# First and second sound in a uniform Bose gas

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We have recently derived two-fluid hydrodynamic equations for a trapped weakly interacting Bose gas. In this paper we use these equations to discuss first and second sound in a uniform Bose gas. These results are shown to agree with the predictions of the usual two-fluid equations of Landau when the thermodynamic functions are evaluated for a weakly interacting gas. In a uniform gas, second sound mainly corresponds to an oscillation of the superfluid (the condensate) and is the low-frequency continuation of the Bogoliubov-Goldstone symmetry-breaking mode. [S1050-2947(97)00812-3]

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## I. INTRODUCTION

One of the most spectacular features [1] exhibited by superfluid $^4$He is the existence of two hydrodynamic sound modes, first and second sound. As first pointed out by Tisza [2], the motion of a Bose condensate as a separate degree of freedom results in a two-fluid hydrodynamics describing the superfluid and normal fluid components [3]. In a classic study, Bogoliubov [4] provided a formal derivation of the two-fluid equations in the Landau form. This derivation is valid at all temperatures and holds equally for Bose gases and liquids. However, the derivation is very complex and still requires some microscopic model in order to complete the specification of the thermodynamic quantities that appear.

In recent work we [5] gave a simple microscopic derivation of two-fluid hydrodynamic equations of motion for a trapped weakly interacting Bose-condensed gas that has the advantage of being very transparent and, moreover, gives explicit expressions for all the coefficients in the equations. In the present paper, we use these two-fluid equations to discuss the first and second sound modes of a *uniform* Bose-condensed gas. In contrast to a Bose-condensed liquid like superfluid $^4$He, the superfluid in a gas corresponds directly to the condensate atoms and the normal fluid corresponds to the noncondensate (or excited) atoms. We find that at temperatures close to $T_{BEC}$, first (second) sound mainly corresponds to an oscillation of the noncondensate (condensate) atoms. We also confirm [6,7] that it is the second sound mode in a uniform gas that is the low-frequency hydrodynamic analog of the collisionless Bogoliubov-Goldstone mode at finite temperatures [8].

These results for a uniform gas are of interest as a basis of comparison with the analogous hydrodynamic oscillations of the condensate and noncondensate in a *nonuniform* trapped Bose gas [5,9]. In particular, Andrews *et al.* have very recently presented results for the propagation of collisionless sound pulses along the $z$ axis of a cigar-shaped trap [10]. Due to the large anisotropy in this trap, one has a hybrid situation in which the trapped gas is approximately uniform along the $z$ axis, but highly inhomogeneous in the radial direction. It is clear that the results for a homogeneous gas are not directly applicable in this situation, but nevertheless, the experimental data for the sound speed appears to be consistent with the Bogoliubov phonon velocity in a uniform gas having a density equal to the peak density in the trap. Anticipating the accessibility of the hydrodynamic regime in future experiments, one would therefore expect to see first and second sound propagation with velocities that are approximately given by the uniform gas results of the present paper. The possibility of such experiments has been an important motivation of our work.

We recall that Ref. [5] is based on (a) a time-dependent Hartree-Fock-Popov equation of motion for the condensate wave function $\Phi(\mathbf{r},t)$ and (b) a set of hydrodynamic equations for the fluctuations of the thermal cloud (noncondensate) based on a kinetic equation that includes the effect on the atoms of the time-dependent self-consistent Hartree-Fock field. The analysis of Zaremba, Griffin, and Nikuni [5] (ZGN) uses the local equilibrium solution of the kinetic equation and thus does not include any hydrodynamic damping, such as that considered by Kirkpatrick and Dorfman [11]. However, it should be emphasized that a local equilibrium description is crucially dependent on collisions between the atoms and thus the hydrodynamic equations are only valid for low-frequency phenomena ($\omega \ll 1/\tau_c$, where $\tau_c$ is the mean time between collisions of atoms in the thermal cloud).

In Sec. II we solve the linearized hydrodynamic two-fluid equations for the coupled superfluid and normal fluid velocity fluctuations derived by ZGN. We exhibit the first and second sound normal modes valid at intermediate temperatures, defined as the temperature regime below $T_{BEC}$ where the interaction energy of an atom is much less than the thermal kinetic energy (i.e., $gn_0 \ll k_BT$; here $n_0$ is the gas density and $g = 4\pi a\hbar^2/m$ is the interaction parameter). The analysis of ZGN is built on a mean-field approximation for the equilibrium properties. As discussed in Sec. III, this simple theory is not valid close to the superfluid transition, where it gives rise to spurious discontinuities in the condensate density. In Sec. IV we discuss the relation between our two-fluid equations written in terms of velocity fluctuations and the standard Landau formulation given in terms of density and entropy fluctuations [1,3]. All previous discussions [12,8,6] of hydrodynamic modes in a dilute Bose gas have used the latter formulation.

## II. COUPLED EQUATIONS FOR SUPERFLUID AND NORMAL FLUID VELOCITIES

When there is no trapping potential, the noncondensate density $\tilde{n}_0$ and condensate density $n_{c0}$ do not depend on position. In this case, one can reduce the linearized two-fluid equations given by Eqs. (12), (15), and (16) of ZGN to two coupled equations for the normal and superfluid local velocities

$$m\frac{\partial^2 \delta\mathbf{v}_S}{\partial t^2}=gn_{c0}\nabla(\nabla\cdot\delta\mathbf{v}_S)+2g\tilde{n}_0\nabla(\nabla\cdot\delta\mathbf{v}_N) \tag{1a}$$

$$m\frac{\partial^2 \delta\mathbf{v}_N}{\partial t^2}=\left(\frac{5}{3}\frac{\tilde{P}_0}{\tilde{n}_0}+2g\tilde{n}_0\right)\nabla(\nabla\cdot\delta\mathbf{v}_N)+2gn_{c0}\nabla(\nabla\cdot\delta\mathbf{v}_S). \tag{1b}$$

We emphasize that these equations are only valid at finite temperatures such that $gn_0 \ll k_BT$. In deriving these equations, we have assumed that the contribution from the first term of Eq. (13) of ZGN is negligible in the long-wavelength limit of interest. These equations can be solved to give the low-frequency hydrodynamic normal modes of a uniform Bose-condensed gas, as will be discussed. We defer discussion of the equilibrium quantities ($n_{c0}$, $\tilde{n}_0$ and the kinetic contribution to the pressure $\tilde{P}_0$) that appear in Eqs. (1a) and (1b) to Sec. III.

Introducing the velocity potentials $\delta\mathbf{v}_S\equiv\nabla\phi_S$ and $\delta\mathbf{v}_N\equiv\nabla\phi_N$, it is easy to see that Eqs. (1a) and (1b) have plane-wave solutions $\phi_{S,N}(\mathbf{r},t)=\phi_{S,N}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ satisfying

$$\left[\omega^2-\frac{gn_{c0}}{m}k^2\right]\phi_S-\left(\frac{2g\tilde{n}_0}{m}k^2\right)\phi_N=0,$$

$$-\left(\frac{2gn_{c0}}{m}k^2\right)\phi_S+\left[\omega^2-\left(\frac{5}{3}\frac{\tilde{P}_0}{m\tilde{n}_0}+\frac{2g\tilde{n}_0}{m}\right)k^2\right]\phi_N=0. \tag{2}$$

The zeros of the secular determinant of this coupled set of equations give two phonon solutions $\omega_{1,2}^2=u_{1,2}^2k^2$, where the velocities are the solution of

$$u^4-u^2\left(\frac{5}{3}\frac{\tilde{P}_0}{m\tilde{n}_0}+\frac{2g\tilde{n}_0}{m}+\frac{gn_{c0}}{m}\right)+\frac{gn_{c0}}{m}\left(\frac{5}{3}\frac{\tilde{P}_0}{m\tilde{n}_0}-\frac{2g\tilde{n}_0}{m}\right)=0. \tag{3}$$

Expanding to second order in the explicit dependence on $g$, the sound velocities are given by

$$u_1^2=\frac{5}{3}\frac{\tilde{P}_0}{m\tilde{n}_0}+\frac{2g\tilde{n}_0}{m}+\frac{gn_{c0}}{m}\epsilon, \tag{4a}$$

$$u_2^2=\frac{gn_{c0}}{m}-\frac{gn_{c0}}{m}\epsilon, \tag{4b}$$

where $\epsilon\equiv 4g\tilde{n}_0/\frac{5}{3}(\tilde{P}_0/\tilde{n}_0)\ll 1$ is the expansion parameter. We note (see Sec. III) that the ratio $\tilde{P}_0/\tilde{n}_0=k_BT[g_{5/2}(z_0)/g_{3/2}(z_0)]$ depends weakly on $g$.

The $\omega_1$ mode in Eq. (4a) clearly corresponds to first sound. Using $\omega=u_1k$ in Eq. (2), one finds to leading order in $g$ that

$$\frac{\phi_N}{\phi_S}\simeq\frac{2}{\epsilon}\gg 1. \tag{5}$$

That is to say, the $\omega_1$ first sound mode corresponds to an *in-phase* oscillation in which the noncondensate velocity amplitude is much larger than that of the condensate. The $\omega_2$ mode in Eq. (4b) is the second sound mode. Using $\omega=u_2k$ in Eq. (2), one finds to leading order in $g$ that

$$-\frac{\phi_S}{\phi_N}\simeq\frac{2}{\epsilon}\frac{\tilde{n}_0}{n_{c0}}\gg 1. \tag{6}$$

Thus, at finite temperatures where Eqs. (1a) and (1b) are valid, second sound in a uniform weakly interacting gas is seen to be an *out-of-phase* oscillation, in which the condensate velocity amplitude is much larger than that of the noncondensate (a similar result was obtained many years ago by Lee and Yang [12]).

## III. EQUILIBRIUM PROPERTIES IN THE HARTREE-FOCK APPROXIMATION

We recall that in deriving Eqs. (1a) and (1b), the dynamics of the noncondensate is described by a semiclassical kinetic equation for a gas of atoms moving in the self-consistent Hartree-Fock (HF) mean field. In the context of the thermal equilibrium properties, this treatment of the noncondensate corresponds to excitations with energy $p^2/2m+2gn_0$, which is a useful finite-temperature approximation [13] to the excitations in the self-consistent Hartree-Fock-Popov (HFP) theory [8]. Referring to [5], we recall that the equilibrium equation for the condensate yields the chemical potential

$$\mu_0=2g\tilde{n}_0+gn_{c0}. \tag{7}$$

Within our simple HF mean-field treatment, the noncondensate density is then given by the well-known formula

$$\tilde{n}_0(T,n_0)=\frac{1}{\Lambda^3}g_{3/2}(z_0), \tag{8}$$

where $(n_0\equiv n_{c0}+\tilde{n}_0)$

$$z_0=e^{\beta(\mu_0-2gn_0)}=e^{-\beta gn_{c0}} \tag{9}$$

is the equilibrium fugacity and $\Lambda=\sqrt{2\pi\hbar^2/mk_BT}$ is the thermal de Broglie wavelength. The associated excited-atom *kinetic* pressure is

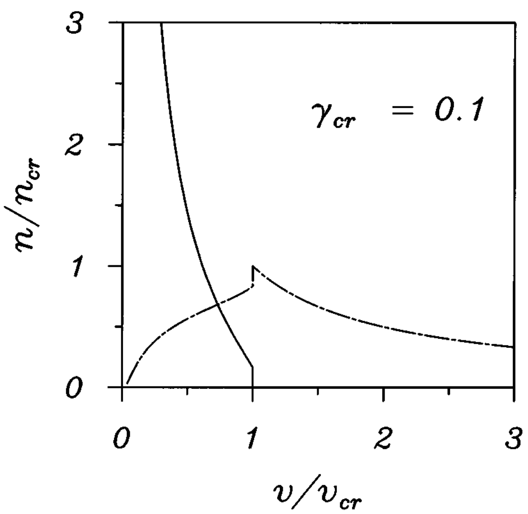


FIG. 1. Density vs volume per particle for a fixed temperature $T$. The long-dash–short-dashed curve corresponds to the noncondensate, the solid curve to the condensate. $\gamma_{cr}$ is the value of $gn_0/k_BT$ at the critical density $n_{cr}=g_{3/2}(1)/\Lambda^3$.

$$\tilde{P}_0(T,n_0)=\frac{1}{\beta\Lambda^3}g_{5/2}(z_0). \quad (10)$$

We note that these HF results are equivalent to the simple model studied in Ref. [14].

Equations (8) and (9) must be solved self-consistently to determine $n_{c0}$ and $\tilde{n}_0$ for a given total density $n_0$. Condensation occurs when the density reaches the critical density $n_{cr}=g_{3/2}(1)/\Lambda^3$. For $n_0<n_{cr}$, the condensate density is zero and Eq. (8) with $\tilde{n}_0=n_0$ determines the equilibrium fugacity. In Fig. 1 we show the equilibrium densities as a function of volume for a fixed temperature. The parameter $\gamma_{cr}\equiv\beta g n_{cr}$ is used to characterize the strength of the interaction. We see that the present level of approximation leads to a discontinuous change in the densities at the transition point [15]. Moreover, below the critical volume $v_{cr}=1/n_{cr}$, $\tilde{n}_0$ decreases as a result of the interactions with the condensate, in contrast to the ideal gas behavior which has the noncondensate maintaining a constant density of $n_{cr}$. Figure 2 gives the total pressure defined as [5]

$$P=\tilde{P}_0+\frac{1}{2}g(n_0^2+2n_0\tilde{n}_0-\tilde{n}_0^2), \quad (11)$$

normalized by the critical pressure $\tilde{P}_{cr}=g_{5/2}(1)/\beta\Lambda^3$ of the ideal gas. The second term in Eq. (11) is the *explicit* interaction contribution, but it should be noted that $\tilde{P}_0$ in (10) also depends on interactions as a result of its dependence on $z_0$. The discontinuous behavior of the noncondensate density leads to an analogous discontinuity in the pressure [15]. In Figs. 3 and 4 we show the corresponding behavior as a function of $T$. It is of interest to note that for a trapped Bose gas, the use of these equilibrium properties in the Thomas-Fermi approximation leads to a similar discontinuous behavior of the equilibrium condensate density, but now as a function of the radial distance from the center of the trap [16].

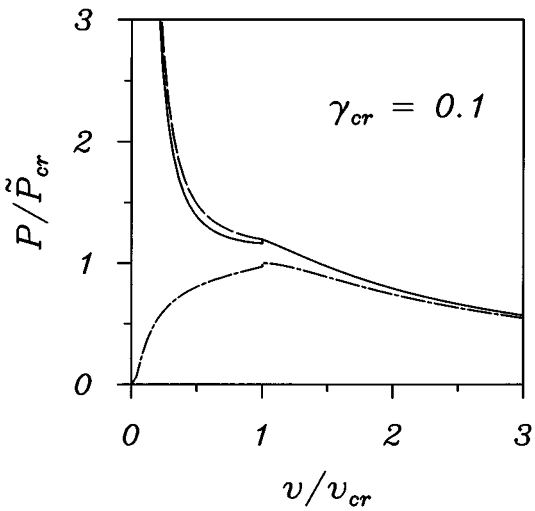


FIG. 2. Pressure isotherms: the solid line is the total pressure according to Eq. (11), the long-dash–short-dashed curve is $\tilde{P}_0$, and the dashed curve corresponds to the usual approximation [20,21] $P\simeq\tilde{P}_{cr}+\frac{1}{2}g(n^2+n_{cr}^2)$.

It is clear that the properties of the weakly interacting gas are nonanalytic functions of the interaction strength $g$ at the transition point within the mean-field approximation described by Eqs. (7)–(10). However, one should not take these features in the Bose-Einstein condensation (BEC) critical region seriously. The simple HF mean-field approximation (as well as the more consistent HFP approximation) for interactions is well known [17,18] not to be valid very close to the transition and the predicted discontinuities exhibited in Figs. 1–4 (characteristic of a first-order transition) are indicative of the limitations of the present simple theory. A correct treatment of this region would require a renormalization-group analysis [19], which is outside the scope of the present paper.

For later purposes, we note that the kinetic pressure $\tilde{P}_0$ in Eq. (10) can be calculated by expanding the fugacity as $z_0\simeq1-\beta g n_{c0}+\cdots$, which yields [using the identity $z\partial g_n(z)/\partial z=g_{n-1}(z)$]

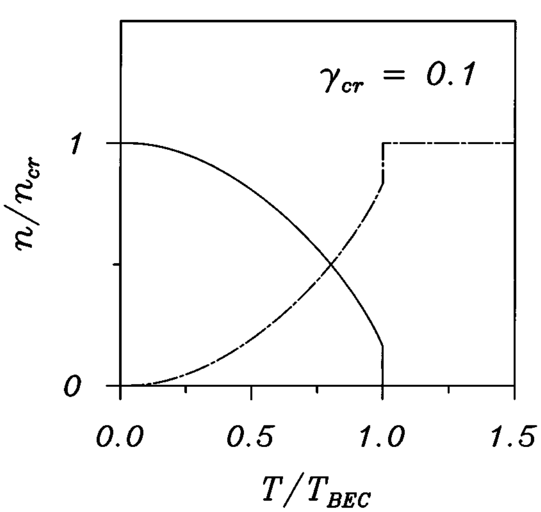


FIG. 3. Same as in Fig. 1, but as a function of $T$ for a fixed density $n_0$. Here $\gamma_{cr}\equiv gn_0/k_BT_{BEC}$.

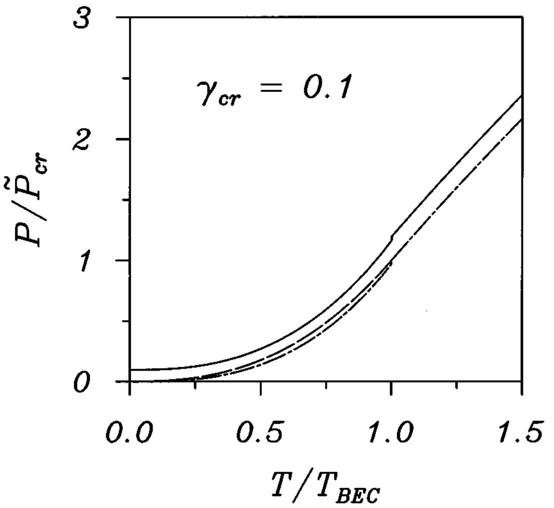


FIG. 4. Normalized pressure as a function of $T$ for a fixed density $n_0$. The solid curve corresponds to Eq. (11) and the long-dash–short-dashed curve is $\tilde{P}_0$. The dashed curve below $T=T_{BEC}$ is the ideal gas result $\tilde{P}_0/\tilde{P}_{cr}=(T/T_{BEC})^{5/2}$.

$$\tilde{P}_0 \simeq \tilde{P}_{cr} - g n_{c0} n_{cr}, \tag{12}$$

where $\tilde{P}_{cr}$ and $n_{cr}$ are the critical pressure and density of the ideal Bose gas introduced earlier. However, a similar perturbative expansion of the noncondensate density $\tilde{n}_0$ in Eq. (8) is not possible since the derivative of $g_{3/2}(z)$ diverges at $z=1$. Indeed, it is this nonperturbative dependence on $g$ that leads to the discontinuities shown in Figs. 1–4.

## IV. RELATION TO STANDARD TWO-FLUID EQUATIONS

First and second sound in a uniform Bose-condensed gas have been previously discussed in the literature [6,8,12]. These earlier treatments start with the usual two-fluid equations of Landau [3]. We recall that these linearized equations are (see Chap.7 of Ref. [1])

$$\frac{\partial \delta n}{\partial t} = -\nabla \cdot \delta \mathbf{j},$$

$$m\frac{\partial \delta \mathbf{v}_S}{\partial t} = -\nabla \delta\mu,$$

$$m\frac{\partial \delta \mathbf{j}}{\partial t} = -\nabla \delta P,$$

$$\frac{\partial \delta s}{\partial t} = -\nabla \cdot (s_0 \delta \mathbf{v}_N), \tag{13}$$

where

$$\delta n(\mathbf{r},t) = \delta \tilde{n}(\mathbf{r},t) + \delta n_c(\mathbf{r},t),$$

$$\delta \mathbf{j}(\mathbf{r},t) = \tilde{n}_0 \delta \mathbf{v}_N + n_{c0} \delta \mathbf{v}_S. \tag{14}$$

$P$ and $s$ are the pressure and entropy density, respectively. ZGN proved that the two-fluid equations that lead to Eqs. (1a) and (1b) are in fact *equivalent* to the two-fluid equations (13) when the thermodynamic functions in the latter are evaluated for the present model of a weakly interacting Bose gas. Using the thermodynamic relation [5] $n_0\delta\mu = \delta P - s_0 \delta T$ to eliminate the chemical potential and defining the entropy per unit mass by $\bar{s} \equiv s/mn = s/\rho$, one can reduce Eqs. (13) to [1,3]

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \nabla^2 \delta P,$$

$$\frac{\partial^2 \delta \bar{s}}{\partial t^2} = \frac{\rho_S}{\rho_N} \bar{s}_0^2 \nabla^2 \delta T. \tag{15}$$

Solving this closed set of equations in terms of the variables $\delta\rho$ and $\delta\bar{s}$, one finds two normal mode solutions $\omega^2 \equiv u^2 k^2$, where $u^2$ is given by the solution of the quadratic equation [3]

$$u^4 - u^2\left[\left.\frac{\partial P}{\partial\rho}\right|_T + \frac{T}{\bar{c}_v}\left(\frac{1}{\rho}\left.\frac{\partial P}{\partial T}\right|_\rho\right)^2 + \frac{\rho_S}{\rho_N}\frac{T\bar{s}_0^2}{\bar{c}_v}\right] + \frac{\rho_S}{\rho_N}\frac{T\bar{s}_0^2}{\bar{c}_v}\left.\frac{\partial P}{\partial \rho}\right|_T = 0. \tag{16}$$

In this equation, $\bar{c}_v$ is the specific heat per unit mass and derivatives of the pressure have been expressed in terms of the independent thermodynamic variables $T$ and $\rho$. Although not immediately apparent, the coefficients in Eq. (16) are in fact consistent with those appearing in Eq. (3).

The problem is thus reduced to evaluation of the various equilibrium thermodynamic functions and derivatives that appear in Eq. (16). For the entropy per unit mass we have the expression [5]

$$\rho_0 \bar{s}_0 T = \frac{5}{2}\tilde{P}_0 + g\,\tilde{n}_0 n_{c0}, \tag{17}$$

from which we obtain

$$\rho_0 \bar{c}_v = \frac{3}{2}\rho_0 \bar{s}_0 + g\left(\frac{3}{2}\tilde{n}_0 + n_{c0}\right)\left.\frac{\partial \tilde{n}}{\partial T}\right|_\rho. \tag{18}$$

From the equation of state (11), we find that

$$\left.\frac{\partial P}{\partial\rho}\right|_T = \frac{g n_0}{m}\left(1 + \left.\frac{\partial \tilde{n}}{\partial n}\right|_T\right) \tag{19}$$

and

$$\left.\frac{\partial P}{\partial T}\right|_\rho = \rho_0 \bar{s}_0 + g n_0 \frac{\partial \tilde{n}}{\partial T^{\rho}}. \tag{20}$$

These quantities have been calculated previously in the limit that the interaction parameter $g$ is regarded as small [6,20,21]. In this situation, $\tilde{P}_0$ in Eq. (10) is approximated by Eq. (12). An additional approximation is typically made whereby $\tilde{n}_0$ is simply replaced by the ideal gas expression $n_{cr}$, in which case $n_{c0}=n_0-n_{cr}$. To the same level of approximation, one finds $\partial\tilde{n}/\partial n|_T=0$ and $\partial\tilde{n}/\partial T|_\rho = 3n_{cr}/2T$.

With these replacements, we also note that the expressions for the pressure and the entropy and energy densities given by ZGN reduce precisely to those of Refs. [20] and [21].

Using these results to calculate the thermodynamic quantities in Eq. (16), the first and second sound velocities are found (after some algebra) to be given by

$$u_1^2=\frac{5}{3}\frac{k_BT}{m}\frac{g_{5/2}(1)}{g_{3/2}(1)}+\frac{2gn_{cr}}{m}-\frac{5}{3}\frac{gn_{c0}}{m}, \tag{21a}$$

$$u_2^2=\frac{gn_{c0}}{m}, \tag{21b}$$

keeping terms to first order in $g$. The leading order terms in Eqs. (21a) and (21b) were obtained from Eq. (16) by this method by Popov (see the last paragraph of Ref. [8]) as well as by Lee and Yang [12]. Precisely the same results follow from Eq. (4) to first order in $g$ when Eq. (12) is again used for the kinetic pressure $\tilde{P}_0$ and $\tilde{n}_0$ is replaced by $n_{cr}$. However, the results given by Eq. (3) are more general than those in Eq. (21), which only keep the leading-order corrections to the properties of a noninteracting gas. As we discussed above, the analysis leading to Eq. (21) ignores any interaction correction to the noncondensate density $\tilde{n}$, which, as can be seen from Fig. 1, becomes significant as the density increases beyond $n_{cr}$.

As we emphasized at the beginning of Sec. II, the analysis of ZGN assumes that $gn_0 \ll k_BT$ and thus our results are not really valid at low-temperatures. To discuss the low temperature region would require a generalization of our work, which is based on a quasiparticle spectrum exhibiting phononlike behavior at long wavelengths (a kinetic equation appropriate to this region has been derived in Ref. [11]). The pioneering work of Lee and Yang [12] did include an analysis of both the low-temperature and high-temperature regions. At low temperatures, they found that the first and second sound modes avoid becoming degenerate by hybridizing and an interchange of the physical meaning of these two modes occurs as a result of this hybridization. While the sound velocities given by Eq. (3) are not really valid at low temperatures, Fig. 5 shows that our results do lead to this expected hybridization of first and second sound in a dilute gas.

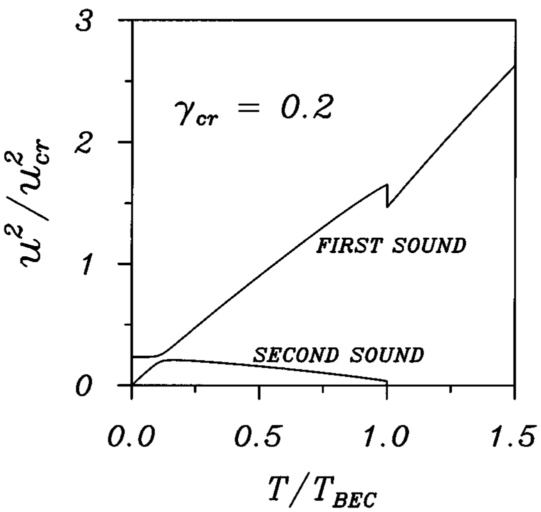


FIG. 5. Squares of the first and second sound velocities (normalized by the first sound velocity of the ideal gas at $T=T_{BEC}$) vs $T/T_{BEC}$. The value of $\gamma_{cr}$ has been increased to more clearly reveal the anticrossing behavior at low temperatures. As discussed in Sec. IV, the low-temperature results indicate only the qualitative behavior. As shown in [12], the $T=0$ limit of the upper branch is the Bogoliubov sound velocity, while the lower branch has a finite limiting value.

## V. CONCLUDING REMARKS

Recently, two-fluid hydrodynamic equations were derived [5] for a trapped, weakly interacting Bose gas. These are given in terms of coupled equations for the superfluid and normal fluid velocity fluctuations. In order to obtain more physical insight into these hydrodynamic equations, we have given in the present paper a detailed analysis for a *uniform* Bose gas. In this case, we proved earlier [5] that our hydrodynamic equations are formally equivalent to the usual Landau two-fluid equations. As the present paper shows, this formal equivalence is somewhat hidden in explicit calculations of the first and second sound velocities. However, as discussed in Sec. IV, our results do reduce (to first order in the interaction $g$) to those found in earlier studies [12,8,6] based on the Landau formulation.

In superfluid $^4$He, one evaluates the equilibrium thermodynamic parameters in Eq. (16) using the phonon-roton excitation spectrum. As is well known [1,3], in superfluid $^4$He, first sound corresponds to an in-phase oscillation in which $\mathbf{v}_N=\mathbf{v}_S$. In contrast, second sound corresponds to an out-of-phase oscillation in which $\rho_n\mathbf{v}_N=-\rho_S\mathbf{v}_S$. The difference between second sound in a dilute Bose gas at finite temperatures and in a liquid is a result of the dominance of the kinetic energy over the interaction energy for atoms in a gas. In both cases, however, we note that the second sound frequency goes to zero (becomes soft) at the superfluid transition. The mode does *not* exist above $T_{BEC}$. Moreover, Eq. (4b) shows that second sound crucially depends on the interaction $g$. It would be absent if we had set $g=0$ in Eqs. (1a) and (1b).

As we have noted, second sound in a dilute gas largely involves an oscillation of the condensate atoms (superfluid density) and is a soft mode that vanishes in the normal phase. We recall that at finite temperatures [8], the generalization of the $T=0$ Bogoliubov phonon gives a velocity formally identical to the first term in Eq. (4b). Thus we conclude that in a weakly interacting Bose-condensed gas at finite temperatures, second sound is the low-frequency (hydrodynamic regime) continuation of the high-frequency (collisionless or mean-field regime) Bogoliubov-Goldstone mode. This was suggested in Refs. [6–8]. The situation is quite different in superfluid $^4$He, where the collisionless phonon spectrum is the continuation of the hydrodynamic first sound mode [7] and there is no high-frequency analog of the second sound branch.

## ACKNOWLEDGMENTS

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