassical regime; thus the weakly nonclassical states do not show up at all in this family.

The material of this paper is arranged as follows. In Sec. II we trace the connection between the descriptions of an operator via its Weyl weight and its Wigner representative, and the diagonal weight $\phi(z)$. This gives us a clear picture of the extent to which $\phi(z)$ can be a singular distribution, and in turn how singular the quantity $\mathcal{P}(I)$ can in principle be. Section III examines the class of centered Gaussian-Wigner distributions. These are fully parametrized by the variance or noise matrix which has to be positive semidefinite and also must obey the uncertainty principle. Among these states the only two qualitatively different ones are the nonsqueezed and squeezed ones. In the former case, both $\phi(z)$ and $\mathcal{P}(I)$ can be computed explicitly, and as expected they are finite nonnegative normalized functions. This is consistent with their being classified as classical states. In contrast, the squeezed states are shown to be strongly nonclassical, and one never sees the weakly nonclassical possibility at all. In Sec. IV we connect our work with recent experimental developments and analyze the effect of nonideal detection on our classification. Section V gives an example of weakly nonclassical states which are naturally outside the Gaussian-Wigner family, and offers some concluding remarks.

II. NATURE OF THE DISTRIBUTIONS $\phi(z)$ AND $\mathcal{P}(I)$

It is useful to begin by recalling the general properties of the diagonal weight $\phi(z)$ and its angular average $\mathcal{P}(I)$, and by giving an indication of the kinds of singular distributions we must be prepared to encounter [10]. This is best done by viewing the set of all possible density matrices $\hat{\rho}$ as a subset of the family of Hilbert-Schmidt (HS) operators. An operator A on Hilbert space is of HS type if

$$\mathrm{Tr}(A^\dagger A)<\infty, \tag{2.1}$$

and among HS operators we have a natural inner product:

$$(A,B)=\mathrm{Tr}(A^\dagger B). \tag{2.2}$$

We deal throughout with systems of one degree of freedom, and with the annihilation and creation operators $\hat{a},\hat{a}^\dagger$ related to Hermitian $\hat{q}$ and $\hat{p}$ in the standard way:

$$\hat{a}=\frac{1}{\sqrt{2}}(\hat{q}+i\hat{p}),\quad \hat{a}^\dagger=\frac{1}{\sqrt{2}}(\hat{q}-i\hat{p}). \tag{2.3}$$

The unitary phase space displacement operators are defined by and have the following properties:

$$D(\sigma,\tau)=\exp(i\sigma\hat{q}-i\tau\hat{p}),\quad -\infty<\sigma,\tau<\infty$$

$$D(\sigma,\tau)^\dagger=D(\sigma,\tau)^{-1}=D(-\sigma,-\tau),$$

$$\mathrm{Tr}[D(\sigma',\tau')^\dagger D(\sigma,\tau)]=2\pi\delta(\sigma'-\sigma)\delta(\tau'-\tau). \tag{2.4}$$

Any HS operator A can be expanded in the form of an operator Fourier integral representation using its ''Weyl weight'' $\widetilde{A}(\sigma,\tau)$ as expansion coefficient [11]:

$$A=\int\int\frac{d\sigma d\tau}{\sqrt{2\pi}}\widetilde{A}(\sigma,\tau)D(\sigma,\tau),$$

$$\widetilde{A}(\sigma,\tau)=\frac{1}{\sqrt{2\pi}}(D(\sigma,\tau),A),$$

$$\mathrm{Tr}(A^\dagger A)=(A,A)=\int\int d\sigma d\tau|\widetilde{A}(\sigma,\tau)|^2. \tag{2.5}$$

Thus the HS property (2.1) of A is translated exactly into the L^2 property of $\widetilde{A}(\sigma,\tau)$ over $\mathcal{R}^2$.

From $\widetilde{A}(\sigma,\tau)$ we pass to the Wigner representative or Wigner distribution $W(q,p)$ of the operator A by a double Fourier transform at the c-number level [12]:

$$W(q,p)=\int\int\frac{d\sigma d\tau}{(2\pi)^{3/2}}\tilde{A}(\sigma,\tau)\exp(i\sigma q-i\tau p). \quad (2.6)$$

Here q and p are canonical coordinates over a classical phase space, and in case A is Hermitian its Wigner representative $W(q,p)$ is real. Now the HS property for A amounts to $W(q,p)$ being an L^2 function over $\mathcal{R}^2$:

$$\mathrm{Tr}(A^\dagger A)=(A,A)=2\pi\int\int dqdp|W(q,p)|^2. \quad (2.7)$$

For density matrices we are also interested in the ordinary trace:

$$\mathrm{Tr}(A)=\sqrt{2\pi}\tilde{A}(0,0)=\int\int dqdpW(q,p). \quad (2.8)$$

It is in the passage from $\tilde{A}(\sigma,\tau)$ or $W(q,p)$ to $\phi(z)$ that the distribution character of the latter shows up. From the diagonal representation

$$A=\int\frac{dxdy}{2\pi}\phi(z)|z\rangle\langle z|, \quad (2.9)$$

where $z=(1/\sqrt{2})(x+iy)$, when we connect up with the previous relations (2.5), (2.6) we get the result

$$\begin{aligned}\phi(z)&=\int\int\frac{d\sigma d\tau}{\sqrt{2\pi}}e^{(1/4)(\sigma^2+\tau^2)}\tilde{A}(\sigma,\tau)e^{i(\sigma x-\tau y)}\\&=\int\int\frac{d\sigma d\tau}{2\pi}e^{(1/4)(\sigma^2+\tau^2)+i(\sigma x-\tau y)}\\&\quad\times\int\int dqdpW(q,p)e^{i(\tau p-\sigma q)}. \end{aligned}\quad (2.10)$$

Thus the most singular kind of $\phi(z)$ is one whose Fourier transform is the increasing Gaussian factor $\exp\frac{1}{4}(\sigma^2+\tau^2)$ times a square integrable function $\tilde{A}(\sigma,\tau)$—this is the worst behavior that can in principle occur. Conversely for a classical state $\tilde{A}(\sigma,\tau)$ must more than overwhelm this exponential factor and moreover yield a non-negative $\phi(z)$.

Let us next see what this situation for $\phi(z)$ entails for its angular average $\mathcal{P}(I)$. We work directly with the Wigner distribution $W(q,p)$ and find after performing the angular integration

$$\begin{aligned}\mathcal{P}(I)&=\int_0^{2\pi}\frac{d\theta}{2\pi}\phi(I^{1/2}e^{i\theta})\\&=\int\int\frac{d\sigma d\tau}{2\pi}e^{(1/4)(\sigma^2+\tau^2)}J_0[\sqrt{2I(\sigma^2+\tau^2)}]\\&\quad\times\int\int dqdpW(q,p)e^{i(\tau p-\sigma q)}. \end{aligned}\quad (2.11)$$

If we substitute $\sigma=\sqrt{2K}\cos\psi$, $\tau=\sqrt{2K}\sin\psi$, we can carry out one more angular integration and bring $\mathcal{P}(I)$ to the following form:

$$\begin{aligned}\mathcal{P}(I)&=\int_0^\infty dKe^{K/2}J_0(2\sqrt{IK})\int\int dqdpW(q,p)\\&\quad\times J_0[\sqrt{2K(q^2+p^2)}]\\&=\int_0^\infty dKe^{K/2}J_0(2\sqrt{IK})\int_0^\infty dLJ_0(2\sqrt{KL})\\&\quad\times\int_0^{2\pi}d\chi W(\sqrt{2L}\cos\chi,\sqrt{2L}\sin\chi). \end{aligned}\quad (2.12)$$

Now just as the relation (2.10) between $\phi(z)$ and $\tilde{A}(\sigma,\tau)$ involved the classical two dimensional Fourier transformation, here one is concerned with the single variable Fourier-Bessel transformation over the half line $(0,\infty)$ which states [13]

$$\int_0^\infty dI|f(I)|^2<\infty\Rightarrow f(I)=\int_0^\infty dKg(K)J_0(2\sqrt{IK}),$$

$$g(K)=\int_0^\infty dIf(I)J_0(2\sqrt{IK}),$$

$$\int_0^\infty dI|f(I)|^2=\int_0^\infty dK|g(K)|^2,$$

$$\int_0^\infty dKJ_0(2\sqrt{LK})J_0(2\sqrt{IK})=\delta(I-L). \quad (2.13)$$

This means that the most singular possible behavior for $\mathcal{P}(I)$ which can in principle occur is that its Fourier-Bessel transform can be the factor $e^{K/2}$ times a square integrable function of K over the domain $(0,\infty)$, namely, the Fourier-Bessel transform of the angular average of $W(q,p)$. The factor $e^{K/2}$ is just the earlier factor $e^{(1/4)(\sigma^2+\tau^2)}$ present in Eq. (2.10); and the situation for $\mathcal{P}(I)$ is marginally better than for $\phi(z)$ since now only the angular average of $\phi(z)$ is involved.

The use of phase space language in describing operators in quantum mechanics leads naturally to an examination of the behaviors of $\phi(z)$ and $\mathcal{P}(I)$ under phase space rotations and translations. As is easy to see, their behavior under rotations is simple:

$$\begin{aligned}W'(q,p)&=W(q\cos\alpha-p\sin\alpha,p\cos\alpha+q\sin\alpha)\Leftrightarrow\phi'(z)\\&=\phi(ze^{i\alpha})\Rightarrow\mathcal{P}'(I)=\mathcal{P}(I). \end{aligned}\quad (2.14)$$

This invariance of $\mathcal{P}(I)$ is as expected. Under translations we have

$$W'(q,p)=W(q-q_0,p-p_0)\Leftrightarrow\phi'(z)=\phi(z-z_0),$$

$$z_0=\frac{1}{\sqrt{2}}(q_0+ip_0). \quad (2.15)$$

However, now $\mathcal{P}'(I)$ is not expressible in terms of $\mathcal{P}(I)$ alone as phase sensitivity is introduced by a translation. Therefore while our threefold classification scheme (1.7) is obviously invariant under phase space rotations, the behavior with respect to translations is much more subtle.

It is evident that the classical states with both $\phi(z)$ and $\mathcal{P}(I)$ nonnegative remain classical under translations. However, a weakly nonclassical state becomes strongly nonclassical for a suitably chosen translation, as the following physical argument shows. At the origin $\mathcal{P}(0)$ reduces to $\phi(0)$ as no angular average remains. If a weakly nonclassical state is given, its $\phi(z)$ must become effectively negative somewhere in the complex plane. By translating the origin to such a point and then computing $\mathcal{P}'(0)$ we see that the resulting state is strongly nonclassical. Following a similar argument we also see that we can recover $\phi(z)$ in its entirety by subjecting the initial state to all possible phase space displacements z_0, $\phi'(z)=\phi(z-z_0)$, and then computing the resulting $\mathcal{P}'(I)$ and collecting the results.

We conclude this section by relating the distribution $\mathcal{P}(I)$ to the photon-number probabilities. Indeed these involve a complete independent set of phase insensitive quantities and their expectation values:

$$f(z^\star,z)=e^{-z^\star z}\frac{(z^\star z)^n}{n!}\leftrightarrow\hat{F}=|n\rangle\langle n|,$$

$$p(n)=\mathrm{Tr}(\hat{\rho}\hat{F})=\langle n|\hat{\rho}|n\rangle=\int_0^\infty dI\mathcal{P}(I)e^{-I}\frac{I^n}{n!}. \quad (2.16)$$

These $p(n)$'s always give well defined normalized probabilities for finding various numbers of photons, whether or not $\mathcal{P}(I)$ is itself a probability density. Formally one can invert the above to get $\mathcal{P}(I)$ in terms of $p(n)$, as indeed one would expect. If we define the generating function $q(K)$ by

$$q(K)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}K^n p(n) \quad (2.17)$$

we see that $q(K)$ converges for all real K and is related to $\mathcal{P}(I)$ by

$$q(K)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}K^n\int_0^\infty dI\mathcal{P}(I)e^{-I}\frac{I^n}{n!}$$
$$=\int_0^\infty dI\mathcal{P}(I)e^{-I}J_0(2\sqrt{IK}). \quad (2.18)$$

Using the formula (2.13) of the Fourier Bessel transformation again we get the inversion

$$\mathcal{P}(I)=e^I\int_0^\infty dKq(K)J_0(2\sqrt{IK}). \quad (2.19)$$

In the classical and weakly nonclassical cases, then, the generating function $q(K)$ is itself well behaved and leads to non-negative $\mathcal{P}(I)$, but in the strongly nonclassical case, it causes $\mathcal{P}(I)$ to be a distribution, or at any rate not a probability.

III. THE CASE OF GAUSSIAN-WIGNER DISTRIBUTIONS

We consider the family of centered Gaussian-Wigner distributions, namely, those which have vanishing means for q and p [14]. The most general such distribution is determined by a real symmetric 2×2 matrix G,

$$W_G(q,p)=\frac{\sqrt{\det G}}{\pi}\exp\left[-(q\ \ p)G\begin{pmatrix}q\\p\end{pmatrix}\right],$$

$$G=\begin{pmatrix}A & B\\ B & C\end{pmatrix}. \quad (3.1)$$

The condition that $W_G(q,p)$ represent a physically realizable quantum mechanical state imposes the following restrictions on G corresponding, respectively, to normalizability and the uncertainty principle [15]:

$$G>0,\quad \text{i.e.,}\ A+C>0,\quad \Delta=\det G=AC-B^2>0, \quad (3.2a)$$

$$G^{-1}+i\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\geqslant0,\quad \text{i.e.,}\ A+C\geqslant0,\quad \Delta\geqslant\Delta^2. \quad (3.2b)$$

Combining these we have the complete set of restrictions on G given by

$$A+C>0,\quad 0<\Delta\leqslant1. \quad (3.3)$$

The noise or variance matrix V is defined and given by

$$V=\begin{pmatrix}(\Delta q)^2 & \Delta(q,p)\\ \Delta(q,p) & (\Delta p)^2\end{pmatrix}=\frac{1}{2}G^{-1}=\frac{1}{2\Delta}\begin{pmatrix}C & -B\\ -B & A\end{pmatrix},$$

$$(\Delta q)^2=\int\int dq\ dp\ q^2\ W_G(q,p),$$

$$\Delta(q,p)=\int\int dq\ dp\ qp\ W_G(q,p),$$

$$(\Delta p)^2=\int\int dq\ dp\ p^2\ W_G(q,p). \quad (3.4)$$

Here the vanishing of the means of q and p has been used. In terms of V, the uncertainty principle appears in the following form [16]:

$$\det V=\frac{1}{4\Delta}\geqslant\frac{1}{4}. \quad (3.5)$$

We can use the covariance of $\phi(z)$ and the invariance of $\mathcal{P}(I)$ under phase space rotations to simplify the situation and to assume without loss of generality that G and V are diagonal. Moreover these rotations do not disturb the threefold classification of states (1.7). Therefore we parametrize G and V using two real positive parameters α and β as follows:

$$V=\frac{1}{2}\begin{pmatrix}\alpha^2 & 0\\ 0 & \beta^2\end{pmatrix},$$

$$G=\begin{pmatrix}1/\alpha^2 & 0\\ 0 & 1/\beta^2\end{pmatrix},\quad \alpha,\beta>0,\quad \alpha\beta\geqslant 1$$

$$W_{(\alpha,\beta)}(q,p)=\frac{1}{\pi\alpha\beta}\exp\left(-\frac{q^2}{\alpha^2}-\frac{p^2}{\beta^2}\right). \tag{3.6}$$

To deal with $\phi(z)$ and $\mathcal{P}(I)$ we need, respectively, the Fourier transform and the angular average of $W_{(\alpha,\beta)}(q,p)$; these are

$$\int\int dq\,dp\,W_{(\alpha,\beta)}(q,p)\exp(i\tau p-i\sigma q)=\exp\left(-\frac{\alpha^2\sigma^2}{4}-\frac{\beta^2\tau^2}{4}\right), \tag{3.7a}$$

$$\int_0^{2\pi}d\chi W_{(\alpha,\beta)}(\sqrt{2L}\cos\chi,\sqrt{2L}\sin\chi)=\frac{2}{\alpha\beta}\exp\left[-L\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right]I_0\left(L\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\right). \tag{3.7b}$$

Here $I_0(w)=J_0(iw)$ is the Bessel function of order zero and imaginary argument.

Returning to the Wigner function $W_{(\alpha,\beta)}(q,p)$, the nonsqueezed case corresponds to both $\alpha,\beta\geqslant 1$, while if one of them becomes less than unity we have a squeezed state. For definiteness in the latter case we take p to be the squeezed variable, so we take $\beta<1$ and $\alpha>1$ maintaining $\alpha\beta\geqslant 1$. Formally we have throughout, on combining Eqs. (2.10) and (3.7a),

$$\phi_{(\alpha,\beta)}(z)=\int\int\frac{d\sigma d\tau}{2\pi}e^{i(\sigma x-\tau y)}\exp\left[-\frac{1}{4}(\alpha^2-1)\sigma^2-\frac{1}{4}(\beta^2-1)\tau^2\right]. \tag{3.8}$$

In the nonsqueezed regime these integrals can be computed and we get expected results:

$$\phi_{(\alpha,\beta)}(z)=\begin{cases}2(\alpha^2-1)^{-1/2}(\beta^2-1)^{-1/2}\exp\left[-\dfrac{x^2}{\alpha^2-1}-\dfrac{y^2}{\beta^2-1}\right], & \alpha,\beta>1\\ \sqrt{2\pi}\delta(x)\sqrt{2}(\beta^2-1)^{-1/2}\exp\left(-\dfrac{y^2}{\beta^2-1}\right), & \alpha=1,\beta>1\\ \sqrt{2\pi}\delta(y)\sqrt{2}(\alpha^2-1)^{-1/2}\exp\left(-\dfrac{x^2}{\alpha^2-1}\right), & \alpha>1,\beta=1\\ 2\pi\delta(x)\delta(y), & \alpha=\beta=1.\end{cases} \tag{3.9}$$

In all these cases the state is classical. However, once β dips below unity, we see from Eq. (3.8) that the Fourier transform of $\phi(z)$ is an increasing Gaussian in the variable τ. This means that $\phi(z)$ has switched abruptly to being a distribution, essentially of the most singular kind that can arise. [Of course, if β continually decreases and squeezing increases, $\phi(z)$ does become more and more singular.] This is consistent with squeezed states being nonclassical. The interesting point is that there is no intermediate regime (among ''Gaussian-Wigner'' states) in which the singularity of $\phi(z)$ is somewhat milder, say involving finite number of derivatives of δ functions.

To follow the behavior of $\mathcal{P}_{(\alpha,\beta)}(I)$ as we pass from the nonsqueezed state to the squeezed regime, and when $\beta<1$ to discriminate between the weakly nonclassical and the strongly nonclassical possibilities, we begin by combining Eqs. (2.12) and (3.7b) to get a formal integral expression for $\mathcal{P}_{(\alpha,\beta)}(I)$:

$$\mathcal{P}_{(\alpha,\beta)}(I)=\frac{2}{\alpha\beta}\int_0^\infty dKe^{K/2}J_0(2\sqrt{IK})\times\int_0^\infty dL\; e^{-L(1/\alpha^2+1/\beta^2)}J_0(2\sqrt{LK})\times I_0\left(L\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\right). \tag{3.10}$$

The first integral, over L, always converges thanks to the asymptotic behaviors of $J_0(z)$ and $I_0(z)$:

$$J_0(z) \underset{z\to+\infty}{\to} \sqrt{\frac{2}{\pi z}}\cos(z-\pi/4),$$

$$I_0(z) \underset{z\to+\infty}{\to} \frac{e^z}{\sqrt{2\pi z}}. \tag{3.11}$$

Moreover, by suitable and permitted analytic continuation of a standard definite integral available in the literature (Ref. [17], p. 711, formula 6.644) we obtain a formula with whose help the L integral can be done explicitly. The requisite formula is, for real parameters a,b,c obeying $a>|c|\geqslant 0, b>0$,

$$\int_0^\infty dx e^{-ax} J_0(2\sqrt{bx}) I_0(cx)$$
$$= \frac{1}{\sqrt{a^2-c^2}} \exp\left(\frac{-ab}{a^2-c^2}\right) I_0\left(\frac{cb}{a^2-c^2}\right). \tag{3.12}$$

Taking $a=1/\alpha^2+1/\beta^2, b=K, c=1/\alpha^2-1/\beta^2$ here and using the result in Eq. (3.10) we get for $\mathcal{P}_{(\alpha,\beta)}(I)$ the single integral

$$\mathcal{P}_{(\alpha,\beta)}(I) = \int_0^\infty dK e^{K/2} J_0(2\sqrt{IK})$$
$$\times e^{-K[(\alpha^2+\beta^2)/4]} I_0\left(\frac{K}{4}(\alpha^2-\beta^2)\right). \tag{3.13}$$

First let us look at the classical nonsqueezed situation. Leaving aside the marginal cases when α or β equals unity, we again use the result (3.12) to evaluate Eq. (3.13) explicitly:

$\alpha,\beta>1$:

$$\mathcal{P}_{(\alpha,\beta)}(I) = 2(\alpha^2-1)^{-1/2}(\beta^2-1)^{-1/2}$$
$$\times \exp\left[-I\left(\frac{1}{\alpha^2-1}+\frac{1}{\beta^2-1}\right)\right]$$
$$\times I_0\left[I\left(\frac{1}{\alpha^2-1}-\frac{1}{\beta^2-1}\right)\right]. \tag{3.14}$$

This is explicitly nonnegative, and is consistent with the state being classical. In this case, we can go further and obtain a closed-form expression for the photon-number probabilities $p_{(\alpha,\beta)}(n)$. We have

$$p_{(\alpha,\beta)}(n) = \int_0^\infty dI \mathcal{P}_{(\alpha,\beta)}(I) e^{-I} \frac{I^n}{n!}$$
$$= \frac{1}{n!} \frac{2}{\sqrt{(\alpha^2-1)(\beta^2-1)}} \int_0^\infty dI e^{-aI} I^n I_0(bI),$$

$$a = 1 + \frac{1}{\alpha^2-1} + \frac{1}{\beta^2-1} = \frac{\alpha^2\beta^2-1}{(\alpha^2-1)(\beta^2-1)},$$

$$b = \frac{(\beta^2-\alpha^2)}{(\alpha^2-1)(\beta^2-1)}. \tag{3.15}$$

The resulting integral is a known one leading to an expression in terms of the hypergeometric function (Ref. [17], p. 711, formula 6.621)

$$\int_0^\infty dx\ e^{-ax} x^n I_0(bx) = \frac{n!}{a^{n+1}} F\left(\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; 1; \frac{b^2}{a^2}\right) \tag{3.16}$$

so the probabilities $p_{(\alpha,\beta)}(n)$ are

$$p_{(\alpha,\beta)}(n) = \frac{2}{\sqrt{(\alpha^2-1)(\beta^2-1)}} \left[\frac{(\alpha^2-1)(\beta^2-1)}{\alpha^2\beta^2-1}\right]^{n+1}$$
$$\times F\left(\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; 1; z\right),$$
$$z = \left(\frac{\alpha^2-\beta^2}{\alpha^2\beta^2-1}\right)^2, \quad \alpha,\beta>1. \tag{3.17}$$

The combination z of α and β does not exceed unity as we have $\alpha,\ \beta>1$:

$$1-z = (\alpha^4-1)(\beta^4-1)/(\alpha^2\beta^2-1)^2. \tag{3.18}$$

It is interesting to note that the result (3.17) for $p_{(\alpha,\beta)}(n)$ is a manifestly nonnegative closed-form expression; in this respect it may be contrasted with the expression given earlier in the literature [8].

Next let us consider the squeezed regime $\beta<1, \alpha\geqslant 1/\beta$. Then the exponential factor $e^{K/2}$ in the integral in Eq. (3.13) overpowers the remaining factors:

$$e^{K/2} e^{-K(\alpha^2+\beta^2)/4} I_0(K(\alpha^2-\beta^2)/4)$$
$$\underset{K\to+\infty}{\to} \frac{1}{\sqrt{\alpha^2-\beta^2}} \sqrt{\frac{2}{\pi K}} e^{K(1-\beta^2)/2}. \tag{3.19}$$

This means that $\mathcal{P}_{(\alpha,\beta)}(I)$ is no longer the Fourier-Bessel transform of a square integrable function of K; it has switched abruptly from being a classical probability density for intensity to being a distribution, essentially as singular as is permitted by the general considerations of the preceding section.

There is thus no regime in which $\mathcal{P}_{(\alpha,\beta)}(I)$ remains ''classical'' while ϕ is not—the weakly nonclassical possibility is not realized at all in the family of Gaussian-Wigner states. Even though $\mathcal{P}_{(\alpha,\beta)}(I)$ is a distribution in the squeezed regime, we can obtain the photon-number probabilities by analytic continuation starting from the result (3.17) in the nonsqueezed case. The justification is the following. At the level of Wigner distributions we know that the probability $p_{(\alpha,\beta)}(n)$ is the phase space integral of the product of

$W_{(\alpha,\beta)}(q,p)$ and the Wigner function $W^{(n)}(q,p)$ for the nth state of the harmonic oscillator [18]:

$$\hat{\rho}=|n\rangle\langle n|\Rightarrow W^{(n)}(q,p)=\frac{(-1)^n}{\pi}e^{-(q^2+p^2)}L_n(2(q^2+p^2)),$$

$$p_{(\alpha,\beta)}(n)=2\pi\int\int dq\,dp\,W_{(\alpha,\beta)}(q,p)W^{(n)}(q,p). \tag{3.20}$$

Here $L_n(\ )$ is the nth order Laguerre polynomial. Using the rotational invariance of $W^{(n)}(q,p)$ and Eq. (3.7b) for the angular average of $W_{(\alpha,\beta)}(q,p)$, we can reduce $p_{(\alpha,\beta)}(n)$ to a single radial phase space integral:

$$p_{(\alpha,\beta)}(n)=\frac{(-1)^n}{\pi}\frac{2}{\alpha\beta}2\pi\int_0^\infty dL \times\exp\left\{-2L-L\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right\} \times L_n(4L)I_0\left(L\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\right). \tag{3.21}$$

This is valid for all α and β subject to the standard restrictions $\alpha,\ \beta>1,\ \alpha\beta\geqslant 1$. Since we have symmetry in α and β, we may assume with no loss of generality that $\alpha\geqslant\beta$. Then the asymptotic behavior (3.11) for $I_0(z)$ as $z\rightarrow\infty$ shows that for large L the integrand here behaves like

$$L^{n-1/2}\exp\{-2L(1+1/\alpha^2)\}. \tag{3.22}$$

Thus the integral (3.21) is absolutely convergent for all α and β, and is in fact analytic in these variables (in the appropriate regions of the complex planes).

Having established this, we may now go back to the closed expression (3.17) valid in the nonsqueezed case and analytically continue it to $\beta<1,\ \alpha\beta\geqslant 1$. Now from Eq. (3.18) we see that the argument z of the hypergeometric function exceeds unity, which lies outside the domain of convergence of the power series expansion of $F((n+1)/2,n/2+1;1;z)$. By analytically continuing to $z>1$, and keeping track of phases generated in switching from (β^2-1) to $(1-\beta^2)$ in the prefactors in Eq. (3.17), we find that in the squeezed regime we have different expressions for $p_{(\alpha,\beta)}(n)$ for even n and for odd n:

$$p_{(\alpha,\beta)}(n)=\frac{2}{\sqrt{\pi}}\frac{[(\alpha^2-1)(1-\beta^2)]^{n+1/2}}{(\alpha^2\beta^2-1)^{n+1}}\frac{1}{z^{(n+1)/2}}\begin{cases}\dfrac{\Gamma(m+1/2)}{m!}F(m+1/2,m+1/2;1/2;1/z), & n=2m\\[2mm] \dfrac{2}{\sqrt{z}}\dfrac{\Gamma(m+3/2)}{m!}F(m+3/2,m+3/2;3/2;1/z), & n=2m+1,\end{cases}$$

$$z=\left(\frac{\alpha^2-\beta^2}{\alpha^2\beta^2-1}\right)^2>1,\quad \alpha>\frac{1}{\beta},\quad \beta<1. \tag{3.23}$$

Once again we have manifestly non-negative closed-form expressions [8].

The actual expressions for the first few probabilities show the general trend. We find after simplification that, as expected, both Eq. (3.17) and Eq. (3.23) give identical functions of α and β:

$$p_{(\alpha,\beta)}(0)=2\times\{(\alpha^2+1)(\beta^2+1)\}^{-1/2}, \tag{3.24}$$

$$p_{(\alpha,\beta)}(1)=2(\alpha^2\beta^2-1)\times\{(\alpha^2+1)(\beta^2+1)\}^{-3/2},$$

$$p_{(\alpha,\beta)}(2)=\{(\alpha^2-\beta^2)^2+2(\alpha^2\beta^2-1)^2\}\times\{(\alpha^2+1)(\beta^2+1)\}^{-5/2},$$

$$p_{(\alpha,\beta)}(3)=(\alpha^2\beta^2-1)\{3(\alpha^2-\beta^2)^2+2(\alpha^2\beta^2-1)^2\}\times\{(\alpha^2+1)(\beta^2+1)\}^{-7/2},$$

$$p_{(\alpha,\beta)}(4)=\frac{1}{4}\{3(\alpha^2-\beta^2)^4+24(\alpha^2-\beta^2)^2(\alpha^2\beta^2-1)^2+8(\alpha^2\beta^2-1)^4\}\times\{(\alpha^2+1)(\beta^2+1)\}^{-9/2},$$

$$p_{(\alpha,\beta)}(5)=\frac{1}{4}(\alpha^2\beta^2-1)\{15(\alpha^2-\beta^2)^4+40(\alpha^2-\beta^2)^2\times(\alpha^2\beta^2-1)^2+8(\alpha^2\beta^2-1)^4\}\times\{(\alpha^2+1)(\beta^2+1)\}^{-11/2},$$

$$p_{(\alpha,\beta)}(6)=\frac{1}{8}\{5(\alpha^2-\beta^2)^6+90(\alpha^2-\beta^2)^4(\alpha^2\beta^2-1)^2+120(\alpha^2-\beta^2)^2(\alpha^2\beta^2-1)^4+16(\alpha^2\beta^2-1)^6\}\times\{(\alpha^2+1)(\beta^2+1)\}^{-13/2}.$$

The appearance of the ‘‘uncertainty principle factor’’ $(\alpha^2\beta^2-1)$ in $p_{(\alpha,\beta)}(n)$ for odd n alone is immediately understandable: when the uncertainty limit is saturated and $\alpha\beta=1$, the Gaussian-Wigner function $W_{(\alpha,1/\alpha)}(q,p)$ describes the squeezed vacuum, for which it is well known that $p_{(\alpha,1/\alpha)}(n)$ vanishes when n is odd [19]. Conversely, even in the nonsqueezed regime, despite the uniform looking expression (3.17), there is a discrimination between the cases of even and odd n which is seen when the hypergeometric function is worked out in detail. In the limit $\alpha=\beta=1$, we have of course just the vacuum state, and then $p_{(1,1)}(n)$ vanishes for all $n \geqslant 1$. This can be seen quite explicitly in the expressions displayed in Eq. (3.24).

IV. CONNECTION WITH RECENT EXPERIMENTS

In this section we discuss our results, making connection with some recent experiments. We begin with the experiment of Munroe *et al.* [20], which shows that the distinction between strong and weak nonclassicality of states is indeed experimentally relevant. It turns out that this experiment reconstructs precisely the information contained in $\mathcal{P}$, and hence it can detect only strong nonclassicality; it cannot distinguish between a classical state and a weakly nonclassical state.

Optical homodyne detection (OHD) is routinely employed to measure the probability distribution

$$P_{\theta}(q_{\theta}) = \langle q_{\theta} | \hat{\rho} | q_{\theta} \rangle = \langle q | \hat{U}(\theta)^{\dagger} \hat{\rho} \hat{U}(\theta) | q \rangle \tag{4.1}$$

of the quadrature component $\hat{q}_{\theta} = (1/\sqrt{2})(\hat{a}^{\dagger} e^{-i\theta} + \hat{a} e^{i\theta})$ in the state $\hat{\rho}$ under consideration. Here $\hat{U}(\theta) = e^{-i\theta \hat{a}^{\dagger} a}$, $\hat{a} = (1/\sqrt{2})(\hat{q} + i\hat{p})$, and θ is the relative phase between the signal $\hat{\rho}$ and the local oscillator of the OHD apparatus. Munroe *et al.* [20] study the situation where this relative phase is random, and hence their OHD apparatus gives the phase-averaged quadrature amplitude distribution.

$$\overline{P(\xi)} = \frac{1}{2\pi} \int_0^{2\pi} d\theta P_{\theta}(q_{\theta} = \xi). \tag{4.2}$$

It is clear that the effect of phase averaging is equivalent to setting to zero the off-diagonal elements of the density operator $\hat{\rho}$ in the Fock basis thus

$$\overline{P(\xi)} = \sum_{n=0}^{\infty} \rho_{nn} |\langle \xi | n \rangle|^2, \tag{4.3}$$

where $\rho_{nn} = \langle n | \hat{\rho} | n \rangle$ and $\langle \xi | n \rangle$ is the configuration space wave function of the Fock state $|n\rangle$. Interestingly, $\overline{P(\xi)}$ captures *all* the information contained in the sequence $\{p_n = \rho_{nn}\}$, as Munroe *et al.* show by analytically inverting Eq. (4.3). See also the detailed analysis of Leonhardt *et al.* [21].

Thus phase-averaged OHD can detect only the strong form of nonclassicality; it cannot distinguish a weakly nonclassical state from a classical state obtained from it by modifying the off-diagonal elements of ρ_{nn} by independent amounts. Further, since $\{p_n\}$ and $\overline{P(\xi)}$ are invertibly related, it follows that the quasiprobability $\mathcal{P}(I)$ and the true probability $\overline{P(\xi)}$ are invertibly related to one another.

Gaussian states have been experimentally most relevant candidates for nonclassical states of light. Indeed, almost all the nonclassical states generated convincingly so far belong to this category, with rare exceptions like the single photon state of Hong and Mandel [22]. We have shown in Sec. III that the centered Gaussian states are either classical or strongly nonclassical. They are never weakly nonclassical and hence their nonclassicality is amenable to phase-averaged OHD. We believe that this is true of noncentered Gaussians as well and we outline here an argument to justify this belief.

The diagonal coherent state distribution function for the centered Gaussian-Wigner function, Eq. (3.8), has a simple behavior under displacement, namely, $\phi(z) \to \phi(z - z_0)$ under a complex displacement z_0 [this is of course true for any $\phi(z)$]. Therefore the distribution properties of $\phi(z)$ do not change under displacement. Further, in the squeezed regime $\beta < 1$ Eq. (3.8) tells us that $\phi(z)$ is singular everywhere in the complex plane, unlike a δ function or finite number of derivatives of a δ function. For such cases the distribution corresponds essentially to the Fourier transform of a polynomial of finite degree and the integral is singular only at selected points. However, here we have the Fourier transform of an exploding Gaussian and therefore the singularity of $\phi(z)$ is spread everywhere in the complex plane. On the other hand, the distribution $\mathcal{P}(I)$, which is the phase-averaged $\phi(z)$, coincides for $I = 0$ with $\phi(0)$ as no averaging remains at that point. Therefore for the above case when $\phi(z)$ is singular everywhere $\mathcal{P}(I)$ is expected to be singular at least at the origin, which suggests that noncentered Gaussian-Wigner distributions are strongly nonclassical.

Recent years have witnessed remarkable progress in quantum state reconstruction using optical homodyne tomography [23]. As a consequence one is now able to map out the quasiprobability—often the Wigner distribution—associated with the quantum mechanical state of the system under consideration or reconstruct the density matrix in the Fock basis. Schiller *et al.* [24] report such a reconstructed density matrix $\rho_{m,n}$ for $m,n \leqslant 6$. The reported values of $p_n = \rho_{n,n}$ for $n = 0$ to 6 are 0.44, 0.07, 0.13, 0.05, 0.06, 0.03, and 0.04 in that order. We have made a least squares fit of these values (with equal weights) to our formulas (3.24), leading to the parameter values $\alpha^2 = 15.3, \beta^2 = 0.265$ in our Gaussian state (3.6). Since $\beta < 1$, the state of Schiller *et al.* is strongly nonclassical: it is a quadrature squeezed state.

For the Gaussian state (3.6) we have the relationship $\mathrm{Tr}(\hat{\rho}^2) = (\alpha\beta)^{-1}$. Thus the Schiller *et al.* state has the value 0.497 for $\mathrm{Tr}(\hat{\rho}^2)$, showing that it is a mixed state. In this experiment the state produced was known to be squeezed vacuum. However, the determined photon statistics corresponds to $\alpha\beta > 1$, rather than the minimum uncertainty value $\alpha\beta = 1$ appropriate for squeezed vacuum. This is due to overall losses in the system.

Following Schiller *et al.* we may collect the losses of various origins and effectively account for them as arising from nonideal detection efficiency $\eta < 1$ in an otherwise ideal lossless system (an ideal detector corresponding to $\eta = 1$). It is of interest to analyze the effect of a nonideal detector on the measurability of nonclassicality. In particular, one may like to ask if nonideal detection will result in a transition across our classical, weakly nonclassical, strongly nonclassical divide. To begin such an analysis, one has to model the inefficient detector in a suitable manner.

Following Caves [25], we may model our lossy detector by the following statement: the *normal ordered* fluctuations (second moments) of the quadrature components are attenuated (phase insensitively) by the factor η (it may be noted that the computation of Schiller *et al.* indeed corresponds to this model). For our Gaussian state (3.6), this model corresponds to the following changes in the parameters:

$$\alpha^2 \to \alpha'^2 = 1 + \eta(\alpha^2 - 1),$$
$$\beta^2 \to \beta'^2 = 1 + \eta(\beta^2 - 1). \quad (4.4)$$

Since $\alpha'^2 - 1$ has the same signature as $\alpha^2 - 1$ and $\beta'^2 - 1$ has the same signature as $\beta^2 - 1$, the classical-nonclassical divide is left unaffected by the lossy detector. Only the degree of nonclassicality (degree of squeezing) gets reduced.

Let the Schiller *et al.* squeezed vacuum have an exponential squeeze parameter $r > 0$ so that $\alpha^2 = e^r$ and $\beta^2 = e^{-r}$ (note that $S_\pm$ in their notation corresponds to $e^{\pm r} - 1$ in our notation). Using these expressions for α, β in Eq. (4.4) one obtains

$$\eta = \frac{(\alpha'^2 - 1)(1 - \beta'^2)}{\alpha'^2 + \beta'^2 - 2}. \quad (4.5)$$

Using in Eq. (4.5) the values $\alpha'^2 = 15.3, \beta'^2 = 0.265$ obtained earlier through least squares fit we deduce $\eta = 0.79$. The slight difference between our values of $\mathrm{Tr}(\hat{\rho}^2)$, η, and those deduced by Schiller *et al.* is probably due to the fact that they used in their computation the reconstructed $\rho_{m,n}$ up to $m = n = 12$.

One could have modeled the inefficient detector by the following statement, instead of the earlier statement: Every photon is detected with a probability $\eta < 1$, irrespective of the presence or otherwise of other photons. This means that $P_\eta(n)$, the measured photon-number distribution (PND), is related to $p(n)$, the actual PND, in the following manner:

$$P_\eta(n) = \sum_{k=0}^{\infty} \binom{n+k}{n} \eta^n (1-\eta)^k p(n+k). \quad (4.6)$$

Substitution of this convolution in Eq. (2.16) shows that the effect of less than ideal detection modeled in this way is to produce the following transformation on the quasiprobability $\mathcal{P}(I)$:

$$\mathcal{P}(I) \to \mathcal{P}'_\eta(I) = \eta^{-1} \mathcal{P}(\eta^{-1} I). \quad (4.7)$$

This simple contraction map on the quasiprobability $\mathcal{P}(I)$ does not affect its pointwise nonnegativity or otherwise, reiterating our earlier conclusion that nonclassicality is not affected by imperfect detection described through Eq. (4.6).

As yet another model of nonideal detection we consider the one discussed in the interesting work of Leonhardt and Paul [26]. This may be described as follows: If $W'_\eta(q,p)$ is the Wigner distribution of a state $\hat{\rho}$ reconstructed in optical homodyne tomography using a detector of efficiency η, and if $W(q,p,s)$ is the s-ordered quasiprobability [27] of the state $\hat{\rho}$, then

$$W'_\eta(q,p) = \eta^{-1} W(\eta^{-1/2} q, \eta^{-1/2} p; -(1-\eta)/\eta). \quad (4.8)$$

Recalling that the Wigner distribution corresponds to $s = 0$ in the Cahill-Glauber scheme of s ordering, we see that Eq. (4.8) involves two distinct operations: (i) change of the s parameter from 0 to $-(1-\eta)/\eta$, and (ii) scaling of the phase space variables by the factor $\eta^{1/2}$. Both these operations clearly respect normalization. The first operation has the following effect on the Gaussian state (3.6):

$$\alpha^2 \to \alpha'^2 = \alpha^2 + (1-\eta)/\eta,$$
$$\beta^2 \to \beta'^2 = \beta^2 + (1-\eta)/\eta, \quad (4.9)$$

and the effect of the second operation is to further modify the parameters in this manner:

$$\alpha'^2 \to \alpha''^2 = \eta \alpha'^2,$$
$$\beta'^2 \to \beta''^2 = \eta \beta'^2. \quad (4.10)$$

Thus the effect of inefficient detection on the Gaussian state (3.6) is to simply make the following change in the parameters of the state: $\alpha^2 \to 1 + \eta(\alpha^2 - 1)$, $\beta^2 \to 1 + \eta(\beta^2 - 1)$. This result is consistent with that of the earlier models.

The content and beauty of the Leonhardt-Paul formula (4.8) attains its naked simplicity when presented in the P-distribution language $(s = +1)$. If $\phi(z)$ is the P distribution of the state $\hat{\rho}$ and if $\phi_\eta(z)$ is the P distribution corresponding to the Wigner distribution reconstructed using a detector of efficiency η, then Eq. (4.8) is equivalent to

$$\phi'_\eta(z) = \eta^{-1} \phi(\eta^{-1/2} z). \quad (4.11)$$

This simple scaling mapping leaves unaffected the pointwise positivity character of $\phi(z)$, rendering our finer classification of the nonclassicality invariant under nonideal detection. Further, since Eq. (4.11) implies Eqs. (4.7) and (4.4), this more detailed model subsumes the two models considered earlier. We may note in passing that under Eq. (4.8) a coherent state $|z\rangle$ simply goes over to another coherent state $|z'\rangle$ with $z' = \eta^{1/2} z$.

In the above models for less than ideal detection, nonclassicality is not altogether destroyed for any state for the following reason: the mode of the detector into which the undetected $(1-\eta)$ fraction of the signal is lost is assumed to be in the vacuum state or equivalently, to be in equilibrium at an effective temperature $T = 0$. If we assume that this mode into which the undetected signal escapes is at an effective temperature $T > 0$ then the effect of a less than ideal detector would be to mix the signal with a thermal state rather than the vacuum state. In that case, nonclassicality can be lost due to inefficient detection, even though in the visible spectrum this can happen only if either η is very close to zero or T is very large, as the following analysis of the modified Leonhardt-Paul model shows.

We assume the mode into which part of the signal is lost to be in a thermal state at temperature T, instead of being in a vacuum state, described by a Wigner distribution

$$W(q_2, p_2) = \frac{\pi}{\kappa^2} e^{-(q_2^2/\kappa^2 + p_2^2/\kappa^2)} \quad (4.12)$$

where κ is related to the temperature and $\kappa = 1$ corresponding to the zero temperature model of Leonhardt and Paul.

An analysis along the lines of Leonhardt and Paul shows that the effect on the phase space distribution functions is given by the following equation corresponding to Eq. (4.8):

$$W'_\eta(q,p)=\eta^{-1}W(\eta^{-1/2}q,\eta^{-1/2}p;\kappa^2[-(1-\eta)]/\eta). \tag{4.13}$$

Using Eq. (4.13) for our Gaussian state (3.6) and setting $\kappa^2=1+\delta^2$ we obtain the following formulas for the change in parameters of the state as a result of inefficient detection with the detector at a finite temperature:

$$\alpha^2\to 1+\eta(\alpha^2-1)+\delta^2(1-\eta),$$

$$\beta^2\to 1+\eta(\beta^2-1)+\delta^2(1-\eta). \tag{4.14}$$

In this case, the signatures of α'^2-1 and β'^2-1 are no longer the same as that for α^2-1 and β^2-1. For the nonclassicality of the state (3.6) to show up in the measurement a minimum amount of squeezing is required; more precisely we need

$$\beta^2<1-\delta^2\left(\frac{1-\eta}{\eta}\right). \tag{4.15}$$

It is not just that the finer classification of the nonclassical states is no longer protected against the inefficiency of detectors, the classical-nonclassical divide is itself no longer invariant. As we have seen above, a nonclassical squeezed Gaussian state with $1>\beta^2>1-\delta^2(1-\eta)/\eta$ will not reveal its nonclassical nature in a measurement of this type. A zero temperature limit $\delta\to 0$ clearly gives us back the results described earlier in this section.

V. CONCLUDING REMARKS

We have examined the class of Gaussian-Wigner distributions for a single-mode radiation field in quantum optics from the point of view of a recently introduced classification of quantum states into three mutually exclusive types—classical, weakly nonclassical, and strongly nonclassical. We have found that only the first and third possibilities arise in this case, corresponding, respectively, to the nonsqueezed and squeezed situations. As shown elsewhere, there is an interesting class of pure states which give physical examples of the weakly nonclassical type. These are superpositions of the number states of the following general type:

$$|\psi\rangle=e^{-(1/2)|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{i\beta(n)}|n\rangle, \tag{5.1}$$

where α is any complex number and $\beta(n)$ is a *nonlinear* function of n. Here the photon-number probabilities are independent of $\beta(n)$ and follow the Poisson distribution, so $\mathcal{P}_\psi(I)$ is a δ function:

$$\mathcal{P}_\psi(I)=\delta(I-\alpha^\star\alpha). \tag{5.2}$$

However, on the basis of Hudson's theorem [28] it turns out that the Wigner function $W_\psi(q,p)$, which is *not* Gaussian, must be negative somewhere, so in turn $\phi(z)$ cannot be nonnegative. This shows that the states (5.1) are weakly nonclassical.

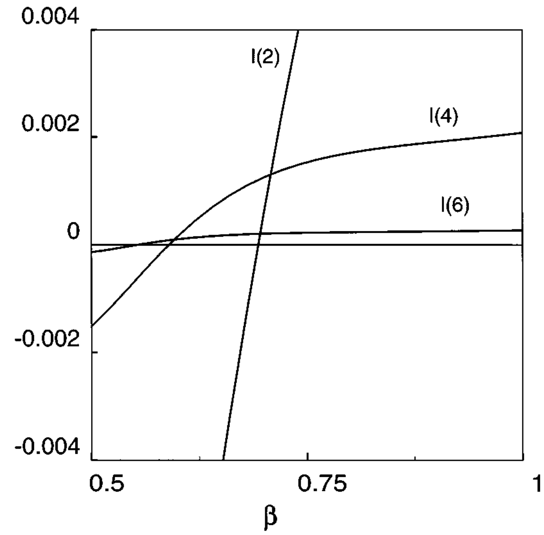

FIG. 1. Violation of the local conditions on photon-number distribution in the squeezed regime.

Our result that the centered Gaussian-Wigner distributions are never weakly nonclassical has an important physical consequence. In the regime $\alpha>1,\beta<1$ which corresponds to *quadrature* squeezing, since $\mathcal{P}(I)$ is not nonnegative the nonclassical nature of the state *must already show up* in properties of the photon-number distribution probabilities $p_{(\alpha,\beta)}(n)$, i.e., via phase insensitive quantities. The simplest such signal, namely, sub-Poissonian statistics, does not, however, display the nonclassicality of the state [8]. We find after simple algebra that the Mandel Q parameter is always nonnegative:

$$Q(\alpha,\beta)=\frac{\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle-\langle\hat{a}^\dagger\hat{a}\rangle^2}{\langle\hat{a}^\dagger\hat{a}\rangle}$$
$$=2\{(\alpha^2-1)^2+(\beta^2-1)^2\}/(\alpha^2+\beta^2-2)^2\geq 0. \tag{5.3}$$

There are, however, (infinitely many) other signatures of a nonclassical photon-number distribution, some of which are local in that they involve only a few contiguous probabilities $p(n)$. For example, we have the result [7]

$$\mathcal{P}(I)\geq 0\Rightarrow l(n)=(n+1)p(n-1)p(n+1)-np(n)^2\geq 0,$$
$$n=1,2,3,\ldots. \tag{5.4}$$

Therefore if any $l(n)$ is negative for some given state, that is evidence for the strongly nonclassical nature of that state. For the states $W_{(\alpha,\beta)}(q,p)$, taking $\alpha=2$, $\frac{1}{2}<\beta<1$ as an example, we do find explicitly as shown in Fig. 1 that $l(2)$, $l(4)$, $l(6)$, ... are negative for some range of values of β before turning positive as β increases; while $l(1)$, $l(3)$, $l(5)$, ... do not display such nonclassical behavior.

The examination of several available models for nonideal detectors presented in Sec. IV leads to the interesting conclusion that the distinction between the different levels of classicality is preserved though the degree of nonclassicality may well be reduced. However, in the case of inefficient detection involving finite nonzero temperature the situation is different and there is a threshold below which nonclassicality escapes detection. It is expected that our conclusions will not be altered drastically if we consider general noncentered Gaussian-Wigner distributions. This aspect and other examples of states and the cases of two or more modes, will be taken up elsewhere.

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