Time-dependent Floquet theory and absence of an adiabatic limit

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Quantum systems subject to time periodic fields of *finite* amplitude λ have conventionally been handled either by low-order perturbation theory, for λ not too large, or by exact diagonalization within a finite basis of N states. An adiabatic limit, as λ is switched on arbitrarily slowly, has been assumed. But the validity of these procedures seems questionable in view of the fact that, as $N \to \infty$, the quasienergy spectrum becomes dense, and numerical calculations show an increasing number of weakly avoided crossings (related in perturbation theory to high-order resonances). This paper deals with the highly nontrivial behavior of the solutions in this limit. The Floquet states, and the associated quasienergies, become highly irregular functions of the amplitude λ. The mathematical radii of convergence of perturbation theory in λ approach zero. There is no adiabatic limit of the wave functions when λ is turned on arbitrarily slowly. However, the quasienergy becomes independent of $\lambda(t)$ in this limit. We introduce a modification of the adiabatic theorem. We explain why, in spite of the pervasive pathologies of the Floquet states in the limit $N \to \infty$, the conventional approaches are appropriate in almost all physically interesting situations. [S1050-2947(97)08811-2]

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I. INTRODUCTION

Physical systems subject to finite time periodic perturbations of amplitude λ and period $T \equiv 2\pi/\omega$ have been studied extensively [1] by making use of the Floquet theorem. This theorem, a consequence of the discrete time translation symmetry of the Hamiltonian, states that there is a complete set of quasiperiodic solutions of the time-dependent Schrödinger equation that, when $t \to t + T$, are simply multiplied by a phase factor $\exp(-i\epsilon_n T)$, where ϵ_n is called the "quasienergy." This phase factor defines ϵ_n only mod ω and so the quasienergy may always be taken to lie in the strip $0 \leq \epsilon_n < \omega$.

For such systems the quasienergies ϵ_n are of comparable interest to the energy levels of time-independent systems. One would expect to use the subscripts n as unambiguous labels of the time-dependent states of a periodically driven system as the magnitude λ of the perturbation is switched on adiabatically, as in the case of time-independent quantum systems. Indeed, there is a substantial literature studying Floquet systems along these lines, usually employing numerical methods, in a *finite* set of basis states [2]. Of special interest are the "avoided crossings" (AC's): regions in the λ-ϵ plane where two quasienergies approach each other as a function of λ and (except for special symmetries) avoid crossing one another.

But there is a difference in principle when a complete *infinite* set of basis states is included. For a spatially confined system, with an infinite number of discrete energy levels for $\lambda = 0$, there is also an infinite number of quasienergies for $\lambda > 0$ and the spectrum fills the fundamental strip densely. In fact, typically it is a dense point spectrum [3]. (For example, in the special case of a particle in a one-dimensional square well and vanishing λ, Weyl [4] has shown that the energy eigenvalues fill the fundamental strip densely and uniformly.) As the number of basis states becomes infinite there is a weakly avoided crossing (WAC) near each point in the λ-ϵ strip. This leads to qualitatively different issues: Do the Floquet state solutions for a given λ converge to well-defined limits? Are the Floquet states and quasienergies well-behaved functions of λ? Is there a well-defined limiting path that the system follows as λ is switched on arbitrarily slowly—i.e., does an adiabatic limit exist? We have examined these questions in a variety of approximate ways and have arrived at a coherent picture, though generally we do not have conclusive mathematical proofs. We find that the radii of convergence of power-series expansions in λ, starting from the unperturbed eigenstates, are zero. Floquet states and their quasienergies are discontinuous functions of λ everywhere; there is no adiabatic limit in the usual sense (though we will propose a useful weakened modification of the adiabatic theorem). At the same time, in the limit of slow switching on of λ the quasienergy remains arbitrarily close to its initial ($\lambda = 0$) value. We will explain the consistency of these results with the well-established success of standard time-dependent perturbation theory and of the adiabatic theorem.

We note that these features will not be seen directly in any numerical study as N becomes large. The effects become increasingly weak very rapidly as the basis size is increased. Although, for a given interval in λ, inclusion of very-highly-lying levels does have a major impact on Floquet states and quasienergies, it is only over an increasingly smaller range of λ, and this becomes at some stage unobservable on the scale of numerical accuracy available to the computer.

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In Sec. II we review elementary Floquet theory in a *finite* basis. The problems arising for an infinite basis are discussed in Sec. III. In Sec. IV we analyze the convergence of the Floquet states as the size of the basis becomes infinite. Section V is devoted to the dependence of the states and quasienergies on λ, including questions of labeling Floquet states and of the existence of an adiabatic limit when $N \to \infty$. The relationship to finite-order time-dependent perturbation theory is discussed in Sec. VI. Our conclusions are summarized in Sec. VII.

II. FLOQUET THEORY IN A FINITE BASIS

In this section we review the elements of time periodic Hamiltonian systems

$$H(t) = H_0 + \lambda H_1(t) \quad \text{with} \quad H_1(t+T) = H_1(t), \quad (2.1)$$

when approximated by a *finite* matrix $H^N(t)$ in the representation of the N lowest states of the time-independent Hamiltonian H_0. In Sec. III we will discuss the considerable problems that occur in the limit $N \to \infty$.

Because of the discrete time translational symmetry of $H^N(t)$, there is a complete set of solutions $\varphi^N(t)$ of the time-dependent Schrödinger equation (with $\hbar = 1$)

$$i \frac{d\varphi^N}{dt} = H^N(t) \varphi^N, \quad (2.2)$$

which are of the Floquet form

$$\varphi_n^N(t) = \exp(-i\epsilon_n^N t) u_n^N(t), \quad (2.3)$$

with a time periodic part

$$u_n^N(t+T) = u_n^N(t) \quad (n=1,2,\ldots,N) \quad (2.4)$$

and "quasienergies" ϵ_n^N, which may be taken to lie in the interval $[0,\omega)$, with $\omega = 2\pi/T$. For given λ one may label the Floquet states in order of increasing quasienergy, the eigenvalue associated with discrete time translation.

Figure 1 shows such a quasienergy spectrum (for finite N) as a function of λ for the example of a free particle in a one-dimensional box ($x \in [-a,a]$) with harmonic driving,

$$H(t) = p^2/2m + \lambda \, \sin\!\left(\frac{\pi x}{2a}\right)\cos(\omega t). \quad (2.5)$$

The quasienergies are continuous functions of λ that do not cross, but show AC's provided there are no symmetries that allow actual crossings. These AC's are abundant in the spectra of time periodic systems and are of central interest in this work. As the basis size is increased, the newly introduced avoided crossings tend to become rapidly weaker (smaller gaps at the crossing and smaller range of λ over which their effects are substantial). They therefore tend to become isolated from one another, and it is useful and meaningful to consider their effects individually, as we do in the following.

As two quasienergy lines pass an isolated WAC at $\lambda = \lambda_1$, the corresponding Floquet states rapidly interchange their forms. At the point $\lambda = \lambda_1$ they are very nearly linear combinations of the two Floquet functions just outside the region of the AC, with amplitudes of equal magnitude. As
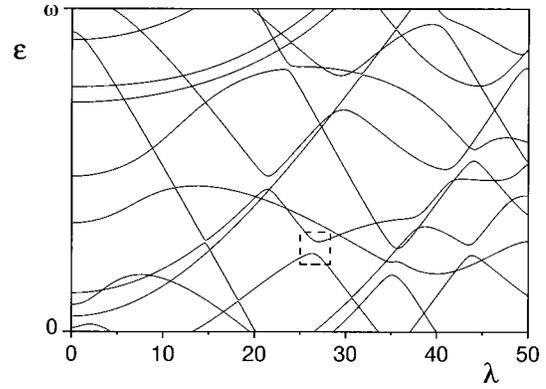

FIG. 1. Quasienergy spectrum as a function of λ for $N=10$ for the free particle in a box with harmonic driving and frequency $\omega = 8.3$ [see Eq. (2.5)]. One finds many avoided crossings, with a typical one marked by the dashed box. There are some strictly *real* crossings, corresponding to states of opposite parity under the combined symmetry operation of spatial inversion plus time translation by half a period $T/2$. Other apparently real crossings are just so weakly avoided that they cannot be resolved.

$|\lambda - \lambda_1|$ grows, this mixing decreases rapidly, as follows. The functions are mixed by no more than a given small relative weight δ, provided that $|\lambda - \lambda_1| > w(\delta)/2$, where

$$w(\delta) = \frac{\Delta\epsilon}{\delta|\sigma_1 - \sigma_2|}. \quad (2.6)$$

Here $\Delta\epsilon$ is the quasienergy splitting at the AC and σ_1 and σ_2 are the slopes $d\epsilon/d\lambda$ of the quasienergies at the crossing point in the absence of the terms connecting these states (see Fig. 1). We neglect the weak influence of all other states near this WAC.

Using standard Floquet state perturbation theory [6] in λ, one finds that the second-order term describes very well the overall behavior of the quasienergies for small λ. However, near $\lambda = \lambda_1$ nearly degenerate perturbation theory is needed, giving

$$\epsilon_{1,2}(\lambda) \approx \frac{1}{2}[(\sigma_1 + \sigma_2)(\lambda - \lambda_1)$$
$$\pm \sqrt{(\sigma_1 - \sigma_2)^2(\lambda - \lambda_1)^2 + (\Delta\epsilon)^2}]. \quad (2.7)$$

It is helpful to consider *complex* values of λ. Then the time evolution is no longer unitary, and quasienergies are complex. They may be considered as the N values of a single N-valued analytic function $\epsilon(\lambda)$, with N Riemann sheets connected at complex branch points [7]. As is the case for real λ, the function $\epsilon(\lambda)$ is defined only mod ω and we choose always $0 \leq \text{Re}\,\epsilon(\lambda) < \omega$. From Eq. (2.7) we see that WAC's for real λ are manifested as branch points near the real axis of $\epsilon(\lambda)$, at

$$\lambda = \lambda_1 \pm \frac{\Delta\epsilon}{|\sigma_1 - \sigma_2|} i \quad (2.8)$$

(see Fig. 2). When λ passes along the real λ axis through λ_1, as discussed above, there is a rapid change in the spatial part
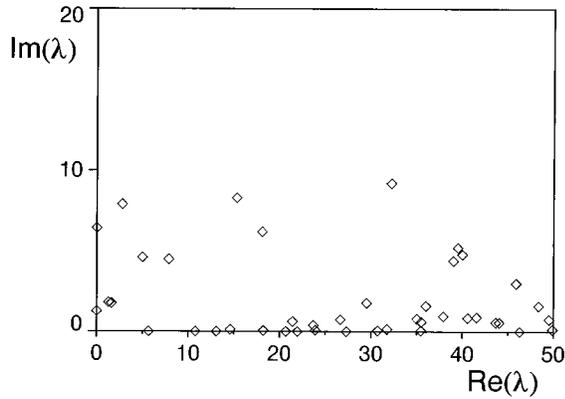
FIG. 2. Position of branch points in the complex λ plane for the system and parameters of Fig. 1. Just one quadrant is shown, as the position of branch points is symmetric with respect to the axes $\text{Re}(\lambda)=0$ and $\text{Im}(\lambda)=0$, since $\epsilon(\lambda^*)=\epsilon^*(\lambda)$ and $\epsilon(-\lambda)=\epsilon(\lambda)$. There are no branch points on the real axis, but they do appear on the imaginary axis, as shown.

of the two Floquet states, which are approximately interchanged. In contrast, on a path starting from the real axis, going in a loop in the complex plane around the branch point (2.8) and back to the real axis, each of the two Floquet states returns approximately to its original spatial dependence. Perturbation expansions in λ have finite radii of convergence. At $\lambda=0$ the eigenstate φ_j with energy E_k defines the quasienergy $\epsilon_j=E_k \pmod{\omega}$ on the jth Riemann sheet of $\epsilon(\lambda)$ (the indices k and j are unequal, in general, because j labels increasing values of ϵ in $[0,\omega)$, whereas k labels increasing energy values E over $[0,\infty)$). Then the radius of convergence $\lambda_{c,j}$ for φ_j and ϵ_j is the magnitude of λ at the branch point nearest the origin on that jth sheet.

In the laboratory the perturbation λH_1 is commonly turned on slowly. This can be characterized, as usual, by a switching factor e^{st} in the interaction ($s>0$)

$$H(t)=H_0+\lambda e^{st}H_1(t) \quad \text{where} \quad H_1(t+T)=H_1(t),$$
$$(2.9)$$

with the initial condition

$$\psi_j^{s,\lambda}(-\infty)=\varphi_j. \qquad (2.10)$$

We are interested in the wave function $\psi_j^{s,\lambda}(t)$ at a specified time, say $t=0$. This system is no longer periodic in time and the solutions of the time-dependent Schrödinger equation corresponding to Eqs. (2.9) and (2.10) are therefore no longer Floquet functions. However, if the turn-on rate s is slow compared to the driving frequency ω, it is useful to describe the solutions at time t in the basis of the Floquet functions (2.3) at the corresponding value of $\lambda(t)=\lambda e^{st}$. Two results are known [8]:

(i) Every AC can be characterized by a rate

$$\widetilde{\xi}=\frac{(\Delta\epsilon)^2}{|\sigma_1-\sigma_2|}\equiv\xi\lambda_1, \qquad (2.11)$$

where we have introduced the explicit factor λ_1 in the final definition for convenience below. The behavior of the wave

function as the AC is crossed then depends on how this compares with the rate of growth of the exponentially increasing amplitude $(1/\lambda_1)d\lambda/dt=s$. For large turn-on parameters $s\gg\xi$ (provided that $s\ll\omega$) the solution will follow closely the initial Floquet state, as if there were no AC. This is the so-called [9] Landau-Zener transition through the AC. On the other hand, in the adiabatic limit for this AC, $s\ll\xi$, it will closely follow the Floquet state that passes the AC continuously. For $s\approx\xi$ the solution, after $\lambda(t)$ has passed λ_1, will be a superposition of the two Floquet states involved in the AC.

(ii) There is an adiabatic theorem: Apart from an overall phase factor, the final state $\psi_j^{s,\lambda}(0)$ converges in the limit $s\to0$ and the limit is the Floquet state of the periodic system that corresponds to the quasienergy found by following the quasienergy ϵ_j of the initial state φ_j as a *continuous* function of λ.

These properties are well established for Floquet systems with a finite basis. In the next section we show that the limit $N\to\infty$ is highly pathological.

III. THE LIMIT $N\to\infty$

The general difficulties raised by an infinite basis are outlined here. A detailed discussion is given in Secs. IV and V.

For definiteness we will restrict the discussion to finite one-dimensional systems with $x\in[-a,a]$ and

$$H(t)=p^2/2m+U(x)+\lambda V(x)\cos(\omega t), \qquad (3.1)$$

where $U(x)$ and $V(x)$ are analytic and bounded. $\text{Re}\lambda$ will be kept uniformly bounded, $\text{Re}\lambda\leq\overline{\lambda}_1$, as N becomes infinite. For $\lambda=0$ the conventional energy spectrum E_k is discrete and for sufficiently high eigenvalues exhibits increasing spacings [10] between successive levels [approximately proportional to the level index, as for the case $U(x)\equiv0$]. We will make repeated essential use of this feature.

In the limit $N\to\infty$ one expects that for every λ the N quasienergies within the finite interval $[0,\omega)$ will form a dense spectrum. In fact, Weyl showed [4] that for $U(x)=\lambda=0$, so that $E_k=(1/2m)(\pi/2a)^2k^2$, the spectrum is uniformly dense, provided only that ω/E_1 is irrational. This is the generic situation [5] for $U(x)$, $\lambda\neq0$. Moreover, Howland [3] has shown that, for Hamiltonians of the form (3.1) (an essential feature of which is the increasing spacing between successive energy levels for sufficiently high eigenvalues) and for most values of the coupling strength λ, the quasienergies have a dense point spectrum.

What happens to the AC's in the limit $N\to\infty$? We find a very simple picture in the complex λ plane: AC's correspond to branch points, as in Fig. 2. With increasing N we add higher-lying states of the unperturbed Hamiltonian to the basis and find that the gaps of newly introduced AC's, as well as the imaginary part of the corresponding branch points (see the Appendix), decrease faster than any power law with N. Therefore, the real λ axis is a line of accumulation for the branch points. Moreover, the branch points with $\text{Im}\lambda>\overline{\lambda}_2$ are finite in number and tend to well-defined limits as $N\to\infty$, for any $\overline{\lambda}_2>0$. The problems with the limit $N\to\infty$ are restricted to the immediate neighborhood of the real λ axis.

For real λ one expects the AC's to be dense in the λ-ϵ plane. In fact, along each quasienergy curve $\epsilon(\lambda)$ obtained within a finite basis approximation one expects a dense set of AC's when $N \to \infty$. This has a number of consequences:

(i) As there will be AC's on each quasienergy curve for arbitrarily small values of λ, the radii of convergence $\lambda_{c,j}$ of the perturbation theory in λ for quasienergies and Floquet states all shrink to zero.

(ii) The increase of the basis size from an initial value N will introduce AC's arbitrarily close to any point on a given quasienergy curve defined with the initial basis of size N. Each of these AC's will make large changes in the corresponding Floquet states over a finite λ interval, strongly admixing and interchanging pairs of states, as described above, and it is therefore by no means clear whether or not Floquet states converge as $N \to \infty$.

(iii) For any *finite* N one can label the Floquet states $\varphi_n^{N,\lambda}$ and quasienergies $\epsilon_n^{N,\lambda}$ in such a way that they are continuous functions of λ (if, as usual, we identify ϵ with $\epsilon + \omega$). If we increase N, we have to rearrange labels for every new AC and it is not clear if there exists a meaningful labeling that tends to a well-defined limit as $N \to \infty$.

(iv) Let the periodic field be switched on over the time interval $-\infty < t \leq 0$, with $\lambda(t) \equiv e^{st}\lambda$. For any *finite* N there is a well-defined adiabatic limit, as $s \to 0$, of the state (up to an overall phase factor) and of the quasienergy at the final time $t = 0$. Let $\xi_{\min}$ be the smallest of the rate parameters ξ, defined by Eq. (2.11), characterizing the relevant AC's in $[0,\lambda]$. Then, for $s \ll \xi_{\min}$ the solution of the Schrödinger equation (2.9) will simply follow the Floquet state corresponding to a continuous quasienergy curve. In the limit $N \to \infty$, however, there will be AC's with arbitrarily small parameters ξ. Therefore, for smaller s more and more of these weak AC's will be passed adiabatically, rather than undergo Landau-Zener transitions, leading to completely different final states $\psi_k^{s,\lambda}(t)$. Thus an adiabatic limit as $s \to 0$ cannot be expected.

IV. CONVERGENCE AND NONCONVERGENCE OF FLOQUET STATES

For systems of type (3.1) we shall present strong arguments that, for a set of full measure in λ in an N-independent interval $0 \leq \lambda \leq \bar{\lambda}$, the Floquet states converge in the limit $N \to \infty$, even though the AC's are dense in the λ-ϵ plane. This does not imply, however, that the limit is a continuous function of λ. For we will also argue that the AC's give rise to infinitely many λ's, of measure zero but dense in any λ interval, for which *none* of the Floquet states of a finite basis converge. This can be stated more precisely as follows.

Proposition I (convergence). For any interval $[\lambda_a, \lambda_b]$, any $\delta > 0$, and any $\eta > 0$, there is an integer $M(\delta, \eta, \lambda_a, \lambda_b)$ with the following properties: For any $M' > M$ and each $m = 1, 2, \ldots, M$ there exist a label $m' \in \{1, 2, \ldots, M'\}$ and an overall phase factor $e^{i\alpha}$ such that

$$|\varphi_m^{M,\lambda}(x,t) - e^{i\alpha}\varphi_{m'}^{M',\lambda}(x,t)| < \delta \qquad (4.1)$$

and

$$|\exp[-i(\epsilon_m^{M,\lambda} - \epsilon_{m'}^{M',\lambda})T] - 1| < 2\delta \qquad (4.2)$$

for all x, for all $t \in [0,T]$, and for all $\lambda \in [\lambda_a, \lambda_b]$, except for a subset of λ of measure $\leq \eta$.

Proposition II (nonconvergence). For any interval $[\lambda_a, \lambda_b]$, any $\delta > 0$, any M, any label $m \in \{1, 2, \ldots, M\}$, sufficiently large $M' > M$, and any label $m' \in \{1, 2, \ldots, M'\}$,

$$\frac{1}{2a}\left|\int_{-a}^{a} dx \; \varphi_m^{M,\lambda}(x,t)^* \varphi_{m'}^{M',\lambda}(x,t)\right| < \delta \qquad (4.3)$$

for a set of λ's dense in $[\lambda_a, \lambda_b]$ and all t. That is, for this dense set of λ's the eigenfunctions within the smaller basis have arbitrarily small overlap with any of those corresponding to the larger basis; they do not converge to a limit.

Proposition I states that as N grows, there is an increasing measure of λ where a finite basis calculation gives the Floquet states and quasienergies correctly within an arbitrarily small error. For many practical purposes this supports the use of a finite basis for describing a Floquet system. Nevertheless, it is important to realize, as stated in proposition II, that even for an arbitrarily large basis size N there are infinitely many λ's, dense in any interval (albeit of total measure zero), where a given Floquet state $\varphi_n^N(t)$ does *not* converge in the limit $N \to \infty$.

We can prove proposition I for any system with the following model property, which, we submit, captures for this purpose the essence of a real Floquet system of type (3.1): For an AC at $\lambda_n \in [\lambda_a, \lambda_b]$ we again define the interval $\lambda_n \pm w_n(\delta_n)/2$, outside of which the admixture of the two unperturbed Floquet states changes by no more than δ_n. Then we assume that we can choose a set $\{\delta_n\}$, with $\sum_{n=1}^{\infty}\delta_n < \delta$ for any chosen $\delta > 0$, such that $\sum_{n=1}^{\infty}w_n(\delta_n)$ converges. This assumption appears to be satisfied in systems of type (3.1), although we have no mathematical proof. Numerical calculations and analytical considerations (see the Appendix) suggest strongly that the gaps $\Delta\epsilon_n$ decrease faster than any negative power of n, due to the increasing spacings of the unperturbed energies E_j. Let us choose δ_n to decrease relatively slowly, as a small power of n, say $\delta_n = \delta/2n^2$. Then the sum over w_n will converge [see Eq. (2.6)] as long as the difference $|\sigma_1 - \sigma_2|$ in quasienergy slopes does not decrease too rapidly with n. Numerical experience suggests this to be the case.

We make an argument based on the Borel-Cantelli lemma [11]. Let us start, as usual, with an approximation to the system given by restriction to a finite number M of spatial basis functions. The new AC's introduced as this is increased to a complete, infinite, basis set are labeled from n_M to ∞. For a large enough choice of the initial basis size M, the partial sum $\sum_{n=n_M}^{\infty}w_n(\delta_n)$, which gives the measure of λ's where a Floquet state might be affected by more than δ when increasing the basis size from M to infinity, can be made smaller than any given η. For all other values of λ, those that are within none of the intervals $\lambda_n \pm w_n(\delta_n)/2$ and therefore constitute a set of measure at least $\lambda_b - \lambda_a - \eta$, the Floquet states are changed by no more than $\sum_{n=1}^{\infty}\delta_n < \delta$. This explains Eq. (4.1) of proposition I.

Equation (4.2) follows at once from Eq. (4.1). The quasienergies can be determined from $e^{i\epsilon_n^M T} = \varphi_n^M(x, t+T)/\varphi_n^M(x,t)$. For Floquet states normalized by $\int_{-a}^{a}dx|\varphi_n^M(x,t)|^2/2a = 1$, we can choose an x and t

where $|\varphi_n^M(x,t)| \gtrsim 1$. Then $\epsilon_n^{M'}$ can be determined from $\varphi_n^{M'}(x,t)$, which differs from $\varphi_n^M(x,t)$ by less than δ [Eq. (4.1)], leading to Eq. (4.2).

Proposition II follows from the plausible, but unproved, assumption that any quasienergy line of a finite basis (M) calculation will show an AC within any given λ interval if the basis size is increased sufficiently. (Failure of this assumption would imply that there do exist finite λ intervals in which a quasienergy line is *never* crossed as $M \to \infty$.) Using that assumption, we can argue straightforwardly: Within any given interval $I = [\lambda', \lambda''] \subset [\lambda_a, \lambda_b]$, for a suitably large basis size M one will eventually find an AC. This crossing changes the n_1th Floquet state $\varphi_{n_1}^M$ by more than some chosen amount, say 40% admixture of orthogonal basis states, over some finite λ interval. Within that interval the two Floquet states of the AC then have an overlap of less than 0.6 with $\varphi_{n_1}^M$. We now select one of these states. With a further increase in the basis size, it will ultimately encounter an AC within the chosen interval and will be changed by more than 40% over some smaller but still finite λ interval. Within that interval for an even larger basis size also the second state will be changed by more than 40% due to an AC. This leads to four Floquet states within a finite λ interval, each with an overlap of less than $(0.6)^2$ with $\varphi_{n_1}^M$. Repeating this argument sufficiently often, one finds a basis size $M_1 > M$ and an interval $I_{n_1} \subset I$ where no Floquet state $\varphi_n^{M_1}$ has an overlap of more than a given δ with $\varphi_{n_1}^M$ of the initial basis size M, leading to proposition II. As stated in proposition I, these λ's, even though they are dense, have a measure that tends to 0 as $M \to \infty$.

We can derive a stronger version of proposition II. Repeating the argument above for the n_2th Floquet state $\varphi_{n_2}^M$ of the initial basis size M, one finds a basis size $M_2 > M_1 > M$ and an interval $I_{n_2} \subset I_{n_1} \subset I$ where *no* Floquet state $\varphi_n^{M_2}$ has an overlap of more than a given δ with $\varphi_{n_2}^M$ nor with $\varphi_{n_1}^M$. This argument can be repeated for all Floquet states φ_n^M of the initial basis. Thus there is a dense set of λ's in any λ interval where *none* of the Floquet states of a finite basis converge [12].

V. FUNCTIONAL DEPENDENCE ON PERTURBATION STRENGTH: ADIABATIC LIMIT

A. Labeling of Floquet states

Proposition II of the preceding section has an immediate consequence: A continuous labeling of the Floquet states $\varphi_n^{N,\lambda}$ and quasienergies $\epsilon_n^{N,\lambda}$ as a function of λ, possible for finite N, is no longer possible in the limit $N \to \infty$. We therefore propose a different way of labeling, useful at least for small λ: We assign the label n to a Floquet state if its overlap with the nth eigenstate $\varphi_n(x,t)$ of H_0 is larger than 50%, in the sense

$$\frac{1}{T}\int_0^T dt \, \frac{1}{2a}\int_{-a}^a dx |\varphi_n(x,t)^* \varphi_n^\lambda(x,t)| > 0.5. \quad (5.1)$$

This procedure will not always find a label for a Floquet state. This is obvious for large λ, where none of the Floquet

states resembles a low-lying unperturbed state. But, even for small λ, in the center region of an AC the overlap with an unperturbed state will be less than 50%.

Since for small λ the quasienergies are very flat as a function of λ and AC's will generically occur only between Floquet states related to states of H_0 that are far apart in energy, their AC's are expected to have a width that decreases faster than any power law as λ goes to zero (Appendix). We therefore expect that, on the interval $[0, \bar{\lambda}]$, the labeling works for a Cantor set of λ values with finite measure less than $\bar{\lambda}$ and that this measure approaches $\bar{\lambda}$ as $\bar{\lambda} \to 0$.

B. Adiabatic turn-on

Even though perturbation theory for Floquet states and quasienergies in λ has zero radius of convergence in the limit $N \to \infty$, we find a simple, strict result for the perturbation expansion of a solution $\psi_k^{s,\lambda}(t)$ of the Schrödinger equation (2.9), where the periodic driving is turned on from $t = -\infty$ by the factor λe^{st} [we remark again that $\psi_k^{s,\lambda}(t)$ is *not* a Floquet state].

Theorem. The perturbation expansion of $\psi_k^{s,\lambda}(t)$ in λ, for $t < \infty$, has an infinite radius of convergence for any $s > 0$, i.e., $\psi_k^{s,\lambda}(t)$ is an entire function of the complex variable λ.

This can be proven by majorizing the perturbation expansion. With the wave function $\chi(t)$ expressed in the interaction picture

$$\chi(t) = e^{iH_0 t}\psi_k^{s,\lambda}(t), \quad (5.2)$$

the Schrödinger equation (2.2) becomes $id\chi(t)/dt = \lambda W(t)\chi(t)$, with $W(t) = e^{iH_0 t}e^{st}V(x)\cos(\omega t)e^{-iH_0 t}$. The familiar formally iterated solution is

$$\chi(t) = \left[1 + (-i\lambda)\int_{-\infty}^t dt_1 W(t_1) \right.$$
$$+ (-i\lambda)^2 \int_{-\infty}^t dt_2 \int_{-\infty}^{t_2} dt_1 W(t_2)W(t_1)$$
$$+ (-i\lambda)^3 \int_{-\infty}^t dt_3 \int_{-\infty}^{t_3} dt_2 \int_{-\infty}^{t_2} dt_1 W(t_3)$$
$$\left. \times W(t_2)W(t_1) + \cdots \right] \chi(t = -\infty). \quad (5.3)$$

Since $e^{-iH_0 t}$ is unitary, it can easily be shown that for any normalized states f and g

$$\langle f|W(t_n)W(t_{n-1})\cdots W(t_2)W(t_1)|g\rangle \lesssim V_{\max}^n e^{s(t_n + \cdots + t_1)}, \quad (5.4)$$

where $V_{\max}$ is the maximum of $|V(x)|$. From this the nth-order term of the perturbation expansion of $\langle f|\chi(t)\rangle$ can be majorized by $(1/n!)(\lambda V_{\max}/s)^n e^{st}$ and thus the expansion (5.3) converges for any $s > 0$. That is, the state that evolves with a given switching-on rate s is uniquely and well defined (in a finite *or* an infinite basis), for an arbitrarily large final interaction strength, in spite of the convergence problems with Floquet states. There *is*, however, an anomaly: There is no well-defined adiabatic limit when $s \to 0$ (see also Sec. III).

Although an adiabatic limit in the usual sense does not exist, we now show that for sufficiently small λ there is a (logarithmically) large window of turn-on parameters $\underline{s} < s < \overline{s}$, where the final state $\psi_k^{s,\lambda}(t)$ is *almost* independent of s.

Proposition III. For any small $\delta > 0$ and large $\eta > 0$ there exists a $\overline{\lambda} > 0$ and an interval $[\underline{s}, \overline{s}]$ with

$$\overline{s}/\underline{s} > \eta \tag{5.5}$$

such that for all $s', s'' \in [\underline{s}, \overline{s}]$, all $\lambda < \overline{\lambda}$, all x, and all states k,

$$\frac{|\psi_k^{s',\lambda}(x,t=0) - e^{i\alpha(s',s'')}\psi_k^{s'',\lambda}(x,t=0)|}{\lambda} < \delta, \tag{5.6}$$

where $e^{i\alpha}$ is a (physically uninteresting) overall phase factor.

That is, for any desired level of convergence (as defined by δ) we can find a range of s of arbitrarily large relative size (arbitrarily large $\overline{s}/\underline{s} = \eta$) by restricting λ to a sufficiently small value.

Consider any finite N and the Floquet state arising out of the unperturbed state φ_k. The perturbation series (5.3) involves first-order terms of the form

$$\chi^{s,(1)} \equiv \lambda \sum_{k'} \frac{V_{k',k}}{E_{k'} - E_k \pm \omega - is} \varphi_{k'}. \tag{5.7}$$

From this it is clear that to achieve independence of s with an accuracy δ, s must be smaller than a value $\overline{s}$ given by

$$\overline{s}\lambda \sum_{k'} \left| \frac{V_{k',k}}{(E_{k'} - E_k \pm \omega)^2} \varphi_{k'} \right| < \delta. \tag{5.8}$$

For small enough λ all higher-order terms up to any finite order n may be neglected (assuming, as we do, that there is no exact resonance $E_{k'} - E_k = \pm m\omega$, $m \leq n$).

However, for any given λ_0, no matter how small, there will exist a sufficiently large basis size such that there are some (weak) AC's associated with *any* quasienergy in the λ interval $[0, \lambda_0]$. Consider for any initial basis size N_0 an arbitrary quasienergy curve $\epsilon_k(\lambda)$, continuous over the interval $0 \leq \lambda \leq \lambda_0$. As the basis size is then increased to a sufficiently large size $N' > N_0$, avoided crossings of $\epsilon_k(\lambda)$ will be introduced within this interval. These AC's will be characterized by rates [see Eq. (2.11)] that we label $\xi_1, \xi_2, \ldots$, in order of increasing energy of the states at $\lambda = 0$ from which the crossing curves arise, which then ensures that they are ordered with decreasing rates: $\xi_{i+1} < \xi_i$. As $N \to \infty$ and $\lambda(t) (\equiv \lambda e^{st})$ grows from 0 to λ all these (infinitely many) weak AC's will be encountered and each of them gives rise to an admixture of a new state into $\psi_k^{s,\lambda}(x,t=0)$ with amplitude (see [9]) smaller than

$$\delta_n = \sqrt{\pi \xi_n / s}. \tag{5.9}$$

To make the total variation in $\psi_k^{s,\lambda}(x,t=0)$, as defined by $\Sigma_n \delta_n$, smaller than a given δ over the whole range of turn-on rates set by Eq. (5.5), we use the following properties of the ξ's in the limit $\lambda \to 0$:

$$\lim_{\lambda \to 0} \xi_1 = 0, \tag{5.10}$$

$$\lim_{\lambda \to 0} \xi_{l+1}/\xi_l = 0. \tag{5.11}$$

Equation (5.10) follows from the fact that for small enough λ the largest AC of ϵ_k^λ in $[0, \lambda]$ will be with an arbitrarily high-lying state k' of H_0 and that the rate ξ_1 characterizing the AC, according to Eq. (2.11) and the Appendix, decreases faster with k' than any power law. Equation (5.11) is due to the fact that this ratio of successive rates is, in the most unfavorable case, due to AC's of neighbouring levels $k' + 1$ and k' of H_0 with ϵ_k^λ. The ratio of the corresponding quasienergy splittings decreases exponentially with $(k'+1)^2 - k'^2 = 2k' + 1$ and goes to zero in the limit $\lambda \to 0$.

From these properties we conclude that for s larger than any $\underline{s} < \overline{s}/\eta$ the variation in $\psi_k^{s,\lambda}(x,t=0)$ due to infinitely many AC's can be made smaller than any given δ for sufficiently small $\overline{\lambda}$. Therefore, for small enough $\overline{\lambda}$ one finds for all $\lambda \leq \overline{\lambda}$ almost adiabatic behavior in a window $[\underline{s}, \overline{s}]$ of turn-on parameters with $\overline{s}/\underline{s}$ arbitrarily large.

C. Conservation of quasienergy

Here we consider again the limit of turning on $\lambda(t)$ arbitrarily slowly from 0 up to some arbitrary λ. We have seen that the *eigenfunction* $\psi_k^{s,\lambda}(0)$ does not have a limit for $s \to 0$. However, we will argue here (but not prove mathematically) that, as $s \to 0$, $\psi_k^{s,\lambda}(0)$ is within arbitrary accuracy a linear combination of Floquet states of the periodic Hamiltonian [Eq. (2.1)] that have *quasienergies* arbitrarily close to the *initial* value $\epsilon_k(\lambda = 0)$, i.e., the *energy* mod ω. Thus, in the limit $s \to 0$ the quasienergy is a conserved quantity.

Proposition IV. For any $\delta > 0$ and any $\lambda > 0$ there exists an $s_0 > 0$ such that for all $s < s_0$

$$\int_{-a}^{a} dx |\psi_k^{s,\lambda}(x,t) - e^{i\epsilon_k T}\psi_k^{s,\lambda}(x,t+T)|^2 < \delta \tag{5.12}$$

for all $t \leq 0$ and for all states k. This is true even though $\lim_{s \to 0} \psi_k^{s,\lambda}(x,t)$ does not exist for any finite time $-\infty < t \leq 0$.

To make this plausible we will use a simplified geometrical picture, in which quasienergies are linear functions of λ. Although all AC's are now represented by actual crossings, they will be traversed dynamically like real AC's with some finite rate parameters ξ. The infinitely many quasienergy lines fall into two classes, according to whether the magnitudes of their slopes are smaller or larger than a specified critical value $\sigma_c = \delta'/2\lambda$. We will further assume in this model that the latter (large slope) class, for any $\delta' > 0$ and any $\lambda > 0$, has only a finite number $K(\delta', \lambda)$ of members.

In the course of turning on the periodic driving from 0 up to λ with a small enough turn-on parameter s, the infinitely many quasienergy lines with slopes $\leq \sigma_c$ will, at most, cause a deviation in quasienergy by $\sigma_c \lambda (= \delta'/2)$ away from the initial value $\epsilon_k(\lambda = 0)$. The additional changes in quasienergy from AC's with the finite number $K(\delta', \lambda)$ of steeper quasienergy lines can also be restricted to be less than $\delta'/2$ by a sufficiently small choice of turn-on rate s: Since the AC's are dense on any quasienergy line, for sufficiently

small s one will be diverted from any of the steep quasienergy lines within any given small quasienergy range, which we choose to be $\delta'/2K$. The total change in quasienergy as λ is increased from 0 to its final value thus can be made smaller than $\delta'/2 + K\delta'/2K = \delta'$ for any given δ'. For $\delta' = \sqrt{\delta}/T$ and within this simplified model we have thus proved proposition IV.

The same arguments should hold for the original Floquet problem, as including the quasienergy dependence on λ at the AC's reduces their steepness and thus further reduces the spread in quasienergy from the above estimates. Also, the overall nonlinear dependence on λ that follows, e.g., from second-order perturbation theory poses no problem for the argument. The assumption that the number $K(\delta',\lambda)$ of slopes larger in magnitude than σ_c is finite remains reasonable since an increase in the basis size introduces Floquet states originating from higher-lying states of H_0, which show a decreasing dependence on λ. However, the problem of defining slopes at all (in the limit $N \to \infty$), as there is no continuous labeling of the quasienergies as a function of λ, will make a mathematical proof difficult.

This proposition leads us to a better understanding of the nature of the state $\psi_k^{s,\lambda}(0)$ in the limit $s \to 0$. As $\psi_k^{s,\lambda}(0)$ is a linear combination of Floquet states with quasienergies closer and closer to ϵ_k, it changes constantly as δ and s go to zero. Thus, while there cannot be an adiabatic limit for the wave function, quasienergy is conserved in the limit $s \to 0$.

VI. STATUS OF TRADITIONAL FINITE-ORDER PERTURBATION THEORY

Nonlinear optics is a major field of science in which traditional, finite-order perturbation theory in the applied electric field (usually to low order) successfully describes experiment. Here we shall show why this well-established theory is consistent with our considerations in spite of our conclusion that, strictly speaking, the radii of convergence $\lambda_{c,j}$ of perturbation theory vanish.

We reiterate first that if the turn-on rate $s \to 0$, perturbative nonlinear optics in fact fails. For small λ this failure is due to near resonances $E_{k'} - E_k \approx \pm n\omega$, generally with very highlying excited states. (This is the reason why, for any finite basis size N, λ_c is finite.)

When the perturbation is turned on as in Eq. (2.9), we have seen that, provided the turn-on rate s is small enough but exceeds a lower limit $\underline{s}$, then as $\lambda \to 0$ the resultant state can be made arbitrarily close to the traditional first-order perturbative solution. A similar result can be derived for the traditional perturbative solution up to any finite order. In typical laboratory situations we have seen that $\underline{s}$ is exponentially and unphysically small. For finite small λ the perturbation expansion is asymptotically convergent.

Finally, we briefly mention the unavoidable effects of line broadening. The quantum system of interest is inevitably in contact with its environment and there are interactions between the many particles that ordinarily constitute the quantum system of interest, so the individual particle states are lifetime broadened. We conjecture that if broadening is characterized by a finite width Γ, then a finite radius of convergence will be restored.

VII. CONCLUSION

There have been two standard approaches to dealing with the behavior of quantum systems subject to strong time periodic fields. One is the use of finite-order perturbation theory (e.g., second- or third-order nonlinear optical susceptibilities) and the other the exact solution of the problem within a finite basis of states. But both of these approaches miss qualitative features of the exact mathematical solutions.

We have shown by a set of ''propositions'' (as opposed to rigorous mathematical proofs) that for a large class of time periodic problems the structure of the exact states and the quasienergy spectrum is remarkably irregular. By ''exact'' we mean here that the complete infinite set of basis states is included. Interaction with the environment is neglected. We have considered the states and quasienergies as functions of the strength λ of the time periodic potential, as the number N of basis states becomes infinite. We have found that in any interval $\lambda_a < \lambda < \lambda_b$, although the states converge to a well-defined limit as $N \to \infty$ for a set of λ with the full measure $\lambda_b - \lambda_a$ of the interval, there is a set of λ, of total measure zero, but *dense* within every finite interval, for which the states do *not* converge. As a result, in contrast to the situation for any finite N, it is impossible to label states and quasienergies continuously as a function of λ. The familiar quasienergy ''dispersion'' curves as functions of λ (as shown, e.g., in Fig. 1) become discontinuous everywhere. One consequence of these discontinuities is the absence of a true adiabatic limit; there is no unique final state to which the system tends as the periodic perturbation is switched on arbitrarily slowly.

But these pathologies, including a radius of convergence of perturbation theory in λ that approaches zero as $N \to \infty$, do not show up under most physically realistic circumstances. In particular, we have explained the familiar and well-established success of ordinary time-dependent perturbation theory in terms of the modified adiabatic theorem and the typical smallness of the parameter s that enters that theorem in practice, as well as the successes of finite basis calculations.

APPENDIX: EXPONENTIAL DECREASE WITH BASIS SIZE N OF NEWLY INTRODUCED QUASIENERGY GAPS

We label the eigenstates of the time-independent Hamiltonian ($\lambda = 0$) by an index j that increases with the unperturbed energy. We consider the solution of the full time-dependent problem in the limited spatial basis of the first N such states. We demonstrate here that for sufficiently small λ, the new AC's introduced by the inclusion of the next basis state (labeled $N+1$) are characterized by quasienergy gaps

that are smaller than a bound that decreases exponentially with N.

As in Sec. II, let us take as the ''Nth-level'' Hamiltonian the representation of $H(t)$ in the basis of the first N unperturbed states:

$$H^N(t) \equiv H_0 + \sum_{j,l=1}^{N} |l\rangle\langle l|V(t)|j\rangle\langle j| \equiv H_0 + V^N(t), \quad (A1)$$

where the unperturbed Hamiltonian is

$$H_0 = \sum_{j=1}^{\infty} E_j^0 |j\rangle\langle j|. \quad (A2)$$

Since H^N has the same time periodicity as the full Hamiltonian, the corresponding time-dependent Schrödinger equation has solutions of the standard Floquet form (2.3),

$$\psi_k^N(x,t) = \exp(-i\epsilon_k^N t)u_k^N(x,t), \quad (A3)$$

with $k=1,2,\ldots,N$, where the functions $u_k^N(x,t)$ are time periodic. Therefore, the solutions formed from the basis of the first N states are of the form

$$|u_k^N\rangle = \sum_{j=1}^{N}\sum_{n=-\infty}^{\infty} a_k^N(j;n)e^{-in\omega t}|j\rangle, \quad (A4)$$

where normalization imposes the restriction

$$\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty} |a_k^N(j;n)|^2 = 1. \quad (A5)$$

Now we include in Eq. (A1) the next highest state $|N+1\rangle$, whose unperturbed energy E_{N+1}^0 can be written as

$$E_{N+1}^0 = \epsilon + M_N\omega, \quad (A6)$$

with ϵ confined to the fundamental stripe, $0 \le \epsilon < \omega$. At some value of the coupling λ the quasienergy of this state ϵ may equal that of one of the solutions labeled k ($1 \le k \le N$) in the basis of the first N states [see Eqs. (A3) and (A4)]. The perturbation of the remaining potential $V - V^N$ turns that into an *avoided* crossing, with a gap given approximately by twice the corresponding matrix element,

$$\Delta = \lambda \sum_{j=1}^{N} \langle N+1|V|j\rangle [a_k^N(j;M_N+1) + a_k^N(j;M_N-1)]. \quad (A7)$$

We now place strong limits on the size of the right-hand side of this equation. Since the state label k and the basis size N will remain fixed, for simplicity of notation we will no longer write the subscript k and superscript N on the coefficients $a(j;n)$. The time-dependent Schrödinger equation for $u(x,t)$ can be rewritten as a set of equations for these coefficients:

$$[n\omega + \epsilon - E_j^0]a(j;n) = \frac{\lambda}{2}\sum_{l=1}^{N}\langle j|V|l\rangle[a(l;n+1)$$
$$+ a(l;n-1)] \quad (A8)$$

for $j=1,2,\ldots,N$. We emphasize that this is the *exact* equation for the time-dependent problem in the finite basis N. It contains all orders of λ and makes no reference to convergence of perturbation theory; there may be arbitrary resonances or near resonances of the time-dependent Hamiltonian between the initial N states. We will draw only on the fact that normalized solutions, satisfying Eq. (A8), exist. From Eq. (A7) we see that we need the coefficients $a(j;n)$ only for the large values of frequency index $n \approx M_N$. The high-lying energies of the unperturbed static Hamiltonian are approximately $E_j^0 \approx j^2\omega_0$ [where $\omega_0 = (1/2m)(\pi/2a)^2$]. Then $M_N\omega + \epsilon = E_{N+1}^0 \approx (N+1)^2\omega_0$ and the factor in square brackets on the left-hand side of Eq. (A8) for the case of interest, $n = M_N$, is greater than $2N\omega_0$ for any value of j (the smallest value occurs for the largest possible index, namely, $j=N$). We use the symbol V_0 to denote the maximum absolute value of the matrix elements of V between *any* two basis states. Then the absolute value of the coefficient $a(j;n)$ is limited by Eq. (A8) to

$$|a(j;M_N)| \le \frac{\lambda V_0}{4N\omega_0}\sum_{i=1}^{N}[|a(i;M_N+1)|+|a(i;M_N-1)|]$$

$$\le \frac{\lambda V_0}{2\sqrt{2N}\omega_0}, \quad (A9)$$

where we have used only the limitation imposed by normalization (A5) on sums over the absolute values of any subset of the coefficients $a(j,n)$ corresponding to a single state u_k^N, namely, $|a_1|+|a_2|+\cdots+|a_m| \le \sqrt{m}$. But we can do much better, essentially by iterating this process. We start with Eq. (A8) for a smaller value of the photon index $n = M_N - p$, use the argument just given to limit the right-hand side for the next higher value $n = M_n - p + 1$, and work back to the value of interest $n = M_N$. We choose the starting integer p as the integer part of $N\omega_0/\omega$ (this gives a substantial improvement only for $N\omega_0/\omega \gg 1$, so we choose N large enough for this to be the case). Then the coefficient in square brackets on the left-hand side of Eq. (A8) is greater than $N\omega_0$. Thus, by exactly the same kind of argument that led to Eq. (A9) we have

$$|a(j;M_N-p)| \le \frac{\lambda V_0}{\sqrt{2N}\omega_0}. \quad (A10)$$

The same limitation holds for $|a(j;M_N-p+2q)|$, with $q=1,2,\ldots,p$, where the coefficient on the left-hand side of Eq. (A8) is even larger. Then we use these maximum values to bound the right-hand side of Eq. (A8) for the next iteration, for the values of n lying between those just limited, $n = M_N - p + 2q - 1$ with $q = 1,2,\ldots,p$ (*not* for $q=0$):

$$|a(j;M_N-p+q)| \le \frac{\lambda V_0}{2N\omega_0}\sum_{i=1}^{N}[|a(i;M_N-p+q+1)|$$
$$+ |a(i;M_N-p+q-1)|]$$

$$\le \frac{1}{\sqrt{2N}}\left[\frac{\lambda V_0}{\omega_0}\right]^2, \quad (A11)$$

where the first inequality comes again directly from Eq. (A8)

with the minimum possible coefficient on the left-hand side, and the final inequality results from substitution of Eq. (A10), which holds for each value of the state index i, into the middle expression. This is repeated, using these bounds to limit the right-hand side of Eq. (A8) for the n values lying between *these*:

$$|a(j;M_N-p+2q)| \leq \frac{1}{\sqrt{2N}} \left[\frac{\lambda V_0}{\omega_0} \right]^3, \qquad \text{(A12)}$$

now for $q=1,2,\ldots,p-1$. We then repeat this $p-2$ times, with the power of $\lambda V_0/\omega_0$ increasing by one and the range of q decreasing by one at each iteration, to obtain

$$|a(j;M_N)| \leq \frac{1}{2N} \left[\frac{\lambda V_0}{\omega_0} \right]^{p+1} \leq \frac{1}{\sqrt{2N}} \left[\frac{\lambda V_0}{\omega_0} \right]^{N\omega_0/\omega}. \tag{A13}$$

Finally, we use this in Eq. (A7) to put limits on the size of the gap:

$$\Delta \leq \lambda V_0 \sqrt{2N} \left[\frac{\lambda V_0}{\omega_0} \right]^{N\omega_0/\omega}. \tag{A14}$$

Therefore, the gap is limited by this to be exponentially decreasing with basis size N, at least for small enough coupling $\lambda < \omega_0/V_0$.

For specific examples of the spatial dependence of the time-dependent potential $V(x)\cos\omega t$ we can construct even tighter limits. There are, in particular, two limiting cases of interest: With the square-well confining potential in the interval $-a<x<a$ we take (i) $V(x)=2V_0\sin(\pi x/2a)$ or (ii) $V(x)=V_0 a\,\delta(x)$. In both cases the only nonzero matrix elements of $V(x)$ between the eigenstates of the square well are of magnitude V_0. In the first instance (sinusoidal potential) these occur only for nearest-neighbor states in the energy ladder $\langle j|V|k\rangle = V_0\delta_{k,j\pm 1}$, whereas for the δ-function potential all pairs of even-spatial-parity states are connected by V_0 regardless of how far apart in energy they are (and all odd-parity states with vanishing wave function at $x=0$ are, of course, totally unaffected by the potential). These are then limiting cases of short- and long-range effects of the spatial potential relative to the energy spectrum of the unperturbed static square-well potential.

For the sinusoidal potential the right-hand side of Eq. (A8) contains only the four coefficients corresponding to $l=j\pm 1$, so that the inequality (A9) becomes

$$|a(j;M_N)| \leq \frac{\lambda V_0}{2N\omega_0}. \tag{A15}$$

The limitation to four coefficients occurs at each stage of the iterative process that led to Eq. (A13), which limit now becomes

$$|a(j;M_N)| \leq \frac{1}{N} \left[\frac{\lambda V_0}{N\omega_0} \right]^{N\omega_0/\omega}. \tag{A16}$$

Note that by choosing N large enough ($\lambda V_0/N\omega_0<1$, as well as $N\omega_0/\omega>1$) we find the ultimate exponential (indeed, powers of $1/N!$) decrease of the gaps for arbitrarily large coupling strength λ for this case.

We also can obtain tighter limits for the δ-function potential. In this case we can rewrite Eq. (A8) as

$$a(j;,n) = \frac{\lambda V_0/2}{n\omega + \epsilon - E_j^0} [A_N(n+1)+A_N(n-1)], \tag{A17}$$

where we have defined

$$A_N(n) = \sum_{i=1}^N a(i;n), \tag{A18}$$

and throughout the analysis of this case the unperturbed eigenstate index j refers only to even-parity states (we have noted above that the odd-parity states are not affected by this potential). Then we can sum Eq. (A17) over the eigenstate index j to find a recursive relationship for the $A_N(n)$,

$$A_N(n) = (\lambda V_0/2\omega_0)S_N(n)[A_N(n+1)+A_N(n-1)], \tag{A19}$$

where we have defined one more sum

$$S_N(n) = \sum_{j=1}^N \left[\frac{n\omega+\epsilon}{\omega_0} - j^2 \right]^{-1} \equiv \sum_{j=1}^N \frac{1}{C_n^2 - j^2} \approx \frac{1}{C_n} \ln \frac{C_n+N}{C_n-N}. \tag{A20}$$

The final approximation on the right-hand side is the Euler-Maclaurin integral estimate for the sum; corrections are of order $1/C_n$. Now, as before, we start by considering the recursion relation (A19) for $n=M_N-p$, with p the integer part of $N\omega_0/\omega$, and limit the right-hand side by the maximum imposed by the normalization condition $|A_N(n)|<\sqrt{N}$, so that

$$|A_N(M_N-p)| < \frac{\lambda V_0\ln(4N+1)}{2\omega_0 \quad \sqrt{2N}}. \tag{A21}$$

We use this on the right-hand side of Eq. (A19) for the next higher value of n, namely, $n=M_N-p+1$, and iterate as before to obtain

$$|A_N(M_N)| < \left[\frac{\lambda V_0 \ln(4N+1)}{2\omega_0\sqrt{2N}} \right]^{N\omega_0/\omega}. \tag{A22}$$

Finally, we put this back into Eq. (A17) for the original coefficient $a(j;M_N)$ to find

$$|a(j;M_N)| < \frac{\lambda V_0}{(2N+1)\omega_0} \left[\frac{\lambda V_0 \ln(4N+1)}{2\omega_0\sqrt{2N}} \right]^{N\omega_0/\omega}. \tag{A23}$$

[1] For a review, see S.-I Chu, Adv. Chem. Phys. **73**, 739 (1989).

[2] There are many such papers, including those found as references in [1] above. A particularly relevant example, demonstrating the effect of choice of size and nature of the finite set of basis states, is H. P. Breuer and M. Holthaus, Z. Phys. D **11**, 1 (1989).

[3] J. S. Howland, Ann. Inst. Henri Poincaré Phys. Theor. **49**, 309 (1989); **49**, 325 (1989).

[4] H. Weyl, Math. Ann. **77**, 313 (1916).

[5] See, e.g., H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory* (Princeton University Press, Princeton, 1981), p. 69.

[6] H. Sambé, Phys. Rev. A **7**, 2203 (1973).

[7] A similarly useful analysis of energy as a function of complex quasimomentum, for electrons in crystalline solids, was made many years ago by one of us: W. Kohn, Phys. Rev. **115**, 809 (1959).

[8] M. Holthaus, Phys. Rev. Lett. **69**, 1596 (1992).

[9] See, e.g., L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Addison-Wesley, Reading, MA, 1958), Sec. 87.

[10] The fundamental difference between periodically driven systems with increasing and those with decreasing spacings between successive energy levels has been emphasized by N. Brenner and S. Fishman, J. Phys. A **28**, 5973 (1995); **29**, 7199 (1996).

[11] See, e.g., P. R. Halmos, *Measure Theory* (Van Nostrand, Princeton, 1950), Sec. 47.

[12] This nonconvergence implies [13] that the corresponding quasienergy spectrum is not a point spectrum for these values of λ. But Howland has shown [3] that the spectrum is also not absolutely continuous. It follows that the quasienergy spectrum is singular continuous for a set of λ that is everywhere dense but of total measure zero. See also the conclusion of G. Casati and I. Guarneri, Commun. Math. Phys. **95**, 121 (1984), that the quantum kicked rotor has singular continuous quasienergy spectra for a nonempty set of driving frequencies.

[13] We are grateful to Shmuel Fishman for pointing this out to us.