Nonlinear amplification of x-ray channeling radiation

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The nonlinear amplification of X-ray channeling radiation in crystal is discussed on the basis of the self-consistent set of the Maxwell and Dirac equations. Both planar and axial channeling for relativistic electron beams are considered. Two stationary regimes of nonlinear amplification are discussed. [S1050-2947(97)00811-1]

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As is known, the channeling radiation of ultrarelativistic electrons and positrons is in the x-ray and γ-ray domain, and its spectral intensity exceeds that of other radiation sources in this frequency range [1,2]. So, stimulated channeling radiation of electrons is of certain interest as a potential source for the short-wave coherent radiation.

The stimulated radiation of channeled particles has been investigated mainly in the linear regime [3–11]. Multiphoton processes for the channeled positron in the external radiation field is considered in [12].

As the absorption coefficients of x-rays and γ rays in crystals are very high, $\Gamma\sim 10^2-10^3\ \mathrm{cm}^{-1}$ [13], and the construction of mirrors in this domain is very problematic, it is necessary to investigate the possibilities of radiation generation in nonlinear regimes. To obtain coherent radiation in crystals it is most appropriate to use electron beams with comparatively low energies ($\varepsilon\leqslant 50$ MeV for planar channeled electrons and $\varepsilon\leqslant 10$ MeV for axial ones). First of all, the states of channeled electrons are most stable in this energy region [14,15], i.e., the scattering of channeled particles on atomic electrons and nucleii of the lattice are suppressed. Then, at these energies a few discrete energy levels in the transverse potential well of the channeled electron are formed that are not equidistant. In this case by means of varying the angle of incidence of the electron beam to the crystal an inverted population of electron states in the transverse potential can be reached [14]. In addition, at low energies it is possible to use electron beams with high densities and increase the population inversion. Because the energy levels are not equidistant the stimulating electromagnetic wave resonantly couples only two energy levels, the physical processes in the above mentioned case of the channeling are similar to those of a two-level atom (two-dimensional ''atom'' in the case of the axial channeling, and one dimensional in the case of planar channeling one) moving with relativistic velocity.

In this work we investigate the possibilities of coherent radiation generation in nonlinear regimes by means of a relativistic channeled electron beam. The study is based on the self-consistent set of the Maxwell equations and the Dirac equation for the channeled electron. Maxwell's equations are solved in the slow varying envelope approximation.

We consider the channeling process for mildly relativistic electrons when the condition

$$\hbar\omega\ll\varepsilon \tag{1}$$

is always fulfilled, where $\hbar$ is the Plank constant, ω is the radiation frequency, and ε is the electron energy. With condition (1) the spin interaction can be neglected and the Dirac equation of quadratic form is reduced to the Klein-Gordon equation

$$\left[i\hbar\frac{\partial}{\partial t}-U(\rho)\right]^2\Psi=\left[c^2\left(\hat{\vec{p}}-\frac{e}{c}\vec{A}\right)^2+m^2c^4\right]\Psi, \tag{2}$$

where e and m are electron charge and mass, respectively, c is the light speed in vacuum, the axis OZ is aligned along the channel axis, $U(\rho)$ is the average potential of the crystal plane ($\rho\equiv x$)

$$U(x)=-U_0ch^{-2}\frac{x}{b}, \tag{3}$$

or of the crystal axis ($\rho\equiv\sqrt{x^2+y^2}$)

$$U(\rho)=-\frac{\alpha}{\rho}, \tag{4}$$

and $\vec{A}$ is the vector potential of the self-consistent electromagnetic wave satisfying the Maxwell equation

$$\Delta\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\frac{4\pi}{c}\vec{j}. \tag{5}$$

The electromagnetic wave has a linear polarization in the (x,z) plane and propagates in the (x,z) plane:

$$\vec{A}=\{A\cos\theta,0,-A\sin\theta\},$$

$$A=\tfrac{1}{2}[A_0(x,z,t)\exp\{i(\omega t-\vec{k}\cdot\vec{r})\}+\mathrm{c.c.}],$$

$$\vec{k}=\{k\sin\theta,0,k\cos\theta\}, \tag{6}$$

where $k=n(\omega)\omega/c$ is the wave number, θ is the angle between $\vec{k}$ and axis OZ, and $A_0(x,z,t)$ is the slowly varying envelope of the wave amplitude. The index of refraction of a

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crystal medium is $n(\omega)\simeq 1$ for the frequency region under consideration (x ray and γ ray).

We make the ansatz

$$\Psi=\sum_{\mu,p_z} a_{\mu,p_z}(z,t)\exp\left\{\frac{i}{\hbar}(p_z z-\varepsilon_{\mu,p_z}t)\right\}|\mu\rangle, \qquad (7)$$

$$\langle\mu|\mu'\rangle=\delta_{\mu\mu'},$$

where $\delta_{\mu\mu'}$ is the Kronecker symbol, μ is the complete set of quantum numbers, $\mu=\{p_y,n\}$ for planar channeling and $\mu=\{m_l,n\}$ for axial channeling, n is the main quantum number, and m_l is the magnetic quantum number; $|\mu\rangle$ are eigenvectors for transverse motion of the channeled particle with a longitudinal momentum p_z and energy ε_{μ,p_z}. The wave function's expansion coefficients $a_{\mu,p_z}(z,t)$ are slowly varying functions so the second-order derivatives will be neglected. Note that the energy spectrum of the planar channeled electron in the potential well (3) has the form

$$\varepsilon_{\perp n}=-\frac{\hbar^2}{2b^2m\gamma}[s-n]^2;\quad n=0,1,\ldots,[s], \qquad (8)$$

$$s=-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2b^2m\gamma U_0}{\hbar^2}},$$

and for the axial channeled electron in the potential (4),

$$\varepsilon_{\perp n}=-\frac{m\gamma\alpha^2}{2\hbar^2}\frac{1}{\left(n+\frac{1}{2}\right)^2},\quad n=0,1,2,\ldots, \qquad (9)$$

where $\gamma=\varepsilon/mc^2$ is the Lorentz factor.

Inserting Eqs. (6) and (7) into Eq. (2) and taking into account the condition (1) as well as the well-known condition for the channeling $|U|\ll\varepsilon_{\mu,p_z}$, we arrive at the following set of equations:

$$\frac{\partial a_{\mu,p_z}}{\partial t}+v_z\frac{\partial a_{\mu,p_z}}{\partial z}=\frac{eA_0(x,z,t)}{2\hbar c}\sum_{\mu'}M_{\mu\mu'}\exp\left\{\frac{i}{\hbar}(\varepsilon_{\mu,p_z}-\varepsilon_{\mu',p_z+\hbar k_z}+\hbar\omega)t\right\}a_{\mu',p_z+\hbar k_z}$$
$$-\frac{eA_0^*(x,z,t)}{2\hbar c}\sum_{\mu'}M^*_{\mu'\mu}\exp\left\{\frac{i}{\hbar}(\varepsilon_{\mu,p_z}-\varepsilon_{\mu',p_z-\hbar k_z}-\hbar\omega)t\right\}a_{\mu',p_z-\hbar k_z}, \qquad (10)$$

where

$$M_{\mu\mu'}=(\Omega_{nn'}\cos\theta-kv_z\sin^2\theta)\langle\mu|x|\mu'\rangle,$$

$$\hbar\Omega_{nn'}=\varepsilon_{\perp n'}-\varepsilon_{\perp n},$$

and v_z is the electron longitudinal velocity.

The selection rules for transitions are determined by matrix elements of the dipole momentum and for the axial channeling they are $\Delta m_l=\pm 1$. For planar channeling the $x_{\mu\mu'}$ is different from zero between states having different parities. For the axial channeling there is degeneracy by the magnetic quantum number and in the case of the wave of linear polarization both states $m_l=\pm 1$ will contribute to the resonant interaction process. As $M_{\mu\mu'}$ is only $|m_l|$ dependent for $\Delta m_l=\pm 1$ transitions, the $m_l=\pm 1$ states are equally populated if the initial populations are also equal.

In the channeling potential (3) the matrix element for the $\mu_0=\{0,0\}\rightarrow\mu=\{0,1\}$ transition in the resonant case equals

$$M_{\mu_0\mu}\equiv M=\frac{\hbar}{2bm\gamma}\frac{\cos\theta-\beta}{1-\beta\cos\theta}(2s-1)\times\left(\frac{s-1}{2}\right)^{1/2}\frac{\Gamma^2(s-1/2)}{\Gamma^2(s)}, \qquad (11)$$

and in the potential (4) for the transition $\mu_0=\{0,0\}\rightarrow\mu=\{\pm 1,1\}$ we have

$$M_{\mu_0\mu}\equiv M=\frac{\alpha}{\hbar}\left(\frac{3}{32}\right)^{1/2}\frac{\cos\theta-\beta}{1-\beta\cos\theta}. \qquad (12)$$

Assuming that the wave is resonant to the μ_0 and μ states (μ_0 is the ground state)

$$\hbar\omega=\varepsilon_{n,p_{0z}}-\varepsilon_{n_0,p_{0z}-\hbar k_z},$$

and keeping in Eq. (10) only resonant terms we have

$$\frac{\partial a_{\mu,p_{0z}}}{\partial t}+v_z\frac{\partial a_{\mu,p_{0z}}}{\partial z}=-\frac{eA_0^*(x,z,t)}{2\hbar c}M\exp(i\Delta\Omega t)a_{\mu_0,p_{0z}-\hbar k_z},$$
$$\frac{\partial a_{\mu_0,p_{0z}-\hbar k_z}}{\partial t}+v_z\frac{\partial a_{\mu_0,p_{0z}-\hbar k_z}}{\partial z}=\frac{eA_0(x,z,t)}{2\hbar c}qM\exp(-i\Delta\Omega t)a_{\mu,p_{0z}}, \qquad (13)$$

where q is the factor connected with the degeneracy ($q=2$ for axial and $q=1$ for planar channeling). The set of equations (13) should be supplemented by the Maxwell equation, which is reduced to

$$\frac{\partial A_0}{\partial t}+\frac{c^2k_z}{\omega}\frac{\partial A_0}{\partial z}+\frac{c^2k_x}{\omega}\frac{\partial A_0}{\partial x}=\frac{4\pi eN_0cqM}{\omega}\exp(i\Delta\Omega t)a_{\mu_0,p_{0z}-\hbar k_z}a^*_{\mu,p_{0z}}. \quad (14)$$

$N_0$is the average density of the electron beam. Introducing new variables

$$\Delta=N_0[q|a_{\mu,p_{0z}}|^2-|a_{\mu,p_{0z}-\hbar k_z}|^2],$$
$$\Pi=eN_0qM\ \exp(i\Delta\Omega t)a_{\mu_0,p_{0z}-\hbar k_z}a^*_{\mu,p_{0z}}, \quad (15)$$

the self-consistent set of equations in the new variables reads

$$\frac{\partial\Pi}{\partial t}+v_z\frac{\partial\Pi}{\partial z}+i\Delta\Omega\Pi=\frac{e^2qM^2}{2\hbar c}A_0(x,z,t)\Delta,$$

$$\frac{\partial\Delta}{\partial t}+v_z\frac{\partial\Delta}{\partial z}=-\frac{1}{\hbar c}(A_0^*\Pi+A_0\Pi^*),$$

$$\frac{\partial A_0}{\partial t}+\frac{c^2k_z}{\omega}\frac{\partial A_0}{\partial z}+\frac{c^2k_x}{\omega}\frac{\partial A_0}{\partial x}=\frac{4\pi c}{\omega}\Pi. \quad (16)$$

These equations yield the conservation laws for the energy of the system and the norm of the wave function:

$$\frac{\partial|A_0|^2}{\partial t}+\frac{c^2k_z}{\omega}\frac{\partial|A_0|^2}{\partial z}+\frac{c^2k_x}{\omega}\frac{\partial|A_0|^2}{\partial x}=-\frac{4\pi\hbar c^2}{\omega}\left(\frac{\partial\Delta}{\partial t}+v_z\frac{\partial\Delta}{\partial z}\right), \quad (17)$$

$$\left(\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}\right)\left(\Delta^2+\frac{4}{e^2qM^2}\left|\Pi\right|^2\right)=0.$$

Let us consider the stationary regime in the case of exact resonance. The electromagnetic wave is assumed to propagate along the OZ axis to achieve the maximal Doppler shift. Then the set of equations (15) and conservation laws (17) are reduced to

$$\frac{\partial\Pi}{\partial z}=\frac{e^2qM^2}{2\hbar cv_z}A_0\Delta,$$

$$\frac{\partial\Delta}{\partial z}=-\frac{2}{\hbar cv_z}A_0\Pi,$$

$$\frac{\partial A_0}{\partial z}=\frac{4\pi}{\omega}\Pi, \quad (18)$$

$$\Delta^2+\frac{4}{e^2qM^2}|\Pi|^2=N_0^2,$$

$$I=I_0+\frac{\hbar\omega v_z}{2}(\Delta_0-\Delta),$$

where I is the wave intensity and I_0 is the initial one. From Eq. (18) we have the following expressions for Π and Δ:

$$\Delta=N_0\cos\left\{\frac{eMq^{1/2}}{\hbar cv_z}\int_0^z A_0dz+\varphi_0\right\},$$

$$\Pi=\frac{eMq^{1/2}}{2}N_0\sin\left\{\frac{eMq^{1/2}}{\hbar cv_z}\int_0^z A_0dz+\varphi_0\right\}, \quad (19)$$

where φ_0 is determined by boundary conditions. Denoting

$$\varphi=\frac{eMq^{1/2}}{\hbar cv_z}\int_0^z A_0dz+\varphi_0, \quad (20)$$

we arrive at the nonlinear pendulum equation

$$\frac{\partial^2\varphi}{\partial z^2}=æ^2\ \sin\varphi, \quad (21)$$

where

$$æ^2=\frac{2\pi e^2qM^2N_0}{\hbar\omega cv_z}. \quad (22)$$

We will consider two regimes of amplification that are determined by initial conditions. For the first regime the initial average dipole momentum of the channeled electron beam is zero but the population of energy levels in the transverse potential is inverted and it is necessary to have an electromagnetic wave with rather high initial intensities. In this case the following boundary conditions are imposed:

$$\Delta|_{z=0}=N_0,\quad \Pi|_{z=0}=0,\quad I|_{z=0}=I_0. \quad (23)$$

The solution of Eq. (21) in this case reads

$$I(z)=I_0\mathrm{dn}^{-2}\left(\frac{æ}{\kappa}z;\kappa\right), \quad (24)$$

$$\kappa=\left(1+\frac{I_0}{N_0\hbar\omega v_z}\right)^{-1/2},$$

where $\mathrm{dn}(z,\kappa)$ is the elliptic Jacobi function,

As is known $\mathrm{dn}(z,\kappa)$ is the periodic function with the period $2K(\kappa)$, where $K(\kappa)$ is the complete elliptic integral of first order. Therefore at the distances $L=\kappa K(\kappa)/æ$ the wave intensity reaches its maximal value, which equals (the interaction length should be equal to half of spatial period of the wave envelope variation)

$$I=I_0+N_0\hbar\omega v_z. \quad (25)$$

Since we have not taken into account the relaxation processes, this consideration is correct only for the distances $L \leqslant c\tau_{\min}$, where $\tau_{\min}$ is the minimum of all relaxation times.

Let us now consider the other regime of wave amplification when the initial average dipole momentum of transition differs from zero. This regime can operate without any initial intensity of electromagnetic wave ($I_0=0$). We will consider the optimal case with maximal possible initial dipole momentum. So, we take

$$\Pi|_{z=0}=eN_0\sqrt{q}|M|/2, \quad \Delta|_{z=0}=0, \quad I|_{z=0}=0. \qquad (26)$$

Then the wave intensity is expressed by the formula

$$I(z)=\frac{N_0\hbar\omega v_z}{2}\left[1-\mathrm{dn}^{-1}\left(i\text{æ}z;\frac{1}{\sqrt{2}}\right)\right]. \qquad (27)$$

As is seen from Eq. (27) in this case the intensity varies periodically with distances, with the maximal value of intensity

$$I_{\max}=\frac{N_0\hbar\omega v_z}{2}.$$

In conclusion, let us estimate both considered regimes. In the first regime the wave has initial intensity I_0 then the intensity varies periodically with distances. To extract maximal energy from the electron beam the interaction length should be equal to half of the spatial period of the wave envelope variation- $L=\kappa K(\kappa)/\text{æ}$. At this condition the intensity value $I=N_0\hbar\omega v_z$ is achieved. In the opposite case of short interaction length $z\ll L$ the wave gain is small:

$$I(z)=I_0(1+\text{æ}^2z^2).$$

The coherent interaction time of channeled particles with electromagnetic radiation is confined by the lifetime of eigenstates of the channeled particles. As is known [15] for the axial channeling of mildly relativistic electrons $\varepsilon\sim3.5$ MeV in flint crystals, the eigenstate width is order of 1 eV, which corresponds to $z\sim1$ μm. For planar channeling this length is a little large. To fulfill the condition $z\sim L/2$ for axial channeled electron beam $\varepsilon\sim1$ MeV ($\text{æ}^2=0.026\omega_p^2/c^2\gamma^3$, where $\omega_p=\sqrt{4\pi N_0e^2/m}$ is the plasma frequency for electron beam) it needs unrealistically high electron currents $j\sim10^{12}$ A/cm^2. Even for very large electron currents $j\sim10^6$ A/cm^2 we have $\text{æ}^2l^2\sim10^{-6}$ so the wave gain is very small in this case.

The second regime is more interesting. It is the regime of amplification without initial population inversion between the operating resonance levels. The radiation intensity in this regime reaches a significant value even at $\text{æ}^2z^2\ll1$ and according to Eq. (27) is equal to

$$I(z)=\frac{N_0\hbar\omega v_z\text{æ}^2z^2}{8}.$$

Then we have $I(z)\sim1$ kW/cm^2 on the frequency $\hbar\omega\sim1$ keV at $\text{æ}^2z^2\sim10^{-6}$ ($j\sim10^6$ A/cm^2) and the energy of electrons $\varepsilon\sim1$ MeV.

However, the maximal electron current that can be used in this process is strongly restricted because of the effects of damaging the crystal as well as the increasing beam divergence and the strong bremsstrahlung background. Those are the heating of the crystal, space charge effects, and collective excitations. Therefore in the actual experimental situation the generation of x-ray channeling radiation is not feasible even in nonlinear regimes.

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