PHYSICAL REVIEW A VOLUME 56, NUMBER 6 DECEMBER 1997

Non-ground-state Bose-Einstein condensates of trapped atoms

V. I. Yukalov, E. P. Yukalova, and V. S. Bagnato
International Centre of Condensed Matter Physics, University of Brasilia, Caixa Postal 04513, Brasilia,
Distrito Federal 70919-970, Brazil
and Instituto de Fisica de São Carlos, Universidade de São Paulo, Caixa Postal 369, São Carlos, São Paulo 13560-970, Brazil

(Received 5 June 1997)

The possibility of creating a Bose condensate of trapped atoms in a non-ground-state is suggested. Such a nonequilibrium Bose condensate can be formed if one first obtains the conventional Bose condensate in the ground state and then transfers the condensed atoms to a non-ground-state by means of a resonance pumping. The properties of ground and non-ground-states are compared and plausible applications of such nonequilibrium condensates are discussed. [S1050-2947(97)03912-7]

PACS number(s): 03.75.Fi

I. INTRODUCTION

The recent realization of Bose-Einstein condensation of dilute atomic gases in magnetic traps [1–3] has opened a rapidly expanding field of studies of condensate properties. There has been a splash of both experimental and theoretical activity on this subject (see Refs. [4,5]).

Atoms trapped in a confining potential possess the discrete spectrum of states. At high temperatures these states are occupied, according to the quantum Bose-Einstein distribution, so that no state is occupied macroscopically. Under the macroscopic occupation of a state one implies that the number of atoms in this state is proportional to the total number of atoms in the system. An important consequence of quantum statistics is that when the system is cooled down below some critical temperature, bosons pile up in the lowest-energy state of a confining potential. The macroscopic population of the quantum-mechanical ground state of a confining potential is the characteristic feature of Bose-Einstein condensation.

A natural question that can be raised is: Is it possible to realize the macroscopic population of some other quantum-mechanical state rather than the ground state or in addition to the latter? That is, can one produce a Bose condensate in a non-ground-state? The answer to this question is interesting in itself. Also, if that is possible, several important applications can be suggested.

For example, recently two overlapping ^{87}Rb condensates in two different ground-state hyperfine levels were created [6]. However, for other atoms the simultaneous creation of two condensates in different ground-state hyperfine levels may be difficult or not feasible [7]. Then the alternative could be the creation of two condensates, one in the ground state and the other in a non-ground-state level.

Realizing the macroscopic population of a non-ground-state of a confining potential could be a way to produce various spatial distributions in the system of coherent atoms. This may find application for atom lasers for which the creation of coherent atomic beams with different spatial modes may be required.

One more possibility of employing such a non-ground-state condensate could be to study relaxation processes in the quantum degenerate regime. The macroscopic occupation of an excited state would clearly result in a nonequilibrium sample. When external forces supporting this state are switched off, the system will relax, returning to the equilibrium by repopulating the discrete levels of the potential. The observation of this process of relaxation in the quantum degenerate regime can provide useful information about this form of quantum matter. Finally, when a system with unusual features is explored, there is always the chance of finding something completely unexpected. In this paper we describe a possible way of transferring the macroscopic number of atoms from the conventional ground-state condensate to a non-ground-state level of the confining potential, thus creating a non-ground-state condensate.

II. RESONANCE PUMPING

Assume that the Bose gas of neutral atoms has been cooled down so that all atoms are in a coherent condensate state. The latter is described by the nonlinear Schrödinger equation, which is often called the Gross-Ginzburg-Pitaevskii equation [8–12]. This equation is written

$$i\hbar\frac{\partial\varphi}{\partial t}=\hat{H}\varphi, \tag{1}$$

where the nonlinear Hamiltonian

$$\hat{H}=H(\varphi)+V_p \tag{2}$$

contains the nonlinear part

$$H(\varphi)=-\frac{\hbar^2}{2m}\nabla^2+U_c(\vec{r})+A|\varphi|^2 \tag{3}$$

and, in general, a time-dependent part V_p related to external fields. The atom-atom interaction is modeled by the s-wave scattering interaction with the amplitude

$$A=(N-1)4\pi\hbar^2\frac{a}{m}, \tag{4}$$

in which N is the number of atoms in the system, a is the s-wave scattering length, and m is the atomic mass. The term

U_c is a confining potential. The wave function is normalized to unity: $(\varphi,\varphi)=1$. Assume that at the initial time $t=0$, all atoms are in the ground state

$$\varphi(\vec{r},0)=\varphi_0(\vec{r}) \tag{5}$$

corresponding to the minimal energy level of the eigenvalue problem

$$H(\varphi_n)\varphi_n=E_n\varphi_n, \tag{6}$$

in which n is a multi-index enumerating quantum state. The chemical potential is incorporated into the notation of the energy levels E_n. Temperature is assumed to be much lower than the condensation temperature, since only then it is possible to condensate almost all atoms in the ground state. Note that the nonlinear Schrödinger equation (1) describes coherent states [13].

Since the atoms are assumed to be initially condensed in the ground state, to transfer them to higher levels one needs to apply an external pumping field that we take in the form

$$V_p=V(\vec{r})\cos\omega t. \tag{7}$$

Such a field can be realized by a special modulation of the magnetic field producing the trap. As far as our aim is to populate a separate energy level, say a particular level p with the energy E_p, we have to choose the frequency of the pumping field (7) satisfying some resonance conditions.

Denote the transition frequencies ω_{mn} by the relation

$$\hbar\omega_{mn}\equiv E_m-E_n \tag{8}$$

and the detuning from the chosen particular transition frequency ω_{p0} as

$$\Delta\omega\equiv\omega-\omega_{p0}. \tag{9}$$

The first evident resonance condition is that the detuning must be small compared to the transition frequency ω_{p0} corresponding to the transition from the ground state, with the energy E_0, to the chosen particular state with the energy E_p,

$$\left|\frac{\Delta\omega}{\omega_{p0}}\right|\ll 1. \tag{10}$$

In addition, it is necessary that the pumping would not influence the neighboring states, that is, the detuning must satisfy the inequalities

$$\left|\frac{\Delta\omega}{\omega_{p+1,p}}\right|\ll 1,\qquad \left|\frac{\Delta\omega}{\omega_{p,p-1}}\right|\ll 1. \tag{11}$$

The resonance conditions (10) and (11) are necessary but not yet sufficient if we intend to populate only one particular level. This goal can be reached only when the pumping does not force transitions between other states, which can be expressed as the inequality

$$\left|\frac{\Delta\omega}{\omega-\omega_{mn}}\right|\ll 1\quad (m\neq p,\ n\neq 0). \tag{12}$$

The conditions (10) and (11) are easy to accomplish, making the detuning sufficiently small. The resonance condition (12) is more restrictive, requiring that the spectrum $\{E_n\}$, defined by the eigenproblem (6), would not be equidistant, as it happens for a simple harmonic oscillator. In fact, if that were the case, then the pumping of atoms from the ground state to the chosen particular state would, at the same time, induce transitions from the latter to another equidistant state and from the latter to higher equidistant states, and so on. In such a case, all atoms will be dispersed over a number of states making it impossible to get a macroscopic population of any of them. Fortunately, because of the nonlinearity, representing atomic interactions, in the Hamiltonian (3), the spectrum $\{E_n\}$ is not equidistant even when the confining potential U_c is harmonic. In addition, we may include the confining potential anharmonic terms and regulate its spectral characteristics by varying anharmonicity parameters [14–16]. Moreover, as we shall show in Sec. III, the spectral properties of the nonlinear Hamiltonian (3) may be essentially modified by varying the intensity of interactions (4), for which it is sufficient to change the number of atoms N. Therefore, it is always possible to prepare the system for which condition (12) holds true. The situation here is similar to the problem of inducing resonant electron transitions in an atom. The latter also contains many electron levels, but, because these are not equidistant, it is practically always possible to induce a resonant transition between a chosen pair of them [17]. The principal difference between the resonant electronic transitions in an atom and atomic transitions in a confining potential is that electronic levels are not equidistant because of the hydrogen-type potential, while the interactions between electrons do not play essential role. In such a case, the resonant electronic transitions can be treated in a linear approximation. Concerning the atoms in a confining potential, if the latter is harmonic, then the main role of making the energy levels nonequidistant is played by the atomic interactions. This makes the problem principally nonlinear and forces one to deal with complicated nonlinear equations.

To describe the time evolution of the system, we have to consider the time-dependent nonlinear Schrödinger equation (1). We present its solution as an expansion

$$\varphi(\vec{r},t)=\sum_n c_n(t)\varphi_n(\vec{r})\exp\left(-\frac{i}{\hbar}E_nt\right) \tag{13}$$

in the basis of the stationary states of the eigenproblem (6). Substituting Eq. (13) into Eq. (1) and taking into account that in the double sum

$$\sum_{k,l} c_k^* c_l(\varphi_m\varphi_k,\varphi_l\varphi_n)\exp(i\omega_{kl}t),$$

the main contribution comes from the term

$$\sum_k |c_k|^2(\varphi_m\varphi_k,\varphi_k\varphi_n)$$

because other terms containing the oscillating factors, being summed up, on average cancel each other. Then, from Eqs. (1) and (13) we have

$$i\hbar\frac{dc_n}{dt}=\sum_m\left(V_{nm}\cos\omega t+\sum_{k(\neq n)}A_{nkm}|c_k|^2\right)c_m e^{i\omega_{nm}t}, \tag{14}$$

with the matrix elements

$$V_{mn}\equiv(\varphi_m,V(\vec{r})\varphi_n),\quad A_{mkn}\equiv A(\varphi_m\varphi_k,\varphi_k\varphi_n).$$

The solution of Eq. (14) must satisfy the normalization

$$\sum_n|c_n(t)|^2=1.$$

Equation (14) is a set of equations for the functions $c_n=c_n(t)$ enumerated by a multi-index n. Let us separate this set into the equation for the ground-state function c_0, the equation for c_p representing a chosen state with $n=p$, and the equations for all other c_k with $k\neq 0,p$. We introduce the notion for the population probability

$$n_j=n_j(t)\equiv|c_j(t)|^2, \tag{15}$$

where the index is either $j=0$, $j=p$, or $j=k\neq 0,p$. Also, we define the parameters

$$\alpha\equiv\frac{1}{\hbar}A_{0p0}=\frac{A}{\hbar}\int|\varphi_0(\vec{r})|^2|\varphi_p(\vec{r})|^2d\vec{r} \tag{16}$$

and

$$\beta\equiv\frac{1}{\hbar}V_{0p}=\frac{1}{\hbar}\int\varphi_0^*(\vec{r})V(\vec{r})\varphi_p(\vec{r})d\vec{r}. \tag{17}$$

The solution to Eq. (14) can be presented as the sum of a guiding center plus a small oscillating ripple around the latter. The equation for the guiding center is obtained from Eq. (14) by averaging its right-hand side over time according to the rule $(1/\tau)\int_0^\tau F(t)dt$, with $\tau\to\infty$. During this averaging the exponential $e^{i\Delta\omega t}$ is treated as a constant since it is a slowly varying factor. Actually, at pure resonance, when $\Delta\omega\to 0$, this exponential $e^{i\Delta\omega t}\to 1$ is exactly one.

After realizing the described procedures, we obtain from Eq. (14) the system of equations

$$\frac{dc_0}{dt}=-i\alpha n_p c_0-\frac{i}{2}\beta e^{i\Delta\omega t}c_p, \tag{18a}$$

$$\frac{dc_p}{dt}=-i\alpha n_0 c_p-\frac{i}{2}\beta^* e^{-i\Delta\omega t}c_0, \tag{18b}$$

$$\frac{dc_k}{dt}=0\quad(k\neq 0,p). \tag{18c}$$

Since the functions c_j are complex, the system of equations (18) must be completed by another system either for the complex conjugate functions c_j^* or for the amplitudes $n_j=|c_j|^2$. The equations for the latter are

$$\frac{dn_0}{dt}=\mathrm{Im}(\beta e^{i\Delta\omega t}c_0^*c_p),$$

$$\frac{dn_p}{dt}=\mathrm{Im}(\beta^* e^{-i\Delta\omega t}c_p^*c_0), \tag{19}$$

and $dn_k/dt=0$ when $k\neq 0,p$. As the initial conditions we have

$$c_0(0)=1,\quad c_p(0)=0,\quad c_k(0)=0. \tag{20}$$

From Eq. (18c), together with the initial conditions (20), it follows that $c_k(t)=0$ for $k\neq 0,p$. Therefore, the normalization condition reads

$$n_0(t)+n_p(t)=1, \tag{21}$$

which demonstrates that the atoms are concentrated in the ground state and p level, preferentially.

Equations (18) and (19) form a system of complicated nonlinear differential equations. This system could be solved by perturbation theory in two limiting cases: either $|\alpha/\beta|\ll 1$ or $|\beta/\alpha|\ll 1$. In the intermediate regime, when $|\alpha/\beta|\sim 1$, perturbation theory is not applicable. A general solution, valid for arbitrary relation between the parameters α and β, can be obtained by employing the method of scale separation [18–20]. This can be done by noticing that the functions c_0 and c_p contain time-dependent imaginary factors absent in $n_0\equiv|c_0|^2$ and $n_p\equiv|c_p|^2$, that is, the time variation of c_0 and c_p is faster than that of n_0 and n_p. Consequently, c_0 and c_p can be classified as fast functions compared to the slower functions n_0 and n_p. Then the system (18) of equations for the fast functions can be approximately solved by keeping the slow functions n_0 and n_p as quasi-integrals of motion. From Eq. (18) we get the equations

$$\frac{d^2c_0}{dt^2}+i(\alpha-\Delta\omega)\frac{dc_0}{dt}+\left[\frac{|\beta|^2}{4}-\alpha n_p(\alpha n_0-\Delta\omega)\right]c_0=0,$$

$$\frac{d^2c_p}{dt^2}+i(\alpha+\Delta\omega)\frac{dc_p}{dt}+\left[\frac{|\beta|^2}{4}-\alpha n_0(\alpha n_p+\Delta\omega)\right]c_p=0, \tag{22}$$

with the initial conditions (20) and

$$\dot{c}_0(0)=-i\alpha n_p,\quad \dot{c}_p(0)=-\frac{i}{2}\beta^*, \tag{23}$$

where the overdot means a time derivative. The solution of Eqs. (22), with n_0 and n_p kept fixed, is written

$$c_0=\left[\cos\frac{\Omega t}{2}+i\frac{\alpha(n_0-n_p)-\Delta\omega}{\Omega}\sin\frac{\Omega t}{2}\right]\times\exp\left\{-\frac{i}{2}(\alpha-\Delta\omega)t\right\},$$

$$c_p=-i\frac{\beta^*}{\Omega}\sin\frac{\Omega t}{2}\exp\left\{-\frac{i}{2}(\alpha+\Delta\omega)t\right\}, \tag{24}$$

with the effective Rabi frequency given by the expression

$$\Omega^2=[\alpha(n_0-n_p)-\Delta\omega]^2+|\beta|^2. \tag{25}$$

Then for the slow functions we obtain

$$n_0 = 1 - \frac{|\beta|^2}{\Omega^2}\sin^2\frac{\Omega t}{2}, \quad n_p = \frac{|\beta|^2}{\Omega^2}\sin^2\frac{\Omega t}{2}. \tag{26}$$

The functions in Eqs. (26) describe the time evolution for the population of the ground state and of the chosen excited state. The form of these functions is similar to that encountered in considering the Rabi oscillations [17]. However, it is worth emphasizing that, contrary to the linear case, which can be recovered by setting $\alpha=0$, expressions (26) are actually the equations for n_0 and n_p since the effective Rabi frequency (25) itself depends on these populations. Because of this, the solution of Eqs. (26) will not result in simple sinusoidal oscillations.

Consider, for example, the case when the detuning is such that it satisfies the relation

$$\alpha + \Delta\omega = 0. \tag{27}$$

Then Eq. (25) gives

$$\Omega = \sqrt{4\alpha^2 n_0^2 + |\beta|^2}. \tag{28}$$

In that case Eq. (26) shows that the ground-state level becomes empty, while the upper resonant level is completely populated, i.e.,

$$n_0(t_k) = 0, \quad n_p(t_k) = 1,$$

at the moments of time

$$t_k = \frac{\pi}{|\beta|}(1+2k) \quad (k=0,1,2,\dots). \tag{29}$$

As far as $n_0 \to 0$, when $t \to t_k$, then the effective Rabi frequency (28) softens, $\Omega \to |\beta|$, and the motion around $t=t_k$ slows down. Hence the system spends more time on the upper level than in the ground state. Furthermore, if at the moment $t=t_k$ we switch off the pumping field (7), then we shall get an inverted system with all atoms being in the nonground-state. Another way of obtaining an inverted system could be by adiabatically varying the detuning, as in the regime of adiabatic passage [17], until we reach the compensation condition (27). The latter certainly makes sense only when the detuning continues to obey the resonance conditions (10)–(12).

If the compensation condition (27) cannot be satisfied, then it is impossible to transfer all atoms from the ground state to the chosen excited state. However, it is always possible to populate these states equally. In fact, consider the resonance case when $\Delta\omega = 0$. Then at the moments

$$t_k^* = \frac{\pi}{2|\beta|}(1+8k) \quad (k=0,1,2,\dots) \tag{30}$$

we have

$$n_0(t_k^*) = n_p(t_k^*) = \frac{1}{2},$$

that is, both states are equally populated.

It is worth paying attention to the following. The characteristic frequency of solutions (24) and (26) is the Rabi frequency (25). Looking back at Eq. (14), we see that, in general, there should exist as well solutions with extra frequencies being combinations of basic frequencies entering Eq. (14). One may ask the question: ‘‘When could such extra frequencies appear?’’

Recall that solutions (24) and (26) correspond to the guiding centers that constitute the main approximation in the method of averaging [21] and in the guiding-center approach [22]. The general form of the population amplitudes c_n is defined by Eq. (14). Denoting the general solution to the latter equation by c_n^{gen}, we may present it as a sum

$$c_n^{gen} = c_n + \sigma_n,$$

in which c_n is given by the guiding centers in Eq. (24) and σ_n is an additional ripple oscillating around the guiding centers. The characteristic frequency of the latter is the Rabi frequency (25). The ripple solution σ_n can in turn be written as a sum of terms with characteristic frequencies that are essentially higher than the Rabi frequency. In this way, the guiding center c_n represents the main harmonic, while the ripple solution σ_n represents a sum of higher harmonics. Averaging the general solution c_n^{gen} over the largest characteristic period corresponding to the higher harmonics gives the guiding center c_n. Therefore, being interested in the average behavior of solutions, one accepts the guiding center as the main approximation. Moreover, not only does the ripple term σ_n oscillate much faster than the guiding center but the amplitude of the former is smaller than that of the latter.

In order to concretize what is said above, let us substitute the general solution $c_n^{gen} = c_n + \sigma_n$ into Eq. (14). We introduce the notation

$$\alpha_{mjn} \equiv \frac{A_{mjn}}{\hbar}, \quad \beta_{mn} \equiv \frac{V_{mn}}{\hbar}.$$

Using Eqs. (18) for the guiding centers, we obtain the equation

$$\frac{d\sigma_n}{dt} = -i\sum_m \left\{ \beta_{nm}\sigma_m \cos\omega t + \sum_{j(\neq n)} \alpha_{njm}[|c_j|^2\sigma_m + (c_j^*\sigma_m + c_j\sigma_m^*)(c_m+\sigma_m)] \right\} e^{i\omega_{nm}t}$$

for the ripple term. Here c_n are the guiding centers defined by Eqs. (18) and (24). As is evident, the equation for the ripple term contains various higher harmonics, as a result of which the ripple solution σ_n oscillates faster than the guiding center.

Now let us explain why the amplitude of the ripple term is smaller than that of the guiding center. Introducing the notation $\varepsilon \equiv \max\{\alpha,\beta\}$, we see from Eq. (25) that the Rabi frequency $\Omega \sim \varepsilon$. Taking this into account and looking at Eq. (24), we conclude that the amplitude of the guiding center c_n is of order unity, $c_n \sim \varepsilon/\Omega \sim 1$.

The ripple solution σ_n can be presented as a sum of harmonics with amplitudes of order ε/Ω_ν, where Ω_ν is a characteristic frequency of a ν harmonic. Among these characteristic frequencies there are various combinations of ω and $\omega_{mn} \pm \omega$. Remember that, according to the quasiresonance condition (10), we have $\omega \sim \omega_{p0} \sim \varepsilon$. Consequently, the characteristic frequencies of harmonics are of order $\Omega_\nu \sim \nu\varepsilon$, with $\nu = 2,3,\ldots$. Thus the corresponding amplitudes are of order $\varepsilon/\Omega_\nu \sim 1/\nu$. Hence the amplitudes of higher harmonics diminish as $1/\nu$ with increasing $\nu = 2,3,\ldots$.

In this way, we see that the guiding centers in Eq. (24) really constitute the main approximation to Eq. (14). In this approximation, the ripple solution, which oscillates faster and has smaller amplitude, can be neglected. If needed, the higher-order harmonics can be taken into account by means of perturbation theory. Such a situation is common for the method of averaging and the guiding-center approach [21,22].

The space-time distribution of atoms is given by the density

$$\rho(\vec{r},t) = \rho_0(\vec{r},t) + \rho_p(\vec{r},t),$$

in which

$$\rho_j(\vec{r},t) = N n_j(t) |\varphi(\vec{r})|^2$$

is a partial density for $j = 0,p$. These densities are normalized to the total number of atoms

$$N = \int \rho(\vec{r},t) d\vec{r}$$

and, respectively, to the number of atoms

$$N_j = \int \rho_j(\vec{r},t) d\vec{r} = N n_j(t)$$

in the corresponding states. Since the spatial dependence of the wave functions for different states is different, we may get condensates with different space distributions. In general, such condensates will coexist, though, if the compensation condition is achieved, a pure non-ground-state condensate can be realized.

In our consideration we have assumed that the system is initially cooled down so that all atoms are condensed in the ground state. The possible admixture of noncondensed thermally excited atoms has been neglected. Such a picture, as is known, is admissible for sufficiently low temperatures below the condensation point. If the temperature is kept low during the process of the resonant pumping, we may continue disregarding thermal excitations. Their role becomes important only after we switch off the pumping field. Since during this pumping the system has acquired additional energy, the latter can be redistributed among the energy levels through some relaxation mechanism. The role of such a mechanism will be played by the interactions between condensed and noncondensed atoms. The thermally excited atoms will form a kind of a heat bath providing the possibility of relaxation to equilibrium.

III. STATIONARY STATES

Stationary states for the nonlinear Hamiltonian (3) are defined by the eigenvalue problem (6). The confining potential, typical of magnetic traps, is well described by the anisotropic harmonic potential

$$U_c(\vec{r}) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \tag{31}$$

It is convenient to pass to dimensionless quantities measured in units of the characteristic oscillator frequency ω_0 and length l_0 given by the expressions

$$\omega_0 \equiv (\omega_x \omega_y \omega_z)^{1/3}, \quad l_0 \equiv \sqrt{\frac{\hbar}{m\omega_0}}. \tag{32}$$

The anisotropy of potential (31) is characterized by the anisotropy parameters

$$\lambda_1 \equiv \frac{\omega_x}{\omega_0}, \quad \lambda_2 \equiv \frac{\omega_y}{\omega_0}, \quad \lambda_3 \equiv \frac{\omega_z}{\omega_0}. \tag{33}$$

We define the dimensionless coordinates

$$x_1 \equiv \frac{x}{l_0}, \quad x_2 \equiv \frac{y}{l_0}, \quad x_3 \equiv \frac{z}{l_0} \tag{34}$$

forming the vector $\vec{x} = \{x_1, x_2, x_3\}$. The dimensionless interaction parameter is

$$g \equiv \frac{mA}{\hbar^2 l_0} = 4\pi(N-1)\frac{a}{l_0}. \tag{35}$$

Introducing the dimensionless Hamiltonian and wave function, respectively,

$$H \equiv \frac{H(\varphi)}{\hbar\omega_0}, \quad \psi(\vec{x}) \equiv l_0^{3/2}\varphi(\vec{r}), \tag{36}$$

we obtain

$$H = \frac{1}{2}\sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial x_i^2} + \lambda_i^2 x^2\right) + g|\psi|^2. \tag{37}$$

Even when the scattering length a is much less than the oscillator length l_0, the interaction parameter (35) can be very large because of the great number of particles N. This situation is similar to that existing for large clusters [23]. Actually, a group of atoms trapped in a confining potential also forms a kind of a cluster. With a large interaction parameter, one cannot apply the standard perturbation theory in powers of g for calculating the eigenvalues of the Hamiltonian (37). Nevertheless, one may employ the renormalized perturbation theory [24–30], which, as has been shown by a number of examples [14–16,24–31], successfully works for arbitrary values of the coupling parameter, as well as for all energy levels, providing good accuracy with the maximal error around 1%. The first step of this approach is to choose an initial approximation containing trial parameters that, in the following steps, will be turned into control functions controlling the convergence of the procedure [24]. These control functions are to be defined from the fixed-point and stability conditions [29,30]. One of the simplest forms of the fixed-point condition is the minimal-sensitivity condition [28], which, for the first-order approximation, is equivalent to the variational condition for an energy functional [27–32].

As an initial approximation, we may take the Hamiltonian

$$H_0=\frac{1}{2}\sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial x_i^2}+u_i^2x_i^2\right), \tag{38}$$

in which the effective frequencies u_i, with $i=1,2,3$, will be control functions. The eigenfunctions of the Hamiltonian (38) are the oscillator wave functions

$$\psi_n(\vec{x})=\prod_{i=1}^{3}\psi_{n_i}(x_i), \quad n\equiv\{n_1,n_2,n_3\},$$

where $n_i=0,1,2,\ldots$. Correspondingly, the eigenvalues of H_0 are given by

$$E_n^{(0)}=\sum_{i=1}^{3}u_i\left(n_i+\frac{1}{2}\right). \tag{39}$$

Perturbation theory is to be constructed with respect to the perturbation $\Delta H\equiv H-H_0$, which is

$$\Delta H=\frac{1}{2}\sum_{i=1}^{3}(\lambda_i^2-u_i^2)x_i^2+g|\psi|^2.$$

The eigenvalues, in the first-order approximation, are

$$E_n^{(1)}(\lambda,g,u)=E_n^{(0)}+\Delta E_n, \tag{40}$$

where, for compactness, the notations $\lambda\equiv\{\lambda_1,\lambda_2,\lambda_3\}$ and $u\equiv\{u_1,u_2,u_3\}$ are accepted and

$$\Delta E_n=(\psi_n,\Delta H\psi_n).$$

The control functions $u_i=u_i(\lambda,g,n)$ can be found from the variational condition

$$\frac{\partial}{\partial u_i}E_n^{(1)}(\lambda,g,u)=0, \tag{41}$$

which is a simple form of the fixed-point condition. Substituting the found u_i into Eq. (40), we have

$$e_n(\lambda,g)\equiv E_n^{(1)}(\lambda,g,u(\lambda,g,n)). \tag{42}$$

For what follows, it is useful to introduce the notation

$$g_n\equiv gJ_n, \quad J_n\equiv\prod_{i=1}^{3}J_{n_i}, \tag{43}$$

in which

$$J_{n_i}=\frac{(|\psi_{n_i}|^2,|\psi_{n_i}|^2)}{\sqrt{u_i(n)}}=\frac{1}{\pi(2^{n_i}n_i!)^2}\int_{-\infty}^{+\infty}H_{n_i}^4(x)\exp(-2x^2)dx,$$

where $H_{n_i}(x)$ is a Hermite polynomial and $u_i(n)\equiv u_i(\lambda,g,n)$. Equation (40), with the notation in Eq. (43), can be written as

$$E_n^{(1)}(\lambda,g,u)=\frac{1}{2}\sum_{i=1}^{3}\left(n_i+\frac{1}{2}\right)\left(u_i+\frac{\lambda_i^2}{u_i}\right)+\sqrt{u_1u_2u_3}\,g_n. \tag{44}$$

Condition (41) results in the equation

$$\left(n_i+\frac{1}{2}\right)(u_i^2-\lambda_i^2)+u_i\sqrt{u_1u_2u_3}\,g_n=0. \tag{45}$$

Using Eqs. (44) and (45), for the spectrum in Eq. (42) we have

$$e_n(\lambda,g)=\frac{1}{6}\sum_{i=1}^{3}\left(n_i+\frac{1}{2}\right)\left(u_i+5\frac{\lambda_i^2}{u_i}\right), \tag{46}$$

where $u_i=u_i(\lambda,g,n)$ are defined by Eq. (45).

To understand better the properties of the spectrum (46), let us consider the weak- and strong-coupling limits. In the weak-coupling limit, when $g_n\to 0$, the solution to Eq. (45) is written

$$u_i\simeq\lambda_i-\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{2\left(n_i+\frac{1}{2}\right)}g_n+\left\{\lambda_i\frac{\sum_{j=1}^{3}\left(n_j+\frac{1}{2}\right)\lambda_j-\left(n_i+\frac{1}{2}\right)\lambda_i}{8\prod_{j=1}^{3}\left(n_j+\frac{1}{2}\right)}\right.$$

$$\left.+\frac{\lambda_1\lambda_2\lambda_3}{4\left(n_i+\frac{1}{2}\right)^2\lambda_i}\right\}g_n^2. \tag{47}$$

Substituting this into Eq. (46), we get

$$e_n(\lambda,g)\simeq\sum_{i=1}^{3}\left(n_i+\frac{1}{2}\right)\lambda_i+\sqrt{\lambda_1\lambda_2\lambda_3}\,g_n$$

$$-\frac{1}{8}\sum_{i=1}^{3}\frac{\lambda_1\lambda_2\lambda_3}{\left(n_i+\frac{1}{2}\right)\lambda_i}g_n^2, \tag{48}$$

as $g_n\to 0$. In the strong-coupling limit, when $g_n\to\infty$, the solution to Eq. (45) reads

$$u_i\simeq\frac{\left(n_i+\frac{1}{2}\right)\lambda_i^2}{\left[\prod_{j=1}^{3}\left(n_j+\frac{1}{2}\right)\lambda_j^2\right]^{1/5}}g_n^{-2/5}. \tag{49}$$

For the spectrum (46) we obtain

$$e_n(\lambda,g)\simeq\frac{5}{2}\left[\prod_{j=1}^{3}\left(n_j+\frac{1}{2}\right)\lambda_j^2\right]^{1/5}g_n^{2/5} \tag{50}$$

as $g_n\to\infty$.

IV. GROUND STATE

The ground state plays a special role for the phenomenon of Bose condensation. Therefore, we pay a little more attention to the case of $n_i=0$. Since $J_0=(2\pi)^{-1/2}$, the effective interaction strength (43) becomes

$$g_0=\frac{g}{(2\pi)^{3/2}}. \tag{51}$$

Equation (46), defining control functions, simplifies to

$$u_i^2 + 2g_0 u_i \sqrt{u_1 u_2 u_3} - \lambda_i^2 = 0 \tag{52}$$

and the spectrum (46) reduces to

$$e_0(\lambda, g) = \frac{1}{12} \sum_{i=1}^{3} \left(u_i + 5\frac{\lambda_i^2}{u_i} \right). \tag{53}$$

In the weak-coupling limit, when $g_0 \to 0$, Eq. (52) yields

$$u_i \simeq \lambda_i - \sqrt{\lambda_1 \lambda_2 \lambda_3}\, g_0 + \left[\frac{\lambda_i}{2} \left(\sum_{j=1}^{3} \lambda_j - \lambda_i \right) + \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_i} \right] g_0^2. \tag{54}$$

For the spectrum (53), we have

$$\begin{aligned} e_0(\lambda, g) \simeq{}& \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) + \sqrt{\lambda_1 \lambda_2 \lambda_3}\, g_0 - \frac{1}{16}(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 \\ &+ \lambda_3 \lambda_1) g_0^2 \end{aligned} \tag{55}$$

as $g_0 \to 0$. In the strong-coupling limit, when $g_0 \to \infty$, for the control functions we get

$$u_i \simeq \frac{\lambda_i^2}{(2\lambda_1 \lambda_2 \lambda_3)^{2/5}} g_0^{-2/5}, \tag{56}$$

and, respectively, for the spectrum we find

$$e_0(\lambda, g) \simeq \frac{5}{4} (2\lambda_1 \lambda_2 \lambda_3)^{2/5} g_0^{2/5}. \tag{57}$$

For an axially symmetric potential, with $\lambda_1 = \lambda_2 \neq \lambda_3$, the strong-coupling limit (57) reduces to that found by Baym and Pethick [33].

For an isotropic potential, for which $\lambda_i = 1$, there is only one control function given by the equation

$$u^2 + 2g_0 u^{5/2} - 1 = 0. \tag{58}$$

Then the ground-state energy (53) becomes

$$e_0(g) = \frac{1}{4} \left(u + \frac{5}{u} \right). \tag{59}$$

In the weak-coupling limit, when $g_0 \to 0$, Eq. (58) gives

$$u \simeq 1 - g_0 + 2g_0^2 \tag{60}$$

and the spectrum (59) is

$$e_0(g) \simeq \frac{3}{2} + g_0 - \frac{3}{4} g_0^2 \tag{61}$$

as $g_0 \to 0$. In the strong-coupling limit, as $g_0 \to \infty$, we have

$$u \simeq (2g_0)^{-2/5} \tag{62}$$

and, respectively,

$$e_0(g) \simeq \frac{5}{4} (2g_0)^{2/5}. \tag{63}$$

For the atoms with negative scattering lengths, as in the case of ^{7}Li or ^{85}Rb, the interaction parameter (35) is negative. All weak-coupling expansions, such as Eqs. (48), (55), and (61), are the same for the negative $g \to -0$. When g increases, the real solution for the spectrum exists until some critical value g_c after which it becomes complex. Thus, for the isotropic potential, the stable ground state is defined for $g_0^c < g_0 < 0$, with

$$g_0^c = -\frac{2}{5^{5/4}} = -0.267\,496. \tag{64}$$

From here, the critical interaction parameter is

$$g_c = (2\pi)^{3/2} g_0^c = -4.212\,960. \tag{65}$$

This, according to notation in Eq. (35), defines the critical number of atoms

$$N_c = 1 + \frac{g_c l_0}{4\pi a} \tag{66}$$

that can condense in a stable state. After $g < g_c$, the spectrum becomes complex. The latter means that the corresponding states are no longer stationary, but quasistationary. The lifetime of a quasistationary state is defined [34,35] as

$$\tau_n(g) = \frac{1}{2\omega_0 |\mathrm{Im}\, e_n(g)|}. \tag{67}$$

The complex spectrum of quasistationary states is defined, as earlier, by Eq. (46), with u_i from Eq. (45). These equations, in the case of negative $g < 0$, have several complex solutions, of which we must choose that continuously branching, at $g = g_c$, from a solution that is real at $g \to -0$. Solving these equations for the isotropic ground state, we find

$$\begin{aligned} \mathrm{Re}\, u &\simeq 0.705\,470 g^{-2/5} + 3.850\,396 g^{-6/5} + 12.402\,511 g^{-2}, \\ \mathrm{Im}\, u &\simeq 2.171\,212 g^{-2/5} + 2.797\,476 g^{-6/5} \end{aligned} \tag{68}$$

for $g \gg 1$. For the real and imaginary parts of the energy we obtain

$$\begin{aligned} \mathrm{Re}\, e_0(g) \simeq{}& 0.169\,198 g^{2/5} + 0.529\,102 g^{-2/5} + 1.443\,899 g^{-6/5} \\ &+ 31.006\,277 g^{-2}, \\ \mathrm{Im}\, e_0(g) \simeq{}& -0.520\,739 g^{2/5} + 1.628\,409 g^{-2/5} \\ &+ 1.049\,054 g^{-6/5}. \end{aligned} \tag{69}$$

Therefore, for the lifetime (67) we have

$$\tau_0(g) \simeq \frac{0.960\,174}{\omega_0 g^{2/5}}. \tag{70}$$

If the number of atoms with a negative scattering length exceeds the critical number given by Eq. (66), then only N_c atoms can form a stable Bose condensate, excessive atoms being expelled out of the condensate during the time (70).

V. EXCITED STATES

The energy of any excited state is defined by Eq. (46). In order to illustrate the relative behaviors of these states, let us write explicitly the corresponding formulas for the first few levels. Consider, for the isotropic case, the energy levels $e_{100}(g)$, $e_{110}(g)$ and $e_{200}(g)$. For these levels, the integrals J_{n_i} entering the notation in Eq. (43) are

$$J_0=\frac{1}{\sqrt{2\pi}}, \quad J_1=\frac{3}{4\sqrt{2\pi}}, \quad J_2=\frac{41}{64\sqrt{2\pi}}.$$

In the weak-coupling limit, the energies of the first three excited states, compared to that of the ground state, behave as

$$e_0(g)\simeq\frac{3}{2}+0.063\,494g, \quad e_{100}(g)\simeq\frac{5}{2}+0.047\,620g,$$
$$e_{110}(g)\simeq\frac{7}{2}+0.035\,715g, \quad e_{200}(g)\simeq\frac{7}{2}+0.040\,676g \tag{71}$$

when $g\to0$. In the strong-coupling limit, we get

$$e_0(g)\simeq0.547\,538g^{2/5}, \quad e_{100(}g\simeq0.607\,943g^{2/5},$$
$$e_{200}(g)\simeq0.632\,193g^{2/5}, \quad e_{110}(g)\simeq0.675\,012g^{2/5} \tag{72}$$

as $g\to\infty$. Notice an interesting and important fact that the energy levels $e_{110}(g)$ and $e_{200}(g)$ cross each other as g varies. In fact, from Eq. (71) it follows that $e_{110}(g)<e_{200}(g)$ at small g, while Eq. (72) shows that $e_{110}(g)>e_{200}(g)$ at large g. This crossing of levels demonstrates that the latter cannot be classified being based on the harmonic-oscillator spectrum. To classify the levels correctly, one has to calculate their energies for each given interaction strength g. At the same time, the strong distortion of the harmonic-oscillator spectrum plays a positive role for our purpose, making the spectrum of interacting atoms nonequidistant, which is necessary for the possibility of transferring the ground-state Bose condensate to a non-ground-state, as explained in Sec. II.

The transition frequency, measured in units of ω_0, is given by the difference

$$\omega_n(g)\equiv e_n(g)-e_0(g). \tag{73}$$

For the low-lying excited states this leads to

$$\omega_{100}\simeq1-0.015\,874g, \quad \omega_{110}\simeq2-0.027\,779g,$$
$$\omega_{200}\simeq2-0.022\,818g \tag{74}$$

in the weak-coupling limit $g\to0$ and to

$$\omega_{100}(g)\simeq0.060\,405g^{2/5}, \quad \omega_{200}(g)\simeq0.084\,655g^{2/5},$$
$$\omega_{110}(g)\simeq0.127\,474g^{2/5} \tag{75}$$

in the strong-coupling limit as $g\to\infty$. Equations (71)–(75) show that a sufficiently strong interaction parameter g makes the spectrum nonequidistant, which favors the possibility of the resonance pumping of atoms from the ground state to an excited level.

The shape of the atomic cloud in a given state can be characterized by the aspect ratio

$$R_{13}\equiv\left(\frac{\langle x_1^2\rangle}{\langle x_3^2\rangle}\right)^{1/2}, \tag{76}$$

in which $\langle x_1^2\rangle$ is a mean-square deviation in the x direction and $\langle x_3^2\rangle$ is a mean-square deviation in the z direction. Similarly, we may define the aspect ratio R_{23}. In the case of cylindrical symmetry, $R_{13}=R_{23}$.

The aspect ratio (76) can be written as

$$R_{13}=\left[\frac{\left(n_1+\frac{1}{2}\right)u_3}{\left(n_3+\frac{1}{2}\right)u_1}\right]^{1/2}. \tag{77}$$

In the weak-coupling limit this yields

$$R_{13}\simeq\left(\frac{\left(n_1+\frac{1}{2}\right)\lambda_3}{\left(n_3+\frac{1}{2}\right)\lambda_1}\right)^{1/2}\left\{1+\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{4(2\pi)^{3/2}}\left[\frac{1}{\left(n_1+\frac{1}{2}\right)\lambda_1}-\frac{1}{\left(n_3+\frac{1}{2}\right)\lambda_3}\right]g\right\} \tag{78}$$

when $g\to0$. For example, for the ground state, Eq. (78) reduces to

$$R_{13}^0\simeq\sqrt{\frac{\lambda_3}{\lambda_1}}\left\{1+\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{2(2\pi)^{3/2}}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_3}\right)g\right\}. \tag{79}$$

In the strong-coupling limit, Eq. (77) gives

$$R_{13}\simeq\frac{\lambda_3}{\lambda_1} \quad (g\to\infty) \tag{80}$$

for all energy levels.

For the ground state, our results agree with the aspect ratio found by other authors [33,36,37]. For excited states, we predict that the aspect ratio in the strong-coupling limit becomes asymptotically, as $g\to\infty$, independent of level numbers, according to Eq. (80).

VI. INTERACTION AMPLITUDE

An important quantity defining whether it is possible to transfer all atoms from the ground state to an upper level, or only half of them, is the interaction amplitude (16). For convenience, let us define the dimensionless amplitude

$$\alpha_n(g)\equiv\frac{\alpha}{\omega_0}, \tag{81}$$

which may be presented as

$$\alpha_n(g)=\sqrt{u_1(0)u_2(0)u_3(0)}\,I_n g,$$

where $u_i(n)\equiv u_i(\lambda,g,n)$ and

$$I_n\equiv\prod_{i=1}^{3} I_{n_i}, \quad n\equiv\{n_1,n_2,n_3\},$$

$$I_{n_i}\equiv\frac{(|\psi_0|^2,|\psi_{n_i}|^2)}{\sqrt{u_i(0)}}=\frac{1}{\pi 2^{n_i}n_i!}\int_{-\infty}^{+\infty}H^2_{n_i}(x)\times\exp\left\{-\frac{u_i(0)+u_i(n)}{u_i(n)}x^2\right\}dx.$$

The last integral can be expressed as

$$I_{n_i}=\frac{(-1)^{n_i}\Gamma\left(n_i+\frac{1}{2}\right)}{\pi n_i!\,\zeta_i^{n_i}}\sqrt{\frac{2\zeta_i-1}{2\zeta_i}}F\left(-n_i,-n_i;\frac{1}{2}-n_i;\zeta_i\right)$$

through the gamma function $\Gamma(\,)$ and the hypergeometric function $F(\,)$, with

$$\zeta_i\equiv\frac{u_i(0)+u_i(n)}{2u_i(0)}.$$

In the weak-coupling limit, using the equality

$$F\left(-n_i,-n_i;\frac{1}{2}-n_i;1\right)=(-1)^{n_i},$$

we find

$$\alpha_n(g)\simeq\frac{\sqrt{2\lambda_1\lambda_2\lambda_3}}{4\pi^3}\left[\prod_{i=1}^{3}\frac{\Gamma\left(n_i+\frac{1}{2}\right)}{n_i!}\right]g \tag{82}$$

as $g\to 0$. In the strong-coupling limit we obtain

$$\alpha_n(g)\simeq(\lambda_1\lambda_2\lambda_3)^{2/5}\left(\frac{64}{\pi^7}\right)^{3/10}\left[\prod_{i=1}^{3}\frac{(-1)^{n_i}}{n_i!}\Gamma\left(n_i+\frac{1}{2}\right)\times\sqrt{\frac{2\zeta_i-1}{2\zeta_i}}F\left(-n_i,-n_i;\frac{1}{2}-n_i;\zeta_i\right)\right]g^{2/5}, \tag{83}$$

where $g\to\infty$ and

$$\zeta_i=\frac{1}{2}+\frac{n_i+\frac{1}{2}}{2\pi\left[8\prod_{i=1}^{3}\left(n_i+\frac{1}{2}\right)J^2_{n_i}\right]^{1/5}}.$$

To compare the interaction amplitudes for the first several levels with the corresponding energies and transition frequencies studied in Sec. V, let us take the isotropic case. Then, in the weak-coupling limit Eq. (82) yields

$$\alpha_{000}(g)\simeq 0.063\,49g, \quad \alpha_{100}(g)\simeq 0.031\,75g,$$

$$\alpha_{110}(g)\simeq 0.015\,87g, \quad \alpha_{200}(g)\simeq 0.023\,81g \tag{84}$$

as $g\to 0$. In the strong-coupling limit, from Eq. (83) we derive

$$\alpha_{000}(g)\simeq 0.219\,02g^{2/5}, \quad \alpha_{100}(g)\simeq 0.183\,04g^{2/5},$$

$$\alpha_{110}(g)\simeq 0.147\,59g^{2/5}, \quad \alpha_{200}(g)\simeq 0.175\,59g^{2/5} \tag{85}$$

as $g\to\infty$.

Comparing the interaction amplitudes (84) and (85) with the transition frequencies (74) and (75), we see that the compensation condition (27) could be accomplished in the weak-coupling limit or in the intermediate region of g, where $\alpha_n(g)\ll\omega_n(g)$. In the strong-coupling limit, $\alpha_n(g)$ becomes of the order of $\omega_n(g)$ and condition (27) cannot be satisfied since $|\Delta\omega|\ll\omega_n(g)$. This means that, with increasing g, that is, with increasing the number of atoms in the ground-state condensate, it becomes more difficult to transfer all these atoms to an upper level. For large atomic clouds, only half of the condensed atoms can be pumped up to an upper level.

VII. DISCUSSION

We suggested a mechanism for creating nonequilibrium Bose condensates in nonground states. The possibility of transferring either all atoms or only some of them to a nonground-state depends on the parameters of the system. In principle, these parameters are changeable and we think that it is possible to adjust them to effectively realize the mechanism suggested. For concreteness, let us look at the parameters typical of magnetic traps used now for condensing atoms in the ground state.

In the JILA trap [1,38] the atoms of ^{87}Rb, with the mass $m=1.445\times10^{-22}$ g and scattering length $a=6\times10^{-7}$ cm, were condensed. The characteristic frequencies of the confining potential are $\omega_x=\omega_y=2\pi\times132$ Hz and $\omega_z=2\pi\times373$ Hz, their geometric mean, defined in Eq. (32), being $\omega_0=2\pi\times187$ Hz. The anisotropy parameters (33) are $\lambda_1=\lambda_2=1/\sqrt{2}$ and $\lambda_3=2$. The oscillator length $l_0=0.788\times10^{-4}$ cm. Therefore, the interaction parameter (35) is

$$g=9.565(N-1)\times10^{-2}.$$

For the number of atoms $N\sim10^3$, one has $g\sim100$, which corresponds to the strong-coupling limit.

In the MIT trap [3,39] a condensate of ^{23}Na, with $m=0.382\times10^{-22}$ g and $a=5\times10^{-7}$ cm, was realized. The characteristic frequencies are $\omega_x=\omega_y=2\pi\times320$ Hz and $\omega_z=2\pi\times18$ Hz, so that their geometric mean is $\omega_0=2\pi\times123$ Hz. The anisotropy parameters are $\lambda_1=\lambda_2=2.610$ and $\lambda_3=0.147$. The oscillator length is $l_0=1.890\times10^{-4}$ cm. Thus the interaction parameter (35) is

$$g=3.324(N-1)\times10^{-2}.$$

For $N\sim(5\times10^5)-(5\times10^6)$, this gives $g\sim10^4-10^5$, which certainly is in the strong-coupling limit.

In the RQI trap [2,40] the atoms of ^{7}Li, having $m=0.116\times10^{-22}$ g and negative scattering length $a=-1.5\times10^{-7}$ cm, were cooled down to the Bose condensation regime. With the characteristic frequencies $\omega_x=2\pi\times150.6$ Hz, $\omega_y=2\pi\times152.6$ Hz, and

$\omega_z = 2\pi \times 131.5$ Hz, the geometric mean frequency is $\omega_0 = 2\pi \times 145$ Hz. The anisotropy parameters (33) are $\lambda_1 = 1.039$, $\lambda_2 = 1.052$, and $\lambda_3 = 0.907$. The oscillator length $l_0 = 3.160 \times 10^{-4}$ cm. The interaction parameter (35) becomes

$$g = -0.597(N-1) \times 10^{-2}.$$

Since the confining potential is almost isotropic ($\lambda_1 \approx \lambda_2 \approx \lambda_3$) we may take the value (65) for the critical interaction strength allowing yet a stable condensate. Then the critical number (66) of atoms allowed in the condensate is $N_c = 707$, which is in agreement with the estimates of other authors [33,37] giving $N_c \sim 10^3$.

In this way, the trapped clouds of ^{87}Rb and ^{23}Na correspond to the strong-coupling limit and that of ^{7}Li to an intermediate regime, with $g \leqslant 4$. Therefore, if the number of condensed atoms in a cloud is $N \geqslant 10^3$, it would be difficult to transfer all of them to an upper level. However, as discussed in Sec. II, it is always possible to transfer half of the ground-state atoms to a non-ground-state level, thus creating two coexisting condensates of the same species in two different quantum states. In addition, it is probably possible to construct specially designed traps allowing all atoms from the ground state to be pumped up to another level, forming a pure non-ground-state Bose condensate. In any case, both these possibilities being realized would produce a different kind of quantum ultracold matter, which may reveal interesting and unexpected properties.

ACKNOWLEDGMENTS

We are grateful to B.W. Shore for very useful discussions. Financial support from the National Science and Technology Development Council of Brazil and from the São Paulo State Research Foundation is appreciated. V.S.B. also acknowledges support from the program PRONEX.

---

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995).

[2] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995).

[3] K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).

[4] *Book of Abstracts of Workshop on Collective Effects in Ultracold Atomic Gases*, 1996 Les Houches Lectures, edited by C. Salomon and F. Castin (Centre de Physique des Houches, Les Houches, 1996).

[5] J. Res. Natl. Inst. Stand. Technol. **101** (4) (1996), special issue on Bose-Einstein condensation, edited by K. Burnett, M. Edwards, and C. W. Clark.

[6] C. J. Myatt, E. A. Burt, R. W. Christ, E. A. Cornell, and C. E. Wieman, Phys. Rev. Lett. **78**, 586 (1997).

[7] S. J. Kokkelmans, H. M. Boesten, and B. J. Verhaar, Phys. Rev. A **55**, 1589 (1997).

[8] E. P. Gross, Phys. Rev. **106**, 161 (1957).

[9] V. L. Ginzburg and L. P. Pitaevskii, J. Exp. Theor. Phys. **7**, 858 (1958).

[10] E. P. Gross, Nuovo Cimento **20**, 454 (1961).

[11] L. P. Pitaevskii, J. Exp. Theor. Phys. **13**, 451 (1961).

[12] E. P. Gross, J. Math. Phys. **4**, 195 (1963).

[13] A. C. Biswas and S. R. Shenoy, Physica B **90**, 265 (1977).

[14] E. P. Yukalova and V. I. Yukalov, Phys. Lett. A **175**, 27 (1993).

[15] E. P. Yukalova and V. I. Yukalov, J. Phys. A **26**, 2011 (1993).

[16] V. I. Yukalov and E. P. Yukalova, Laser Phys. **5**, 154 (1995).

[17] B. W. Shore, *The Theory of Coherent Atomic Excitation* (Wiley, New York, 1990).

[18] V. I. Yukalov, Phys. Rev. Lett. **75**, 3000 (1995).

[19] V. I. Yukalov, Laser Phys. **5**, 970 (1995).

[20] V. I. Yukalov, Phys. Rev. B **53**, 9232 (1996).

[21] N. N. Bogolubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Gordon and Breach, New York, 1961).

[22] N. G. Van Kampen, Phys. Rep. **124**, 69 (1985).

[23] V. I. Yukalov and E. P. Yukalova, Physica A **223**, 15 (1996).

[24] V. I. Yukalov, Mosc. Univ. Phys. Bull. **31**, 10 (1976).

[25] W. E. Caswell, Ann. Phys. (N.Y.) **123**, 152 (1979).

[26] I. Halliday and P. Surranyi, Phys. Rev. D **21**, 1529 (1980).

[27] J. Killingbeck, J. Phys. A **14**, 1005 (1981).

[28] P. M. Stivenson, Phys. Rev. D **23**, 2916 (1981).

[29] V. I. Yukalov, J. Math. Phys. **32**, 1235 (1991).

[30] V. I. Yukalov, J. Math. Phys. **33**, 3994 (1992).

[31] I. D. Feranchuk, L. I. Komarov, I. V. Nichipor, and A. P. Ulyanenkov, Ann. Phys. (N.Y.) **238**, 370 (1995).

[32] R. J. Dodd, J. Res. Natl. Inst. Stand. Technol. **101**, 545 (1996).

[33] G. Baym and C. J. Pethik, Phys. Rev. Lett. **76**, 6 (1996).

[34] R. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

[35] V. I. Melnikov, Phys. Rep. **209**, 1 (1991).

[36] M. Edwards, R. J. Dodd, C. W. Clark, P. A. Ruprecht, and K. Burnett, Phys. Rev. A **53**, 1950 (1996).

[37] F. Dalfovo and S. Stringari, Phys. Rev. A **53**, 2477 (1996).

[38] J. R. Ensher, D. S. Jin, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **77**, 4984 (1996).

[39] M. O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, and W. Ketterle, Phys. Rev. Lett. **77**, 416 (1996).

[40] C. C. Bradley, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. **78**, 985 (1997).