Second-harmonic generation and the conservation of orbital angular momentum with high-order Laguerre-Gaussian modes

J. Courtial, K. Dholakia, L. Allen, and M. J. Padgett
School of Physics and Astronomy, University of St. Andrews, Fife KY16 9SS, Scotland
(Received 16 June 1997)

Laguerre-Gaussian modes of various order are frequency doubled. The azimuthal phase structure of the second-harmonic light is measured directly by interfering the beam with its mirror image. We show that the orbital angular momentum per photon is doubled, so conserving the orbital angular momentum in the light beam. The frequency-doubled output beam is shown to have a Gegenbauer-Gaussian amplitude distribution at the beam waist. The beam can be described as a summation of Laguerre-Gaussian modes that interfere so that it changes form with propagation, but the distribution at the beam waist is reproduced in the far field. [S1050-2947(97)03311-8]

PACS number(s): 42.65.Ky, 42.60.Jf

It was predicted in 1992 that monochromatic beams with an azimuthal phase term $e^{il\phi}$, of which Laguerre-Gaussian laser modes are an example, have a well-defined orbital angular momentum of $l\hbar$ per photon [1]. This orbital angular momentum is associated with the azimuthal component of the Poynting vector [2] and is quite distinct from the spin angular momentum associated with circular polarization. The transfer of this orbital angular momentum to a microscopic particle has been demonstrated recently [3]. Microscopic particles held within a Laguerre-Gaussian laser beam have been rotated and the orbital angular momentum quantified by a comparison of this rotation to that induced by the spin angular momentum. The results confirm that the orbital angular momentum is $l\hbar$ per photon [4,5]. Other work has explored theoretically the interaction of these beams with atomic systems [6,7].

Laguerre-Gaussian modes [8] are characterized by two indices l and p, where l is the number of 2π cycles in phase around the circumference and $p+1$ the number of radial nodes. The amplitude u_p^l of such a mode in cylindrical coordinates is given by

$$u_p^l(r,\phi,z) \propto \exp(-ikr^2/2R)\exp(-r^2/w^2) \times \exp[-i(2p+l+1)\psi] \times \exp(-il\phi)(-1)^p(r\sqrt{2}/w)^l L_p^l(2r^2/w^2), \quad (1)$$

where r is the distance from the beam axis, ϕ the azimuthal angle, z the distance from the beam waist, k the wave number of the light, w the radius for which the Gaussian term falls to $1/e$ of its on-axis value, z_r is the Rayleigh range, $L_p^l(x)$ an associated Laguerre polynomial, and $(2p+l+1)\psi$ is the Gouy phase, where $\psi = \arctan(z/z_r)$.

In 1996 we reported the mode transformation that occurs when a $p=0$ Laguerre-Gaussian mode is frequency doubled [9]. Modes with $p=0$ have a single-annular-ring intensity distribution. They frequency double to give a pure Laguerre-Gaussian mode also with $p=0$, but with an azimuthal index of $2l$, that is, with twice the orbital angular momentum per photon. This mode transformation is readily understood in terms of the spiraling of the Poynting vector, which has the same form for both the fundamental and second-harmonic beams, and was shown to be consistent with conservation of orbital angular momentum within the light beams. In our earlier work the azimuthal phase structure of the frequency-doubled mode was not measured directly. It was deduced by converting the second-harmonic mode into the corresponding Hermite-Gaussian mode by means of a cylindrical lens mode converter [10].

In this paper we confirm our earlier results by the direct measurement of the azimuthal phase structure of the frequency-doubled beams and extend the results to include frequency doubling of the multiringed, $p>0$, Laguerre-Gaussian modes. We show that this results in beams that possess a well-defined orbital angular momentum, but are not simple Laguerre-Gaussian modes.

At the beam waist, $z=0$, the amplitude of a Laguerre-Gaussian mode simplifies to

$$u_p^l(r,\phi,z=0) \propto e^{-r^2/w_0^2} e^{-il\phi}(-1)^p(r\sqrt{2}/w_0)^l L_p^l(2r^2/w_0^2). \quad (2)$$

For $p=0$, the associated Laguerre polynomial is a constant and independent of r. In second-harmonic generation, the amplitude of the frequency-doubled field is proportional to the square of the incident field [11]. It follows that for a $p=0$ mode the second-harmonic beam is also a Laguerre-Gaussian mode that has undergone the following transformations: $k \rightarrow 2k$, frequency doubling; $w_0 \rightarrow w_0/\sqrt{2}$, reduction of the beam waist; $p=0 \rightarrow p=0$, the amplitude distribution remains single ringed; and $l \rightarrow 2l$, the angular momentum per photon is doubled. When $p>0$ the square of the incident field can no longer be described in terms of a single Laguerre-Gaussian mode. However, the azimuthal phase structure is still of the form $\exp(il\phi)$.

The experimental arrangement for generating the frequency-doubled beams, as well as the subsequent analysis of their intensity and phase structure, is shown in Fig. 1. An intracavity cross wire is used to generate a variety of Hermite-Gaussian modes with indices m and n from a diode-pumped Nd:YAG laser (where YAG denotes yttrium aluminum garnet) operating at 1064 nm producing a linearly polarized output power of ≈ 100 mW. Each mode is converted

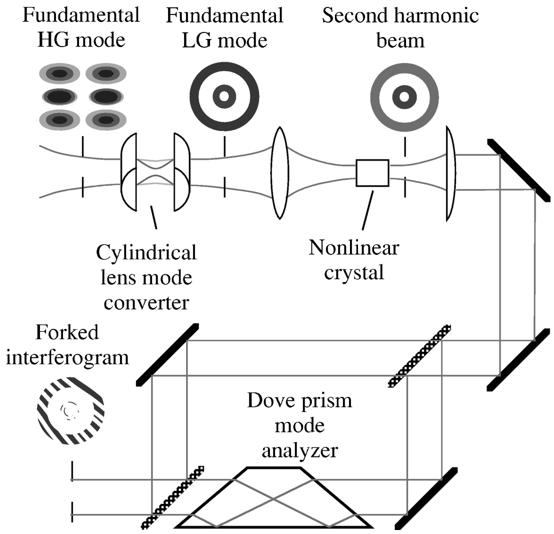

FIG. 1. Experimental apparatus for the generation and analysis of the second-harmonic beams.

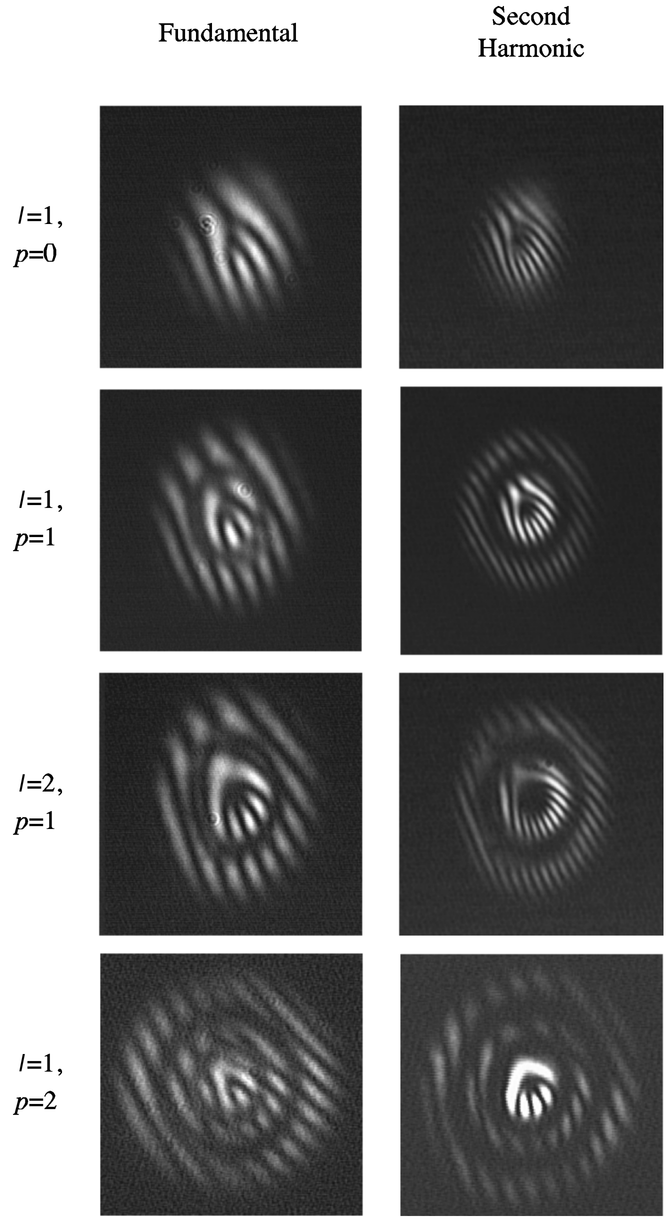

FIG. 2. Forked interferograms derived from a variety of Laguerre-Gaussian beams and their second-harmonic counterparts.

into the corresponding Laguerre-Gaussian mode by means of a cylindrical lens mode converter, giving the transformations $l=m-n$ and $p=\min(m,n)$ [10]. The Laguerre-Gaussian modes are then frequency doubled using a 10-mm-long crystal of potassium titanyl phosphate (KTP), angle tuned to give phase matching for the second harmonic at 532 nm. The efficiency of the process is maximized by focusing the incident beam such that its Rayleigh range is comparable to the length of the crystal [12]. Filters placed after the crystal allow either the fundamental or the second-harmonic beam to be selected and imaged onto a charge coupled device array detector. The azimuthal phase structure of the beams can be measured directly using a mode analyzer [13] based on a Dove prism, which allows the interference pattern between the beam and its own mirror image to be obtained. The azimuthal phase component of the beam gives rise to forked interference fringes and the l index of the beam can be inferred directly from the number of fringes on either side of the fork by dividing the number of additional fringes by 2.

Figure 2 shows the forked interferograms obtained for a variety of fundamental Laguerre-Gaussian beams and their second-harmonic counterparts. As in our previous work [9], these results confirm that the azimuthal index l of the beam is doubled in the second-harmonic process. However, here the azimuthal phase has been measured directly and we observe that the doubling of l in the second-harmonic process holds for Laguerre-Gaussian modes of any order of l and p. Again, this is consistent with the conservation of orbital angular momentum within the light beams.

The less than perfect mode converter introduces residual astigmatism into the beam, which manifests itself as a slight ellipticity in the observed images. Corrected radial profiles for the fundamental and second-harmonic beams are obtained by scaling the images to make them symmetrical and averaging the profiles over 80 azimuths. These profiles are then fitted to the predicted field distributions of a Laguerre-Gaussian and its square, respectively, with the amplitude and beam diameter as the fit variables. Figure 3 shows that the corrected profiles are in good agreement with those predicted. The square of a Laguerre polynomial is a Gegenbauer polynomial; consequently, we refer to the amplitude distribution of a frequency-doubled Laguerre-Gaussian mode at the beam waist as a Gegenbauer-Gaussian distribution.

Pure Laguerre-Gaussian modes propagate without changing their form, with a beam divergence dictated by the size of the beam waist and the corresponding Rayleigh range. As discussed, the second-harmonic beam for $p>0$ can no longer be described as a simple Laguerre-Gaussian mode. Insight as to how it can be described can, however, be gained from the experimental evidence: We find that the distribution of intensity in the far field is the same as that in the plane of the nonlinear process at $z=0$. For a monochromatic beam the far-field amplitude distribution is simply the Fourier transform of the distribution at the beam waist [14]. When the Fourier transform of light consisting of the square of Laguerre-Gaussians at z is taken, it is found that the resulting distribution can be described as a superposition of a number

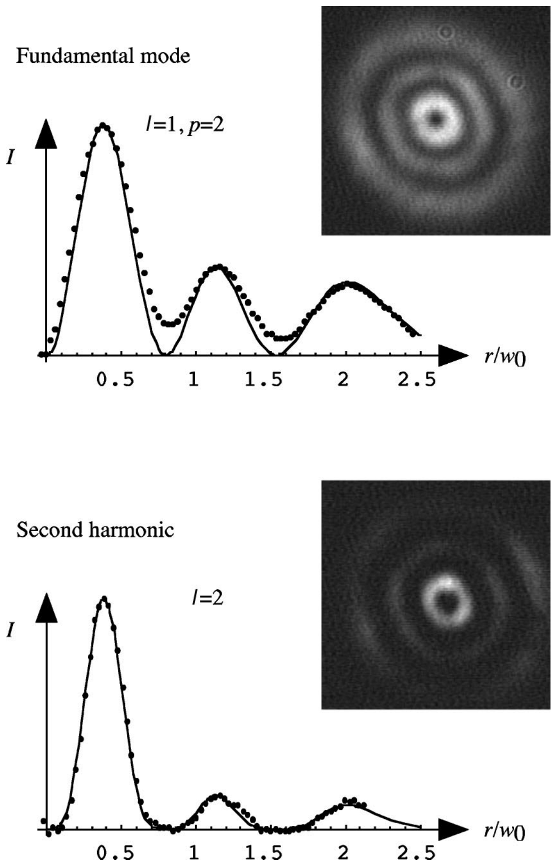

FIG. 3. Corrected radial intensity profiles for the fundamental and second-harmonic beams compared with theory, together with photographs of the observed images recorded in the plane of the frequency-doubling crystal.

of Laguerre-Gaussian modes all with the same index $2l$ but with $p^{(2\omega)}=0,2,\ldots,2p$. For $z=0$, this summation reduces to the square of a Laguerre-Gaussian of the same form as in Eq. (2).

Just as a Laguerre-Gaussian mode is a solution of the paraxial Helmholtz equation, so, too, is any sum of Laguerre-Gaussian modes, and as the second-harmonic beam propagates, interference occurs between the constituent modes. Although the modes all have the same Rayleigh range, the p indices give rise to a differing Gouy phase shift between the modes, which leads to an intensity distribution that changes form with propagation. It is only in the far field, where all the Gouy phase shifts differ by multiples of 2π, that the Gegenbauer-Gaussian distribution is reproduced. We see that the behavior of the $p=0$ modes analyzed in our previous paper [9] is simply a limiting case of the general behavior; when $p=0$ there is only one Laguerre polynomial involved and so only one Laguerre-Gaussian distribution.

Rather than investigate the explicit summation of Laguerre-Gaussian modes to determine the beam distribution as it propagates, we have implemented an algorithm based on the Fourier expansion of the beam in k space into a series of plane waves [15]. The propagation of each plane-wave component to a subsequent plane results in a well-defined change in phase. After propagation a summation of the individual plane-wave components followed by an inverse transform gives the new phase and amplitude distribution of the

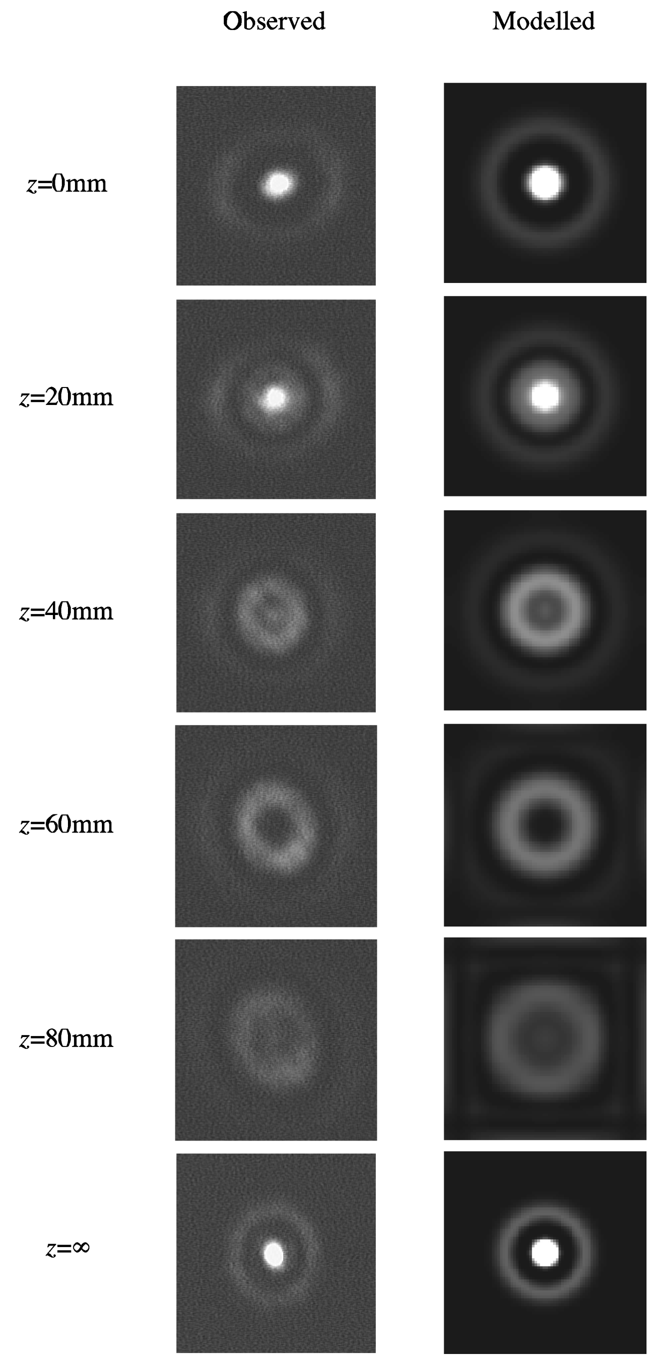

FIG. 4. Observed and modeled intensity distribution of a frequency-doubled Laguerre-Gaussian $(p=1,\ l=0)$ mode as it propagates from the beam waist.

beam. The algorithm is limited only by the array size of the Fourier transforms, which restricts the maximum allowed divergence and physical diameter of the beam. Figure 4 shows the observed and predicted intensity distributions for a frequency-doubled Laguerre-Gaussian mode with $p=1$ and $l=0$ as it propagates from the beam waist to the far field. We see that the intensity distribution of the second-harmonic beam reproduces itself in the far field, but varies at all intermediate positions. We observe similar behavior for all frequency-doubled Laguerre-Gaussian modes with $p>0$.

In this paper we extend our earlier work on the frequency doubling of Laguerre-Gaussian modes to $p>0$ and show that the previously investigated case of $p=0$ is a limiting case of the general behavior. The azimuthal phase of the frequency-doubled modes is measured directly and the azimuthal phase index is shown to become doubled for all modes. This corresponds to a doubling of the orbital angular momentum per photon during the second-harmonic generation process. The second-harmonic amplitude distribution may be described as a Gegenbauer-Gaussian beam that, while reproducing its intensity distribution in the far field, varies its distribution at all intermediate positions. Nevertheless, the frequency-doubled beams are still well defined and symmetrical about the beam axis. The nonlinearity of the second-harmonic generation process means that these beams have a radial intensity distribution in which a higher proportion of the energy is closer to the beam axis than in Laguerre-Gaussian modes. Such beams therefore allow the use of lower-aperture optical components in experiments involving the orbital angular momentum of light.

This work was supported by EPSRC. K. D. acknowledges the financial support of the Royal Society of Edinburgh. M. J. P. would like to thank the Royal Society (London) for financial support. It is a pleasure to express our thanks to Stephen M. Barnett for his analytic solution of the relevant Fourier transform.

[1] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).
[2] M. Padgett and L. Allen, Opt. Commun. **121**, 36 (1995).
[3] H. He, M. E. J. Friese, N. R. Heckenberg, and H. Rubinsztein-Dunlop, Phys. Rev. Lett. **75**, 826 (1995).
[4] M. E. J. Friese, J. Enger, H. Rubinsztein-Dunlop, and N. R. Heckenberg, Phys. Rev. A **54**, 1593 (1996).
[5] N. B. Simpson, K. Dholakia, L. Allen, and M. J. Padgett, Opt. Lett. **22**, 52 (1997).
[6] L. Allen, M. Babiker, W. K. Lai, and V. E. Lembessis, Phys. Rev. A **54**, 4259 (1996).
[7] W. K. Lai, M. Babiker, and L. Allen, Opt. Commun. **133**, 487 (1997).
[8] A. E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986), Sec. 17.5.
[9] K. Dholakia, N. B. Simpson, M. J. Padgett, and L. Allen, Phys. Rev. A **54**, R3742 (1996).
[10] M. W. Beijersbergen, L. Allen, H. E. L. O. van der Veen, and J. P. Woerdman, Opt. Commun. **96**, 123 (1993).
[11] A. Yariv, *Optical Electronics*, 3rd ed. (Holt, Rinehart and Winston, New York, 1985), Chap. 8.
[12] G. D. Boyd and D. A. Kleinmann, J. Appl. Phys. **39**, 3597 (1968).
[13] M. Harris, C. A. Hill, and J. M. Vaughan, Opt. Commun. **106**, 161 (1994).
[14] E. Hecht and A. Zajac, *Optics* (Addison-Wesley, Reading, MA, 1974), Sec. 11.3.3.
[15] E. A. Sziklas and A. E. Siegman, Appl. Opt. **14**, 1874 (1975).