# Vortex states for the quantized radiation field

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We study the properties of wave-packet states with vortex structure. In particular we present the characteristics of a two-mode radiation field characterized by a configuration-space wave function $\psi_v=(x-iy)^m e^{-(x^2+y^2)/2\sigma^2}$. We show the generation of such states by the interaction of $\Lambda$ systems with squeezed radiation in a two-mode cavity. The two modes are correlated due to entanglement, which also produces the mixed state of each mode even though the state of the two-mode field is a pure state. [S1050-2947(97)04011-0]

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## I. INTRODUCTION

In recent years there has been phenomenal interest in the study of wave packets of a quantum system [1]. The Gaussian wave packets occupy a central place in such studies. In the case of radiation field, the Gaussian wave packets are in fact minimum-uncertainty states and describe both the coherent states [2] and the squeezed states [3] of the radiation field. In this paper we study the wave-packet states that have a vortex structure, i.e., states described by

$$\psi_v(x,y)\sim(x-iy)^m e^{-(x^2+y^2)/2\sigma^2}, \tag{1}$$

where $m$ is an integer. Note that $|\psi_v|^2=0$ at $x=y=0$. A vortex of order $m$ exists at the origin since the circulation of the argument of $\psi_v$ along a closed contour containing $x=y=0$ is $2\pi m$,

$$\oint \vec{\nabla}(\arg\psi_v)\cdot d\vec{l}=2\pi m. \tag{2}$$

In Fig. 1 we illustrate the probability distribution associated with Eq. (1). Berry [4] introduced the concept of a defect or vortex on the optical wave field, i.e., he considered the optical field distribution $\mathcal{E}(x,y)$ with the structure (1). A large body of literature has been devoted to the study of optical vortices on *classical* wave fields [5]. In contrast, we consider quantum systems characterized by the wave function given by Eq. (1). Our quantum system could be a two-dimensional harmonic oscillator or a two-mode radiation field. We study in detail various properties of, say, a field in the state (1). We show how the state (1) can be generated in cavity QED experiments.

The paper is organized as follows. In Sec. II we introduce a state of a two-mode quantized field that has vortex-type singularity in the configuration-space-like representation. We consider the transformation of the vortex structure to a circular basis. In Sec. III we investigate various properties of the vortex state such as antibunching, squeezing, mode-mode correlations, and photon number distribution. A method of generation of the proposed vortex state is outlined in Sec. IV. In Sec. V we determine the single-mode reduced density matrix. Although we study all this in the context of a two-mode field, the results are applicable to all systems that can be reduced to a system of two harmonic oscillators.

## II. VORTEX STATES FOR THE QUANTIZED RADIATION FIELD

In this section we give an operator version of Eq. (1). This will also shed light on the relation of Eq. (1) to other well-known states as well as on the methods that can generate such states. Note that the exponential part in Eq. (1) is related to the squeezed state $|\psi\rangle$ for the two mode radiation field defined by the direct product of the squeezed $a$-mode state $|\psi_a\rangle$ and the squeezed $b$-mode state $|\psi_b\rangle$,

$$\begin{aligned}|\psi\rangle&=\exp[\zeta\{a^{\dagger 2}+b^{\dagger 2}-a^2-b^2\}]|0,0\rangle\\&=\exp[\zeta\{a^{\dagger 2}-a^2\}]|0\rangle\exp[\zeta\{b^{\dagger 2}-b^2\}]|0\rangle\\&\equiv|\psi_a\rangle|\psi_b\rangle,\end{aligned} \tag{3}$$

where $a,b$ are the bosonic annihilation operators for the two-field modes and $\zeta$ is the squeezing parameter. First we evaluate the expression $\langle x,y|\psi\rangle$, where $|x,y\rangle$ is the eigenvector of

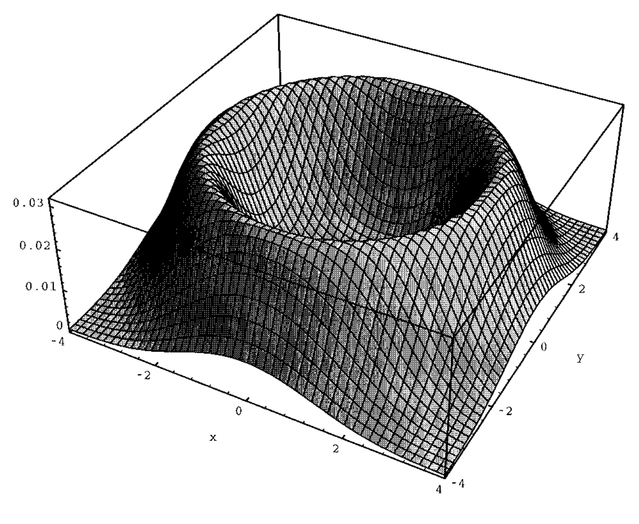


FIG. 1. Intensity distribution for the vortex state given by Eq. (1) for $\sigma=\sqrt{2}$ and $m=4$.

 

$(a+a^\dagger)/\sqrt{2}$ and $(b+b^\dagger)/\sqrt{2}$ with eigenvalues $x$ and $y$. To that end, we use the disentangling theorem [3]

$$\exp[\zeta(a^{\dagger 2}-a^2)]|0,0\rangle = \exp\left[\frac{\xi}{2}a^{\dagger 2}\right]\exp\left\{-\ln[\cosh(2\zeta)]\left(a^\dagger a+\frac{1}{2}\right)\right\} \times\exp\left[\frac{\xi}{2}a^2\right]; \quad \xi=\tanh(2\zeta) \tag{4}$$

to write the exponential in Eq. (3) as a product of the exponentials and make use of the fact that the operators $a$ and $b$ acting on their respective vacuum states give zero. It then follows that

$$|\psi_a\rangle=\frac{1}{\sqrt{\cosh(2\zeta)}}\exp\left[\frac{\xi}{2}a^{\dagger 2}\right]|0\rangle, \tag{5}$$

with a similar expression for $|\psi_b\rangle$. Now, on taking the scalar product of Eq. (5) with $|x\rangle$ and using the relation

$$\langle x|n\rangle=\frac{\exp(-x^2/2)}{\sqrt{2^n n!\sqrt{\pi}}}H_n(x), \tag{6}$$

where $H_n(x)$ is the Hermite ploynomial, it follows that

$$\psi_a(x)\equiv\langle x|\psi\rangle=\frac{\exp(-x^2/2)}{\sqrt{\sqrt{\pi}\cosh(2\zeta)}}\sum_{m=0}^{\infty}\left(\frac{\xi}{4}\right)^m\frac{1}{m!}H_{2m}(x). \tag{7}$$

The sum in Eq. (7) can be evaluated by using the integral representation

$$H_m(x)=\frac{2^m}{\sqrt{\pi}}\int_{-\infty}^{\infty}(x+it)^n\exp(-t^2)dt \tag{8}$$

of the Hermite polynomial. It is found that

$$\psi_a(x)\equiv\langle x|\psi\rangle=\frac{1}{\sqrt{\sigma\sqrt{\pi}}}\exp(-x^2/2\sigma^2), \tag{9}$$

where

$$\sigma=\exp(2\zeta). \tag{10}$$

The expression for $\langle x,y|\psi\rangle$ now follows by taking the scalar product of Eq. (3) with $|x,y\rangle$ and using Eq. (9) and the similar expression for $\psi_b(y)$,

$$\psi(x,y)\equiv\langle x,y|\psi\rangle=\psi_a(x)\psi_b(y)=\frac{1}{\sqrt{\pi}\sigma}\exp\left[-\frac{x^2+y^2}{2\sigma^2}\right]. \tag{11}$$

We next prove that the *vortex state* (1) can be expressed as

$$|\psi_v\rangle=A(a^\dagger-ib^\dagger)^m|\psi\rangle, \tag{12}$$

where $A$ is the normalization constant. Let us first evaluate $\psi_v(x,y)$. To that end, consider the expression

$$I=\langle x|\exp[\alpha a^{\dagger 2}+\beta a^\dagger|0\rangle = \exp[-\beta^2/4\alpha]\langle x|\exp[\alpha(a^\dagger+\beta/2\alpha)^2]|0\rangle. \tag{13}$$

On expanding the exponential in Eq. (13) and using Eq. (6) we get

$$I=\frac{\exp[-\beta^2/4\alpha]}{\sqrt{\sqrt{\pi}}}\exp(-x^2/2)\sum_{m=0}^{\infty}\sum_{n=0}^{2m}\frac{\alpha^m}{m!}\left(\frac{\beta}{\alpha}\right)^{2m-n} \times\frac{(2m)!}{(2m-n)!n!}\sqrt{\frac{1}{2^n}}H_n(x). \tag{14}$$

The sum in Eq. (14) can be evaluated by using the integral representation (8) of the Hermite polynomial. That leads to the expression

$$I=\frac{1}{\sqrt{\sqrt{\pi}}}\frac{1}{\sqrt{1+\xi}}\exp[-x^2/2\sigma^2] \times\exp[-\beta^2/2(1+\xi)]\exp[x\sqrt{2}\beta/(1+\xi)]. \tag{15}$$

The expression for the state (12) in the $x,y$ representation can now be found by noting that

$$\langle x,y|\psi_v\rangle=\frac{A}{\cosh(2\xi)}\frac{d^m}{d\beta^m} \times\exp\left[\frac{\xi}{2}(a^{\dagger 2}+b^{\dagger 2})+\beta(a^\dagger-ib^\dagger)\right]\Bigg|_{\beta=0}. \tag{16}$$

On using Eq. (15) in Eq. (16) it can be shown that

$$\psi_v(x,y)=\frac{A}{\sigma\sqrt{\pi}}\frac{2^{m/2}}{(1+\xi)^m}(x-iy)^m\exp\left[-\frac{x^2+y^2}{2\sigma^2}\right]. \tag{17}$$

On imposing the normalization condition $\int\int|\psi_v(x,y)|^2dx\,dy=1$ we get

$$A=\frac{2^{-m/2}(1+\xi)^m}{\sqrt{m!}\sigma^m},\quad \xi=\frac{\sigma^2-1}{\sigma^2+1}. \tag{18}$$

Hence the normalized wave function $\psi_v(x,y)$ for the vortex state is given by

$$\psi_v(x,y)=\sqrt{\frac{1}{m!\pi\sigma^{2m+2}}}(x-iy)^m \times\exp\left[-\frac{x^2+y^2}{2\sigma^2}\right],$$

$$\int\int|\psi_v(x,y)|^2dx\,dy=1. \tag{19}$$

The operator version of the vortex state is given by Eqs. (12), (3), and (18). The position of the vortex can be shifted by using

$$\psi(x,y)=\sqrt{\frac{1}{\pi\sigma}}\exp\left[-\frac{(x-x_0)^2+(y-y_0)^2}{2\sigma^2}\right]. \quad (20)$$

In that case the vortex state can be defined by

$$|\psi_v\rangle=A'(a^\dagger-ib^\dagger)^m|\psi\rangle, \quad (21)$$

where $A'$ is a normalization constant.

### Vortex state in the circular basis

It is instructive to express the vortex state in the circular basis defined by the operators $c,d$, where

$$c=\frac{a+ib}{\sqrt{2}}, \quad d=\frac{a-ib}{\sqrt{2}}. \quad (22)$$

It should be noted that by use of a polarizing beam splitter that splits a circularly polarized light into two orthogonal plane-polarized components, one can go from basis $c,d$ to $a,b$. In the circular basis the expression (3) for $|\psi\rangle$ reads

$$|\psi\rangle=\exp[2\zeta(c^\dagger d^\dagger-cd)]|0,0\rangle_{c,d}$$
$$=\frac{1}{\cosh(2\zeta)}\sum_{k=0}^{\infty}\xi^k|k,k\rangle_{c,d}, \quad (23)$$

where

$$\xi=\tanh(2\zeta)\equiv\frac{\sigma^2-1}{\sigma^2+1} \quad (24)$$

and $|j,k\rangle_{c,d}$ is the Fock state of modes $c$ and $d$ with $j$ photons in mode $c$ and $k$ photons in mode $d$. Hence the vortex state (12) in the circular basis is given by

$$|\psi_v\rangle=A\frac{2^{m/2}}{\cosh(2\zeta)}\sum_{k=0}^{\infty}\xi^k\sqrt{\frac{(m+k)!}{k!}}|m+k,k\rangle_{c,d}, \quad (25)$$

where $A$ is given by Eq. (18). Equation (25) shows that the vortex state is obtained from the two-mode squeezed vacuum (23) in the circular basis by the addition of $m$ photons of type $c$ [6,7]. For $\xi=0$, i.e., in the case of no squeezing, it follows from Eq. (25) that

$$|\psi_v\rangle\to|m,0\rangle_{c,d}. \quad (26)$$

It is thus interesting that a Fock state of a circularly polarized field is a *vortex state when viewed in the space of the linearly polarized orthogonal components*. Note, however, that a circularly polarized field in a coherent state is *not* a vortex state. In analogy with Eq. (17), we can derive the expression for the vortex state in the basis $|X,Y\rangle\equiv|X\rangle|Y\rangle$, where $|X\rangle,|Y\rangle$ are the eigenstates of $(c+c^\dagger)/\sqrt{2}$ and $(d+d^\dagger)/\sqrt{2}$, respectively. It is shown in the Appendix that

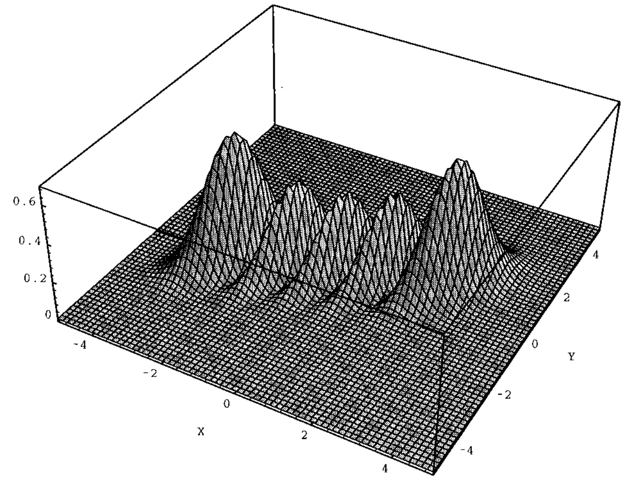


FIG. 2. Intensity distribution for the vortex state in the circular basis for $\sigma=\sqrt{2}$ and $m=4$.

$$\langle X,Y|\psi_v\rangle=\frac{e^{im\pi/2}}{2^{m/2}\pi^{1/4}\sqrt{\Gamma(m+\frac{1}{2})}}\exp\left[\frac{X^2-Y^2}{2}\right]$$
$$\times\exp\left[-\frac{(X-Y\xi)^2}{1-\xi^2}\right]H_m\left(\frac{X-Y\xi}{\sqrt{1-\xi^2}}\right). \quad (27)$$

The locus of points where the wave function vanishes is now a straight line $(X-\xi Y)/\sqrt{1-\xi^2}=x_0$, where $x_0$ is a real zero of the Hermite polynomial $H_m(x)$. For $m$ odd, $x_0=0$ is one of the zeros. It is interesting to observe that the phase of the state in the new basis is just $[(\pi/2)m]$. The intensity distribution $|\psi_v(X,Y)|^2$ is plotted in Fig. 2. A comparison of Figs. 1 and 2 shows the change in intensity distribution while going from one basis to the other.

Next we examine various properties of the vortex state of the quantized field.

## III. PROPERTIES OF VORTEX STATES

In this section we list the expressions for the mean values of the field operators and those for various quasiprobabilities and investigate the nonclassical properties of the field.

### A. Mean values $\langle x^j y^k\rangle_v$

First we give the expression for the average of the product of arbitrary powers of $x$ and $y$. By using Eq. (19) we get

$$\langle x^j y^k\rangle_v=\frac{1}{m!\pi\sigma^{2m+2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2)^m x^j y^k$$
$$\times\exp\left[-\frac{x^2+y^2}{\sigma^2}\right]dx\,dy. \quad (28)$$

By letting $x\to -x$ and $y\to -y$ in Eq. (28) it follows that

$$\langle x^j y^k\rangle_v=(-1)^j\langle x^j y^k\rangle_v, \quad \langle x^j y^k\rangle_v=(-1)^k\langle x^j y^k\rangle_v. \quad (29)$$

Hence

$$\langle x^j y^k \rangle_v = 0, \quad \text{if } j \text{ or } k \text{ is odd.} \tag{30}$$

For $j$ and $k$ even, the integrals in Eq. (28) can be evaluated by the binomial expansion of $(x^2+y^2)^m$ to get

$$\langle x^{2j} y^{2k} \rangle_v = \frac{\sigma^{2(j+k)}\Gamma(m+j+k+1)\Gamma(j+\frac{1}{2})\Gamma(k+\frac{1}{2})}{\pi\Gamma(j+k+1)\Gamma(m+1)}. \tag{31}$$

Hence it follows that the averages in the vortex $|\psi_v\rangle$ and the nonsingular state $|\psi\rangle$ (which corresponds to $m=0$) are related by

$$\langle x^{2j} y^{2k} \rangle_v = \frac{\Gamma(m+j+k+1)}{\Gamma(j+k+1)\Gamma(m+1)} \langle x^{2j} y^{2k} \rangle_0, \tag{32}$$

where

$$\langle x^{2j} y^{2k} \rangle_0 = \frac{\sigma^{2(j+k)}\Gamma(j+k+1)\Gamma(j+\frac{1}{2})\Gamma(k+\frac{1}{2})}{\pi\Gamma(j+k+1)} \tag{33}$$

is the average in the squeezed state. In particular

$$\langle x^2 \rangle = \frac{(m+1)\sigma^2}{2}. \tag{34}$$

It is clear from Eq. (34) that the variance of the $x$ or $y$ distribution increases with $m$ and hence the vortex state has a wider dispersion than a nonsingular state in $x$-$y$ space. There is squeezing in $x$ if $(m+1)\sigma^2 < 1$.

Let us examine also the squeezing properties [8] of the operator $p_x$ conjugate to $x$. It is straightforward to see that $\langle p_x \rangle = 0$ and

$$\langle p_x^2 \rangle = \frac{m+1}{2\sigma^2}. \tag{35}$$

Thus there is squeezing in $p_x$ if $m+1 < \sigma^2$. However, the uncertainty product given by

$$\langle \Delta x_x^2 \rangle \langle \Delta p_x^2 \rangle = \frac{(m+1)^2}{4} \tag{36}$$

is independent of $\sigma$. The state for $m=0$ is the minimum-uncertainty state. In the case of $m=0$, there is always squeezing either in $x$ or in $p_x$ if $\sigma \neq 1$.

### B. Photon correlations and photon statistics

The mean value of a product involving arbitrary powers of $x$ and $y$, determined in the preceding subsection, cannot provide information about an arbitrary field observable. That task is achieved by the mean value of the product of arbitrary powers of the field annihilation and creation operators. In the present case it is simple to determine the wave function in the coherent-state representation and hence the mean value of an antinormally ordered product. Thus, if $Q(\alpha,\alpha^*;\beta,\beta^*) \equiv |\langle \alpha,\beta|\psi_v\rangle|^2/\pi^2$, then the mean value $\langle a^j a^{\dagger k} b^p b^{\dagger q} \rangle$ is given by

$$\langle a^j a^{\dagger k} b^p b^{\dagger q} \rangle = \int\int d^2\alpha\, d^2\beta\, \alpha^j \beta^p \alpha^{*k} \beta^{*q} \times Q(\alpha,\alpha^*;\beta,\beta^*). \tag{37}$$

It can be shown that

$$Q(\alpha,\alpha^*;\beta,\beta^*) = \frac{(1-\xi^2)^{m+1}}{2^m \pi^2 m!} |\alpha + i\beta|^{2m} \times \exp\left[\frac{\xi}{2}(\alpha^2+\beta^2+\text{c.c.}) - |\alpha|^2 - |\beta|^2\right], \tag{38}$$

where $\xi$ is given by Eq. (24). Thus the mean values can be obtained from the moments of the Gaussian distribution. Here we list some of the important mean values that yield fluctuations and correlations:

$$\langle a^\dagger a \rangle = \langle b^\dagger b \rangle = \frac{1}{2(1-\xi^2)}[m + \xi^2(m+2)], \tag{39}$$

$$\langle a^\dagger a b^\dagger b \rangle = \frac{1}{4(1-\xi^2)^2}[\xi^4(m^2+3m+4) + 4m\xi^2 + m(m-1)], \tag{40}$$

$$\langle a^\dagger a b^\dagger b \rangle - \langle a^\dagger a \rangle \langle b^\dagger b \rangle = \frac{-m}{4(1-\xi^2)^2}[1 + \xi^4 + 2m\xi^2], \tag{41}$$

$$\langle a^{\dagger 2} a^2 \rangle = \langle b^{\dagger 2} b^2 \rangle = \langle a^\dagger a b^\dagger b \rangle + \frac{(m+1)[(m+1)\xi^2 + \xi^4]}{(1-\xi^2)^2}. \tag{42}$$

We use the preceding results to investigate the photon statistics of the field. First we determine the intensity fluctuation in the $a$ mode by evaluating the function $g^{(2)}$ defined as

$$g^{(2)} = \frac{\langle a^{\dagger 2} a^2 \rangle - \langle a^\dagger a \rangle^2}{\langle a^\dagger a \rangle^2}. \tag{43}$$

The state is nonclassical if the photon number distribution is sub-Poissonian, i.e., if $g^{(2)} < 0$. By using Eqs. (39)–(42) and (24) the evaluation of Eq. (43) leads to

$$g^{(2)} = \frac{(m^2+5m+5/2)(\sigma^8+1) - 2(4m+1)\sigma^2(\sigma^4+1) - (2m^2+2m+1)\sigma^4}{2[m(\sigma^4+1) + (\sigma^2-1)^2]^2}. \tag{44}$$

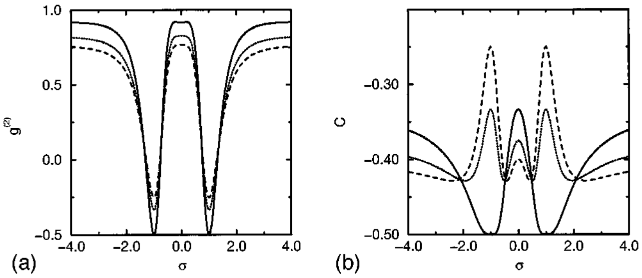


FIG. 3. (a) Intensity correlation function $g^{(2)}$ as a function of $\sigma$ for $m=2$ (solid curve), $m=3$ (dot curve), and $m=4$ (dash curve). (b) Anticorrelation coefficient $C$ as a function of $\sigma$ for $m=2$ (solid curve), $m=3$ (dot curve) and $m=4$ (dash curve).

This shows that the individual modes may exhibit sub-Poissonian statistics for certain values of $\sigma$ and $m$, as shown in Fig. 3(a). We further note that the two modes are anticorrelated in the vortex state, as can be seen by calculating the correlation coefficient

$$C\equiv\frac{\langle a^{\dagger}ab^{\dagger}b\rangle-\langle a^{\dagger}a\rangle\langle b^{\dagger}b\rangle}{\langle a^{\dagger}a\rangle\langle b^{\dagger}b\rangle}$$
$$=-m[m(\sigma^4-1)^2+(\sigma^4+1)^2+4\sigma^4]$$
$$\times\{2[m(\sigma^4+1)+(\sigma^2-1)^2]^2\}^{-1} \qquad (45)$$

using Eqs. (37), (39) and (24). This anticorrelation aspect is apparent from Fig. 3(b).

### C. Photon number distribution

The photon number distribution $N(p,q)$ of the two modes may be found by evaluating

$$\langle p,q|\psi_v\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle p,q|x,y\rangle\langle x,y|\psi_v\rangle dx\ dy,$$

$$N(p,q)\equiv|\langle p,q|\psi_v\rangle|^2, \qquad (46)$$

where $|p,q\rangle$ represents the Fock state of the two-mode field and $\langle p|x\rangle$ and $\langle x,y|\psi_v\rangle$ are given by Eqs. (6) and (19). The distribution $N(p,q)$ can be evaluated directly from Eq. (46). The results of numerical computations are shown in Fig. 4. Note the zeros in $N(p,q)$. These arise from the zeros of the number distribution associated with the squeezed vacuum as well as the cutoff provided by the prefactor in Eq. (12),

$$\langle p,q|\psi_v\rangle\equiv A\langle p,q|(a^{+}-ib^{+})^m|\psi\rangle. \qquad (47)$$

For $m=2$ we get

$$\langle p,q|\psi_v\rangle=A[\sqrt{p(p-1)}\langle p-2,q|\psi\rangle-\sqrt{q(q-1)}\langle p,q-2|\psi\rangle$$
$$-2i\sqrt{pq}\langle p-1,q-1|\psi\rangle], \qquad (48)$$

which, for instance, will be zero for $p=1$, $q=0$.

## IV. GENERATION OF THE VORTEX STATE

Here we outline a method of producing the vortex state. It can be produced by a variety of state-reduction techniques [9]. For example, consider a three-level $\Lambda$ system interacting

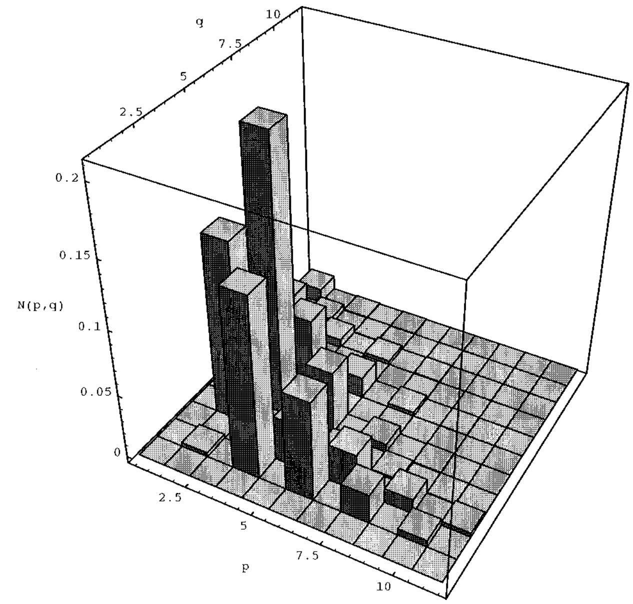


FIG. 4. Photon number distribution $N(p,q)$ as a function of $p$ and $q$ for $\sigma=\sqrt{2}$ and $m=4$.

with two circularly polarized fields on resonance (Fig. 5). That interaction is governed by the Hamiltonian

$$H\propto|1\rangle\langle 3|(a+ib)+|3\rangle\langle 1|(a^{\dagger}-ib^{\dagger})$$
$$+|1\rangle\langle 2|(a-ib)+|2\rangle\langle 1|(a^{\dagger}+ib^{\dagger}). \qquad (49)$$

We let the system, prepared in the state $|\psi_0\rangle$, evolve under the influence of Eq. (49) and think of a conditional measurement, i.e., we determine the state of the field when the atom is detected in the state $|3\rangle$. For short times first-order perturbation theory shows that the state of the field is $(a^{\dagger}-ib^{\dagger})|\psi_0\rangle$. If we consider a succession of $m$ atoms through a bimodal cavity and if we detect all the atoms in state $|3\rangle$, then the state of the field is reduced to $(a^{\dagger}-ib^{\dagger})^m|\psi_0\rangle$, which is the desired vortex state.

We also note that the vortex state (1) is an eigenstate of the $z$ component of the angular-momentum operator $L_Z\equiv xp_y-yp_x\equiv-i\hbar[x(\partial/\partial y)-y(\partial/\partial x)]$, i.e.,

$$L_Z\psi_v(x,y)\equiv-m\hbar\psi_v(x,y). \qquad (50)$$

This then suggests another possibility of realizing states such as Eq. (1), say, by *using coupling between harmonic oscillators* [10].

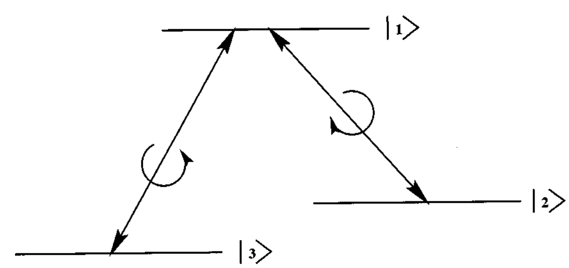


FIG. 5. Scheme for producing the vortex state involving a $\Lambda$ system.

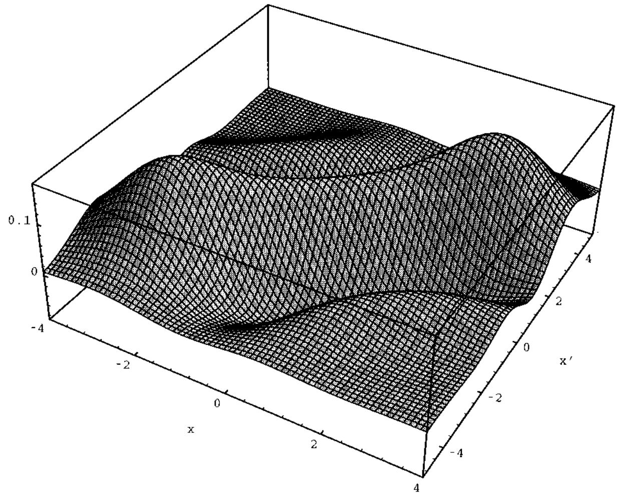


FIG. 6. Correlation function $\rho_a(x,x')$ as a function of $x$ and $x'$ for $\sigma=\sqrt{2}$ and $m=4$.

## V. THE REDUCED STATE FOR EACH MODE IN THE VORTEX STATE

In this section we determine the reduced density matrix $\rho_a$ of the $a$ mode defined by $\rho_a \equiv \mathrm{Tr}_b|\psi_v\rangle\langle\psi_v|$. The single-mode state is a mixed state even though the two-mode state is a pure state. It might be noted that this is also the case with the two-mode squeezed vacuum state defined by Eq. (3).

### A. Spatial coherence of the *a* mode

First we determine the correlation function in the $x$ space of mode $a$ by evaluating the off-diagonal element $\langle x_1|\rho_a|x_2\rangle$. It can be found by integrating the expression for $\langle x_1,y|\rho|x_2,y\rangle \equiv \langle x_1,y|\psi_v\rangle\langle\psi_v|x_2,y\rangle$ over $y$,

$$\langle x_1|\rho_a|x_2\rangle = \int_{-\infty}^{\infty}\langle x_1,y|\psi_v\rangle\langle\psi_v|x_2,y\rangle dy, \tag{51}$$

where $\langle x,y|\psi_v\rangle$ is given by Eq. (19). Using Eq. (19), Eq. (51) reads

$$\langle x_1|\rho_a|x_2\rangle = \left(\frac{1-\xi}{1+\xi}\right)^{m+1}\frac{1}{\pi m!}\exp\left[-\frac{1-\xi}{1+\xi}(x_1^2+x_2^2)/2\right]I, \tag{52}$$

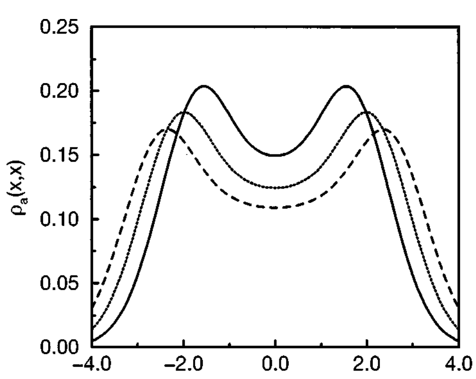


FIG. 7. Coordinate-space distribution $\rho_a(x,x)$ as a function of $x$ for $\sigma=\sqrt{2}$ and $m=2$ (solid curve), $m=3$ (dot curve), and $m=4$ (dash curve).

where

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dy (x_1 - iy)^m (x_2 + iy)^m \exp\left[-\frac{1-\xi}{1+\xi}y^2\right] \\ &= \frac{d^{2m}}{d\lambda^m d\mu^m}\int_{-\infty}^{\infty} dy\ \exp[\lambda x_1 + \mu x_2]\exp[-iy(\lambda-\mu)] \\ &\quad \times \exp\left[-\frac{1-\xi}{1+\xi}y^2\right]\Bigg|_{\lambda=\mu=0} \\ &= \sqrt{\frac{1-\xi}{1+\xi}}\frac{1}{\sqrt{\pi}}\frac{d^{2m}}{d\lambda^m d\mu^m}\exp[\lambda x_1 + \mu x_2] \\ &\quad \times \exp\left[-\left(\frac{1+\xi}{1-\xi}\right) - \frac{(\lambda-\mu)^2}{4}\right]\Bigg|_{\lambda=\mu=0}. \end{aligned} \tag{53}$$

By changing the variables $\lambda,\mu$ to $u,v$, where

$$u = \left(\frac{1+\xi}{1-\xi}\right)\left(\frac{\lambda-\mu}{2}\right),\quad v = \left(\frac{1+\xi}{1-\xi}\right)\left(\frac{\lambda+\mu}{2}\right), \tag{54}$$

Eq. (53) for $I$ leads to

$$\begin{aligned} I &= \left(\frac{1-\xi}{1+\xi}\right)^{m/2}\frac{\sqrt{\pi}}{2^{2m}}\left(\frac{d^2}{dv^2} - \frac{d^2}{du^2}\right)^m \\ &\quad \times \exp\left[v(x_1+x_2)\sqrt{\frac{1+\xi}{1-\xi}}\right]\exp\left[u(x_1-x_2)\sqrt{\frac{1+\xi}{1-\xi}}\right] \\ &\quad \times \exp[-u^2]\big|_{u=v=0} \\ &= \left(\frac{1-\xi}{1+\xi}\right)^{m/2}\frac{\sqrt{\pi}}{2^{2m}}\left((x_1+x_2)^2\frac{1+\xi}{1-\xi} - \frac{d^2}{du^2}\right)^m \\ &\quad \times \exp\left[-\left(p - \frac{x_1-x_2}{2}\sqrt{\frac{1+\xi}{1-\xi}}\right)^2\right] \\ &\quad \times \exp\left[\frac{(x_1-x_2)^2}{4}\frac{1+\xi}{1-\xi}\right]\Bigg|_{u=v=0}. \end{aligned} \tag{55}$$

It is now possible to derive the expression for $\langle x_1|\rho_a|x_2\rangle$ from Eq. (55),

$$\begin{aligned} \langle x_1|\rho_a|x_2\rangle &= \frac{\exp[-(x_1^2+x_2^2)/2\sigma^2]}{2^{2m}\sigma^{2m+1}m!\sqrt{\pi}} \\ &\quad \times \sum_{k=0}^{m}\left[(-1)^k\frac{\sigma^{2k}m!}{(m-k)!k!}\right. \\ &\quad \left.\times (x_1+x_2)^{2(m-k)}H_{2k}\left(\frac{x_1-x_2}{2\sigma}\right)\right]. \end{aligned} \tag{56}$$

The results of the numerical evaluation of Eq. (56) are plotted in Fig. 6 for $\sigma^2=2$ and $m=4$. The function $\rho_a(x_1,x_2) \equiv \langle x_1|\rho_a|x_2\rangle$ for $m=0$, when there is no singularity, is clearly a two-dimensional Gaussian centered at $x_1=x_2=0$. The plots of $\rho_a(x_1,x_2)$ for $m\neq 0$ exhibit hills and valleys. For $x_1=x_2=x$ Eq. (56) reduces to

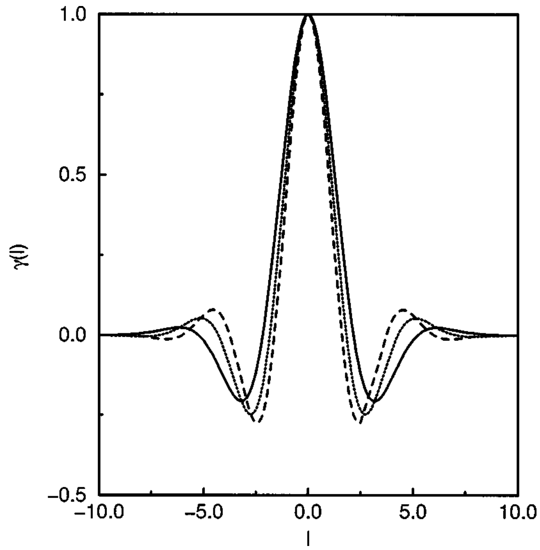


FIG. 8. Single-mode coherence function $\gamma(l)$ [Eq. (57)] as a function of $l$ for $\sigma=\sqrt{2}$ and $m=2$ (solid curve), $m=3$ (dot curve), and $m=4$ (dash curve).

$$\rho_a(x,x)\equiv\langle x|\rho|x\rangle=\frac{\exp[-x^2/\sigma^2]}{\sigma^{2m+1}\pi}\sum_{k=0}^{m}\left[\frac{\sigma^{2k}}{(m-k)!k!}x^{2(m-k)}\times\Gamma\left(k+\frac{1}{2}\right)\right]. \quad (57)$$

The behavior of $\rho_a(x,x)$ is exhibited in Fig. 7. It shows that the maximum exhibited by $\rho_a(x,x)$ at $x=0$ for $m=0$ splits into two for $m\neq 0$, leading to a dip at $x=0$. The bimodal distribution arises from the entanglement of $a$ and $b$ modes [Eq. (12)].

We define spatial coherence function $\gamma(l)$ for the single mode by

$$\gamma(l)\equiv\int \rho_a(x,x+l)dx, \quad (57')$$

which on using Eq. (55) can be reduced to

$$\gamma(l)=\frac{e^{-l^2/4\sigma^2}}{(2\sigma)^{2m+1}m!\sqrt{\pi}}\sum_{k=0}^{m}(-1)^k\frac{m!\sigma^{2k}}{(m-k)!k!}\times(4\sigma^2)^{m-k+1/2}\Gamma(m-k+1/2)H_{2k}(l/2\sigma). \quad (58)$$

To see the nature of the correlation between two points of $x$ space $\gamma(l)$ has been displayed in Fig. 8 for various values of $m$. The minimum in the coherence function can be taken as a signature of the vortex character. The spatial coherence function $\gamma(l)$ is related to the momentum-space representation in the following manner:

$$\begin{aligned}\gamma(l)&=\int dx\ \rho(x,x+l)\\ &\equiv\int dx\ \rho(k_1,k_2)e^{ik_1x}e^{-ik_2(x+l)}dk_1dk_2\\ &\equiv 2\pi\int \rho(k_1,k_2)\,\delta(k_1-k_2)e^{-ik_2l}dk_1dk_2\\ &\equiv 2\pi\int \rho(k,k)e^{-ikl}dk. \end{aligned} \quad (59)$$

The expression for $\rho(k,k)$ can be obtained by taking Fourier transform of Eq. (59) and using Eq. (58). The final result is

$$\rho(k,k)=4\pi e^{-\sigma^2k^2}\sum_{t=0}^{m}\frac{\sigma^{2t+1}k^{2t}}{(m-t)!t!}\Gamma(m-t+\tfrac{1}{2}). \quad (60)$$

The behavior of $\rho(k,k)$ is shown in Fig. 9. The vortex state leads to a bifurcation of momentum distribution.

### B. $Q$ function for the $a$ mode

The $Q$ function $\langle\alpha|\rho_a|\alpha\rangle$ of $\rho_a$ for mode $a$ may be obtained from

$$Q_a(\alpha,\alpha^*)=\langle\alpha|\rho_a|\alpha\rangle/\pi=\int Q(\alpha,\alpha^*,\beta,\beta^*)d^2\beta. \quad (61)$$

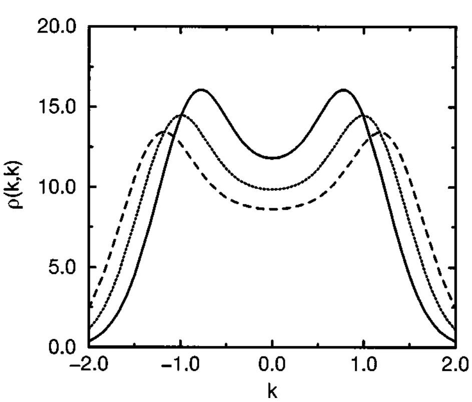


FIG. 9. Momentum-space representation of the wave function $\rho(k,k)$ for $\sigma=\sqrt{2}$ and $m=2$ (solid curve), $m=3$ (dot curve), and $m=4$ (dash curve).

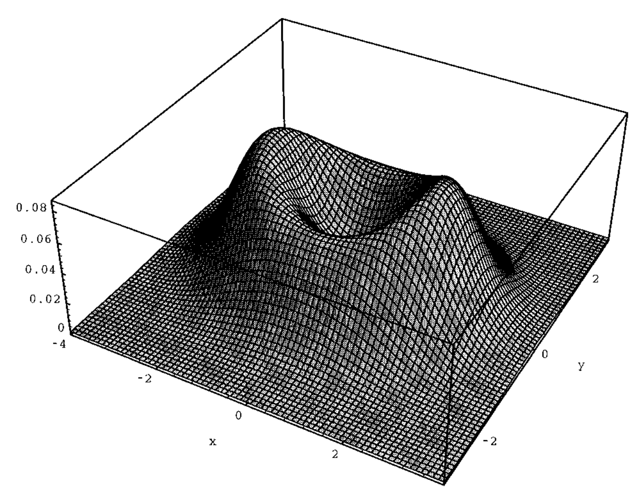


FIG. 10. $Q_a(x,y)$ as a function of real and imaginary parts of $\alpha\equiv x+iy$ for $\sigma=\sqrt{2}$ and $m=4$.

The evaluation of Eq. (61) can be done as follows. By using Eq. (38), Eq. (61) can be written as

$$Q_a(\alpha,\alpha^*)=\frac{(1-\xi^2)^{m+1}}{2^m\pi^2 m!}\exp\left[\frac{\xi}{2}(\alpha^2+\alpha^{*2})-|\alpha|^2\right]\mathcal{I}, \qquad (62)$$

where

$$\mathcal{I}=\frac{d^{2m}}{d\lambda^m d\mu^m}\exp[\alpha\lambda+\alpha^*\mu]\int \times\exp\left[\frac{\xi(\beta^2+\beta^{*2})}{2}+i(\beta\lambda-\beta^*\mu)-|\beta|^2\right]\Bigg|_{\lambda=\mu=0} d^2\beta. \qquad (63)$$

The integral in Eq. (63) can be evaluated by substituting $\beta=x+iy$ to obtain

$$\mathcal{I}=\frac{\pi}{\sqrt{1-\xi^2}}\frac{d^{2m}}{d\lambda^m d\mu^m}\exp[(\alpha\lambda+\alpha^*\mu)] \times\exp\left[-\frac{(\lambda-\mu)^2}{4(1-\xi)}\right]\exp\left[-\frac{(\lambda+\mu)^2}{4(1+\xi)}\right]\Bigg|_{\lambda=\mu=0}, \qquad (64)$$

which on simplification leads to

$$Q_a(\alpha,\alpha^*)=\frac{(1-\xi^2)^{m+1/2}}{2^m\pi}\exp\left[-|\alpha|^2+\frac{\xi}{2}(\alpha^2+\alpha^{*2})\right] \times\sum_{k=0}^{m}\frac{m!}{(m-k)!(k!)^2}\left(\frac{\xi}{2}\right)^k\left|H_k\left(\sqrt{\frac{1-\xi^2}{2\xi}}\,\alpha\right)\right|^2. \qquad (65)$$

The form (65) immediately leads to the expression of the density matrix in the operator form

$$\rho_a=\frac{(1-\xi^2)^{m+1/2}}{2^m\pi}\exp\left(\frac{\xi}{2}a^{\dagger 2}\right)\sum_{k=0}^{m}\frac{m!}{(m-k)!(k!)^2}\left(\frac{\xi}{2}\right)^k H_k\left(\sqrt{\frac{1-\xi^2}{2\xi}}\,a^\dagger\right)|0\rangle\langle 0|H_k\left(\sqrt{\frac{1-\xi^2}{2\xi}}\,a\right)\exp\left(\frac{\xi}{2}a^2\right). \qquad (66)$$

We have plotted $Q_a(x,y)$ in Fig. 10 as a function of the real and imaginary parts of $\alpha\equiv x+iy$ for $\sigma^2=2$ and $m=4$. The function $Q_a(x,y)$ has a maximum centered at $\alpha=0$ for $m=0$. For nonzero values of $m$, there is instead a valley at $\alpha=0$, which is a signature of the singularity. The function $Q_a(x,y)$ is thus qualitatively different for states with and without vortices.

## VI. CONCLUSION

We have introduced wave-packet states with vortex structure. We study the properties of radiation fields in such states, which we show can be produced by the interaction of $\Lambda$ systems in a two-mode cavity containing squeezed radiation. We have also presented detailed results for the distributions associated with one of the modes. It should be borne in mind that the results obtained in this paper are valid for any quantum system that can be effectively described by a two-dimensional harmonic oscillator.

## APPENDIX

In this appendix we derive the equation (27) for $\langle X,Y|\psi_v\rangle$, where $|X\rangle,|Y\rangle$ are the eigenstates of the operators $(c+c^\dagger)/\sqrt{2}$ and $(d+d^\dagger)/\sqrt{2}$, respectively. The circular representation operators $c$ and $d$ are related to operators $a$ and $b$ by Eq. (22). Using Eqs. (22) and (23), the expression (12) for the vortex state reads

$$|\psi_v\rangle=A2^{m/2}c^{\dagger m}\exp[2\zeta(c^\dagger d^\dagger-cd)]|0,0\rangle_{c,d} \equiv Ac^{\dagger m}2^{m/2}\exp[\xi c^\dagger d^\dagger]|0,0\rangle_{c,d} \equiv A2^{m/2}\sum_{k=0}^{\infty}\xi^k\sqrt{\frac{(m+k)!}{k!}}\,|m+k,k\rangle_{c,d}, \qquad (A1)$$

where $A$ is the normalization constant given by Eq. (18) and $\xi$ is given by Eq. (24). On using Eq. (48), Eq. (A1) leads to

$$\langle X,Y|\psi_v\rangle=2^{m/2}A\sum_{k=0}^{\infty}\left(\frac{\xi}{2}\right)^k\frac{1}{k!}H_{m+k}(X)H_k(Y) \times\exp\left[-\frac{X^2+Y^2}{2}\right]. \qquad (A2)$$

The summation in Eq. (A2) can be performed by using the integral representation of the Hermite polynomials given by Eq. (8) to arrive at the expression

$$\langle X,Y|\psi_v\rangle=A_0\exp[(X^2-Y^2)/2] \times\exp[-(X-\xi Y)^2/(1-\xi^2)]H_m\left(\frac{X-\xi Y}{\sqrt{1-\xi^2}}\right), \qquad (A3)$$

where $A_0$ is the normalization constant, which can be shown to be given by

$$A_0^2=\frac{1}{\Gamma(m+1/2)2^m\sqrt{\pi}}. \qquad (A4)$$

Expression (A3) is the desired equation (27).

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