Wave functions of a time-dependent harmonic oscillator with and without a singular perturbation

I. A. Pedrosa
Departamento de Física CCEN, Universidade Federal da Paraíba, Caixa Postal 5008, 58.059–970, J. Pessoa PB, Brazil

G. P. Serra and I. Guedes
Departamento de Física, Centro de Ciências, Universidade Federal do Ceará, Caixa Postal 6030, Campus do Pici 60.450–760, Fortaleza CE, Brazil

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We use the Lewis and Riesenfeld invariant method to obtain the exact Schrödinger wave functions for a time-dependent harmonic oscillator with and without an inverse quadratic potential. As a particular case we also obtain the wave functions for the Caldirola-Kanai oscillator. [S1050-2947(97)07710-X]

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I. INTRODUCTION

Widespread attention has been paid in the past few years to the study of time-dependent harmonic oscillators [1–10]. The time-dependent harmonic oscillator has attracted considerable interest because it gives a good example of an exactly solved model, and has applications in different areas of physics. For instance, in molecular physics, quantum chemistry, quantum optics, plasma physics, and quantum field theory, many quantum-mechanical effects are treated phenomenologically by means of the time-dependent parameters in the Hamiltonian of a general time-dependent oscillator [11–15].

There are several methods to study time-dependent oscillators [3,7,8]. However, for these systems Lewis and Riesenfeld (LR) [1,2] have introduced an important quantum-mechanical invariant and found the exact quantum states in terms of the invariant eigenstates up to some explicitly time-dependent phase. Since then, numerous variants and applications of the LR invariant method have been introduced and used [3–10].

In this paper, we use the LR invariant method and an unitary transformation to obtain the exact Schrödinger wave functions for a time-dependent harmonic oscillator with and without an inverse quadratic potential. As a particular case, we also find the wave functions for the well-known Caldirola-Kanai oscillator.

This paper is organized as follows. In Sec. II we briefly review the LR invariant method for the time-dependent oscillator. In Sec. III we find the wave functions for the time-dependent harmonic oscillator with and without an inverse quadratic potential. We also find the wave functions for the Caldirola-Kanai oscillator. Finally, some concluding remarks are added in Sec. IV.

II. EXACT INVARIANTS AND THE SCHRÖDINGER EQUATION

Consider a time-dependent harmonic oscillator described by the Hamiltonian

$$H(t)=[p^2/2M(t)]+\tfrac{1}{2}M(t)w^2(t)q^2, \tag{1}$$

whose mass and angular frequency depend on time explicitly, and the variables p and q are canonical coordinates with $[q,p]=i\hbar$. From Eq. (1) we obtain the equation of motion

$$\ddot{q}+\gamma(t)\dot{q}+w^2(t)q=0, \tag{2}$$

where

$$\gamma(t)=(d/dt)\ln[M(t)]. \tag{3}$$

Now it is known that an invariant for Eq. (1) is given by [3,5]

$$I(t)=\tfrac{1}{2}[(q/p)^2+(\rho p-M\dot{\rho}q)^2], \tag{4}$$

where $q(t)$ satisfies Eq. (2) and $\rho(t)$ is a c-number quantity satisfying the auxiliary equation

$$\ddot{\rho}+\gamma(t)\dot{\rho}+w^2(t)\rho=(1/M^2\rho^3). \tag{5}$$

The invariant $I(t)$ satisfies the equation

$$(dI/dt)=(\partial I/\partial t)+(1/i\hbar)[I,H]=0, \tag{6}$$

and can be considered Hermitian if we choose only the real solutions of Eq. (5). Its eigenfunctions, denoted by $\phi_n(q,t)$, are assumed to form a complete orthonormal set with time-independent eigenvalues λ_n. Thus

$$I\phi_n(q,t)=\lambda_n\phi_n(q,t), \tag{7}$$

with $\langle\phi_n,\phi_{n'}\rangle=\delta_{n,n'}$.

Next consider the time-dependent Schrödinger equation

$$i\hbar(\partial/\partial t)\psi(q,t)=H(t)\psi(q,t), \tag{8}$$

with

$$H(t)=-[\hbar^2/2M(t)]\ (\partial/\partial q^2)+\tfrac{1}{2}M(t)w^2(t)q^2, \tag{9}$$

where $p=-i\hbar\partial/\partial q$ has been used. Lewis and Riesenfeld showed that the solutions $\psi_n(q,t)$ to the Schrödinger equation (8) are related to $\phi_n(q,t)$ by the relation

$$\psi_n(q,t)=e^{i\alpha_n(t)}\phi_n(q,t), \tag{10}$$

where the phase functions $\alpha_n(t)$ satisty the equation

$$\hbar[d\alpha_n(t)/dt]=\langle \phi_n|i\hbar(\partial/\partial t)-H(t)|\phi_n\rangle. \tag{11}$$

Then, since each $\psi_n(q,t)$ satisfies the Schrödinger equation, the general solution of Eq. (8) may be written as

$$\psi(q,t)=\sum_n c_n e^{i\alpha_n(t)}\phi_n(q,t), \tag{12}$$

where the coeficients c_n are time independent.

III. EXACT SCHRÖDINGER WAVE FUNCTIONS

A. Harmonic oscillator with time-dependent mass and frequency

To obtain the wave functions for the time-dependent oscillator (1) we proceed as follows. Consider the unitary transformation

$$\phi_n'(q,t)=\mathcal{U}\phi_n(q,t), \tag{13}$$

with

$$\mathcal{U}=\exp\{-[iM(t)\dot{\rho}/2\hbar\rho]q^2\}. \tag{14}$$

Under this unitary transformation the eigenvalue equation (7) is mapped into

$$I'\phi_n'(q,t)=\lambda_n\phi_n'(q,t), \tag{15}$$

with

$$I'=\mathcal{U}I\mathcal{U}^{+}=-(\hbar^2/2)\ (\rho^2\partial^2/\partial q^2)+\tfrac{1}{2}\ (q^2/\rho^2). \tag{16}$$

If we now define a new variable $\sigma=q/\rho$, we can write the eigenvalue equation in the form

$$[-(\hbar^2/2)(\partial^2/\partial\sigma^2)+(\sigma^2/2)]\varphi_n(\sigma)=\lambda_n\varphi_n(\sigma) \tag{17}$$

or

$$I'\varphi_n(\sigma)=\lambda_n\varphi_n(\sigma), \tag{18}$$

where

$$\phi_n'(q,t)=(1/\rho^{1/2})\varphi_n(\sigma)=(1/\rho^{1/2})\varphi_n(q/\rho). \tag{19}$$

The factor $\rho^{1/2}$ is introduced into Eq. (18), so that the normalization condition

$$\int \phi'^{*}_n(q,t)\phi_n'(q,t)dq=\int \varphi^{*}_n(\sigma)\varphi_n(\sigma)d\sigma=1 \tag{20}$$

holds. Now Eq. (17) is an ordinary one-dimensional Schrödinger equation whose solution is given by

$$\varphi(\sigma)=\left[\frac{1}{\pi^{1/2}\hbar^{1/2}n!2^n}\right]^{1/2}\exp\left[-\frac{\sigma^2}{2\hbar}\right]H_n\left[\left(\frac{1}{\hbar}\right)^{1/2}\sigma\right], \tag{21}$$

where

$$\lambda_n=\hbar(n+\tfrac{1}{2}), \tag{22}$$

and H_n is the usual Hermite polynomial of order n. Thus, by using Eqs. (13), (14), (19), and (21) we find that

$$\phi_n(q,t)=\left[\frac{1}{\pi^{1/2}\hbar^{1/2}n!2^n\rho}\right]^{1/2}\exp\left[\frac{iM(t)}{2\hbar}\left(\frac{\dot{\rho}}{\rho}+\frac{i}{M(t)\rho^2}\right)q^2\right] \times H_n[(1/\hbar)^{1/2}(q/\rho)]. \tag{23}$$

There remains the problem of finding the phases $\alpha_n(t)$ which satisfy Eq. (11). Carrying out the unitary transformation $\mathcal{U}$, the right-hand side of Eq. (11) becomes

$$\hbar\dot{\alpha}_n(t)=\left\langle \phi_n'\left|i\hbar\frac{\partial}{\partial t}+i\hbar\frac{\dot{\rho}}{\rho}q\frac{\partial}{\partial q}+i\hbar\frac{\dot{\rho}}{2\rho}-\frac{I'}{M\rho^2}\right|\phi_n'\right\rangle, \tag{24}$$

where we have used the auxiliary equation (5) to eliminate $w^2(t)$ from $H(t)$. Next substituting Eq. (19) into Eq. (24), we obtain that

$$\hbar\dot{\alpha}_n(t)=\langle\varphi_n|-(I'/M\rho^2)|\varphi_n\rangle; \tag{25}$$

using Eq. (18) and the normalization of φ_n we find that

$$\alpha_n(t)=-\left(n+\frac{1}{2}\right)\int_0^t\frac{1}{M(t')\rho^2}\,dt'. \tag{26}$$

Finally, using Eqs. (10) and (23) we find that the exact solution of the Schrödinger equation (8) is

$$\psi_n(q,t)=\exp[i\alpha_n(t)][1/\pi^{1/2}\hbar^{1/2}n!2^n\rho]^{1/2} \times\exp\left[\frac{iM(t)}{2\hbar}\left(\frac{\dot{\rho}}{\rho}+\frac{i}{M(t)\rho^2}\right)q^2\right]H_n\left[\left(\frac{1}{\hbar}\right)^{1/2}\frac{q}{\rho}\right], \tag{27}$$

where the phase functions $\alpha_n(t)$ are given by Eq. (26).

When the mass is constant, i.e., $M(t)=m$, our new wave function (27) reduces to those obtained in Refs. [3,6,7]. On the other hand, for the general case where the mass is also time dependent, the Schrödinger wave function (27) agrees with that of Ref. [8] by setting $\rho^2(t)=g_-(t)/w_I$. Here we would like to observe that the result (27) is different from those obtained in Ref. [6]. This is because the wave function obtained in Ref. [6] satisfies the Schrödinger equation only when the mass is constant. However, for the case where mass and frequency are both time dependent the wave function of Ref. [6] is not correct, i.e., it does not satisfy the Schrödinger equation. We also note that when $M(t)=m$ and $w(t)=w_0$ are both constant and $\rho(t)=(1/mw_0)^{1/2}$, which is a particular solution of Eq. (5), result (27) becomes the well-known wave function for the time-dependent harmonic oscillator of mass m and frequency w_0.

B. Caldirola-Kanai oscillator

For the case where the frequency $w(t)=w_0$ is constant and the mass $M(t)$ is given by

$$M(t)=me^{\gamma t}, \tag{28}$$

Hamiltonian (1) becomes

$$H(t)=(p^2/2m)e^{-\gamma t}+\tfrac{1}{2}mw_0^2e^{\gamma t}q^2, \tag{29}$$

with $\gamma=$const [see Eq. (3)]. Hamiltonian (29) is the Caldirola-Kanai oscillator which is one of the most typical time-dependent quantum systems whose exact quantum states are well known [16–19].

Let us now consider a particular solution of Eq. (5) given by

$$\rho(t)=[e^{-\gamma t/2}/(m\Omega)^{1/2}], \tag{30}$$

with

$$\Omega^2=w_0^2-(\gamma^2/4). \tag{31}$$

Here we shall consider only the case where $\Omega>0$. Then using Eqs. (28) and (30) in Eq. (26), we find that the phase functions are given by

$$\alpha_n(t)=-\Omega(n+\tfrac{1}{2})t. \tag{32}$$

Thus by using Eqs. (28), (30), and (32) in Eq. (27), we obtain that

$$\psi_n(q,t)=\left[\frac{(m\Omega)^{1/2}}{\pi^{1/2}\hbar^{1/2}n!2^n}\right]^{1/2}\exp\left[\frac{\gamma}{4}-i\Omega\left(n+\frac{1}{2}\right)\right]t \times\exp\left[-\frac{m}{2\hbar}\left(\Omega+\frac{i\gamma}{4}\right)e^{\gamma t}q^2\right]H_n\left[\left(\frac{m\Omega}{\hbar}\right)^{1/2}qe^{\gamma t/2}\right], \tag{33}$$

which is the exact wave function for the Caldirola-Kanai oscillator [16–18].

C. Time-dependent harmonic oscillator with an inverse quadratic potential

We now consider the time-dependent harmonic oscillator described by the Hamiltonian

$$H(t)=[p^2/2M(t)]+\tfrac{1}{2}M(t)w^2(t)q^2+[g/2M(t)q^2], \tag{34}$$

where g is an arbitrary constant which could be zero. From Eq. (34) we obtain the equation of motion

$$\ddot{q}+\gamma(t)\dot{q}+w^2(t)q=(g/M^2q^3), \tag{35}$$

where $\gamma(t)$ is defined by Eq. (3). For this case, an exact invariant is given by [5]

$$I(t)=\tfrac{1}{2}[(q/\rho)^2+(\rho p-M\dot{\rho}q)^2+(\rho^2g/q^2)], \tag{36}$$

where $q(t)$ and $\rho(t)$ satisty, respectively, Eqs. (35) and (5).

In what follows we wish to solve the Schrödinger equation (8) with $H(t)$ given by [see Eq. (34)]

$$H(t)=-\frac{\hbar^2}{2M(t)}\frac{\partial^2}{\partial q^2}+\frac{1}{2}M(t)w^2(t)q^2+\frac{g}{2M(t)q^2}. \tag{37}$$

To this end, we proceed as in Sec. III A. Thus by using Eqs. (13) and (14) the eigenvalue problem (7) with $I(t)$ given by Eq. (36) is mapped into Eq. (15) where $I'(t)$ is now given by

$$I'=\mathcal{U}I\mathcal{U}^+=-\frac{\hbar^2}{2}\rho^2\frac{\partial^2}{\partial q^2}+\frac{1}{2}\frac{q^2}{\rho^2}+\frac{\rho^2g}{2q^2}. \tag{38}$$

In terms of σ we can rewrite the eigenvalue equation for I' as

$$\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2}+\frac{\sigma^2}{2}+\frac{q}{2\sigma^2}\right]\varphi_n(\sigma)=\lambda_n\varphi_n(\sigma), \tag{39}$$

where $\varphi_n(\sigma)$ is defined by Eq. (19), and λ_n are constant eigenvalues to be determined. Now the solutions for the time-independent Schrödinger equation (39) are [20]

$$\varphi_n(\sigma)=\left(\frac{4}{\hbar}\right)^{1/4}\left(\frac{2\Gamma(n+1)}{\Gamma(a+n+1)}\right)^{1/2}\left(\frac{\sigma^2}{\hbar}\right)^{(2a+1)/4} \times\exp(-\sigma^2/2\hbar)L_n^a(\sigma^2/\hbar), \tag{40}$$

with

$$\lambda_n=\hbar(2n+a+1), \tag{41}$$

$$a=\tfrac{1}{2}[1+(4g/\hbar^2)], \tag{42}$$

where L_n^a denotes generalized Laguerre polynomials. Here we remark that $g>-\hbar^2/4m$ to avoid ‘‘the fall to the center’’ [3]. Then by using Eqs. (13), (14), (19), and (40), we find that

$$\phi_n(q,t)=\left(\frac{2\Gamma(n+1)}{\Gamma(a+n+1)}\right)^{1/2}\left(\frac{1}{\hbar\rho^2}\right)^{(a+1)/2}q^{a+(1/2)} \times\exp\left[\frac{iM(t)}{2\hbar}\left(\frac{\dot{\rho}}{\rho}+\frac{i}{M(t)\rho^2}\right)q^2\right]L_n^a\left(\frac{1}{\hbar}\frac{q^2}{\rho^2}\right). \tag{43}$$

On the other hand, following the same steps as those of Sec. III A, we convert Eq. (11) with $H(t)$ given by Eq. (37) into the form of Eq. (25) with I' given by Eq. (38). Then using Eqs. (39) and (40) and the normalization of φ_n we find that

$$\alpha(t)=-(2n+a+1)\int_0^t\frac{1}{M(t')\rho^2}dt'. \tag{44}$$

Finally, using Eqs. (10) and (43) we find that the exact solution of the Schrödinger equation for the time-dependent oscillator with an inverse quadratic potential is

$$\psi_n(q,t)=e^{i\alpha_n(t)}\left(\frac{2\Gamma(n+1)}{\Gamma(a+n+1)}\right)^{1/2}\left(\frac{1}{\hbar\rho^2}\right)^{(a+1)/2}q^{a+(1/2)} \times\exp\left[\frac{iM(t)}{2\hbar}\left(\frac{\dot{\rho}}{\rho}+\frac{1}{M(t)\rho^2}\right)q^2\right]L_n^a\left(\frac{1}{\hbar}\frac{q^2}{\rho^2}\right), \tag{45}$$

where the phase functions are given by (44).

When the mass is constant, i.e., $M(t)=m$ the new wave function (45) agrees with that obtained in Ref. [3]. We also note that when $g=0$ the solutions ψ_n of Eq. (45) reduce to the solutions of the harmonic oscillator with time-dependent mass and frequency ψ_{2n+1}.

D. Caldirola-Kanai oscillator with an inverse quadratic potential

Let us now consider the Hamiltonian (34) with constant frequency and mass $M(t)$ given by Eq. (28). For this case we obtain that

$$H(t)=(p^2/2m)e^{-\gamma t}+\tfrac{1}{2}mw_o e^{\gamma t}q^2+(ge^{-\gamma t}/2mq^2), \qquad (46)$$

which is the Caldirola-Kanai oscillator with an inverse quadratic potential.

To obtain the wave function associated with Eq. (46) we proceed as follows. Consider the particular solution of Eq. (5) given by Eq. (30). Then, using Eqs. (28) and (30) into Eq. (44), we find that

$$\alpha_n(t)=-\Omega(2n+a+1)t. \qquad (47)$$

Thus by Eqs. (28), (30), and (47) in Eq. (45), we obtain that

$$\psi_n(q,t)=\left(\frac{2\Gamma(n+1)}{\Gamma(a+n+1)}\right)^{1/2}\left(\frac{m\Omega}{\hbar}e^{\gamma t}\right)^{(a+1)/2}q^{a+(1/2)} \times\exp[-i\Omega(2n+a+1)t] \times\exp\left[-\frac{m}{2\hbar}\left(\Omega+\frac{i\gamma}{2}\right)e^{\gamma t}q^2\right]L_n^a\left(\frac{m\Omega}{\hbar}q^2e^{\gamma t}\right), \qquad (48)$$

which is the exact wave function for Caldirola-Kanai oscillator (46). Here we would like to remark that as far as we know the Hamiltonian (46) and the wave function (48) have not yet been exhibited in the literature.

IV. CONCLUDING REMARKS

In this paper we have used a unitary transformation and the LR invariant method in the Schrödinger picture to obtain the exact wave functions for the time-dependent harmonic oscillator with and without an inverse quadratic potential. We also have seen that our results are in agreement with those obtained by other authors which have used different approaches, such as the LR invariant method in the Heinsenberg picture and the path integral method. Futhermore, we have obtained the wave functions for the Caldirola-Kanai oscillator with and without an inverse quadratic potential. For this case, we would like to observe that the key point to find the exact wave functions is the particular solution given by Eq. (30) for the nonlinear equation (5). Here let us recall that there still remains the difficult task of solving the nonlinear equation (5).

In conclusion, we would like to make some comments. As we have already mentioned, the time-dependent harmonic oscillator has been the subject of much investigation. In particular, harmonic oscillators with time-dependent mass have been employed in quantum optics in order to describe the electromagnetic field intensities in a Fabry-Pérot cavity [21], in quantum physics to study quantum tunelling effects [22,23], and in cosmology to study the behavior of a scalar field in an anisotropic universe [11,24]. We also mention that the connection between quantum-mechanical solutions and classical solutions of the harmonic oscillator with and without a singular perturbation has been studied by various authors [4,8–10,20,25,26]. Furthermore, Kim *et al.* [27] used the LR invariant method and the Heisenberg picture approach to obtain the exact quantum motion of a time-dependent forced harmonic oscillator. Thus it seems that it would not be any problem to investigate the time-dependent forced oscillator using the procedure of the present paper, i.e., the LR invariant method and the Schrödinger picture approach. We hope to report on this possibility in a future paper. Finally, we wish to point out that wave functions (27) were also obtained in Ref. [28].

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