Linear response of a nonrelativistic hydrogenlike atom to a single-mode radiation field. II. Electric dipole approximation: General formalism

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The interaction of a low-intensity laser field with a nonrelativistic one-electron atom is considered to the first order of perturbation theory and in the electric dipole approximation. The radiation field is turned on adiabatically modifying the initial unperturbed atomic state, which is either an angular-momentum eigenstate $|nlm\rangle$ or a Stark state $|nn_em\rangle$. The first-order correction to the wave function is expressed both in the length and velocity gauges in terms of a vector function called the linear-response vector and depending on the field-free energy eigenstate. We derive in a unifying manner the linear-response vectors in the position representation, as closed-form contour integrals, starting from a unique generating function built up with the Coulomb Green function. The linear-response vectors are then evaluated in momentum space via a Fourier transformation: they are obtained as integral representations and also in explicit form, as generalized hypergeometric functions. With reference to the static limit, we complement earlier results and find the reduced linear-response vectors in momentum space, as Fourier transforms of their coordinate-representation counterparts. The low-frequency behavior of the length-gauge first-order correction to the ground-state wave function is established in coordinate as well as in momentum space. We finally point out the high-frequency limit of the linear response in the velocity gauge. [S1050-2947(97)03511-7]

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I. INTRODUCTION

In this work we resume the problem of the linear response of a nonrelativistic hydrogenic atom to a classical single-mode radiation field in the *electric dipole approximation* (EDA). An account of the physical system has been given in our preceding paper [1], which will be referred to in the following as I. Although the basics will not be repeated here, we mention that an important issue which determines the extent of the topic is the choice of the initial atomic state.

Beyond all question, our survey of previous work begins with the classic paper written by Podolsky in the early years of quantum mechanics [2]: on the assumption that the atom is initially in its ground state, he calculated the Sturmian-function expansion of the first-order perturbed wave function in the velocity gauge. Interest in this problem has been revived by the outstanding result of Luban, Nudler, and Freund [3], who obtained the same correction in closed form. Florescu and Marian [4] made use thereafter of the Coulomb Green function (CGF) to get compact solutions also in the case of the first excited $|nlm\rangle$ states (with $n=2$ and $n=3$). Besides, we have introduced in this preliminary work the important concept of linear-response vector (LRV) associated to a given unperturbed energy eigenfunction. We have subsequently reported a closed-form solution in the velocity gauge for an arbitrary $|nlm\rangle$ state (spherical bound state), evaluating also its static limit [5]. Florescu [6] has employed this general result to find the LRV's corresponding in the velocity gauge to the Coulomb scattering waves by means of their partial-wave series. Sturmian-function expansions of the linear-response corrections to both negative- and positive-energy spherical eigenstates have then been reported in the length gauge by Maquet, Martin, and Véniard [7]. We have presented in compact form the length-gauge LRV for a $|nlm\rangle$ state, as well as the Coulomb Sturmian-function expansions of the LRV's in both gauges mentioned above [8]. In addition, starting from the LRV's for a spherical state $|nlm\rangle$, we have succeeded in deriving those for a parabolic state $|nn_em\rangle$ by an operator method that makes no *explicit* use of the relation between such states. The same idea has been applied in the static limit. Unlike our method in Ref. [8], it is precisely the relation between spherical and parabolic stationary states that has been exploited by Florescu and Pătraşcu [9] in their independent derivation of the velocity-gauge first-order correction to a Stark state $|nn_em\rangle$. Along Podolsky's line of argument, by using appropriate boundary conditions, Câmpeanu and Florescu [10] have solved the inhomogeneous differential equations pertaining to the velocity-gauge LRV associated with an arbitrary $|nlm\rangle$ state. Furthermore, the same method has been employed by Florescu, Halasz, and Marinescu [11] to evaluate in the velocity gauge the quadratic atomic response to a uniform harmonic electric field, from a stationary spherical state, with emphasis on the ground-state case.

Meanwhile, most of the results enumerated above have been applied to physical problems especially by Florescu and co-workers. The importance of the LRV's as intermediates for calculating in the EDA the two-photon transition amplitudes of an electron in the Coulomb field of a fixed nucleus justifies the following brief overview. The invariant amplitudes for $1s\rightarrow ns$ and $1s\rightarrow nd$ two-photon absorption reported in Ref. [4] have been evaluated and analyzed numerically by Florescu, Pătraşcu, and Stoican [12]. In the meantime, employing the velocity-gauge LRV's for the incoming Coulomb scattering wave derived in Ref. [6], Florescu and Djamo [13] have calculated the matrix element for two-photon bremsstrahlung in the Coulomb field. It is worth stressing that by use of the LRV method, we have established general formulas for the amplitudes of bound-bound two-photon transitions between spherical states, as well as between parabolic ones [14]. Special attention has been paid to photon scattering from the atomic ground state. We have also obtained and discussed the matrix element of a bound-free two-photon transition from an arbitrary $|nlm\rangle$ state [15]. Quite recently, Yakhontov and Jungmann [16] have used linear-response functions to evaluate dynamic polarizabilities in hydrogenic ns states.

We conclude this overview by mentioning two applications of a different kind. The first one refers to the use by Cionga and Florescu [17] of the velocity-gauge LRV for a $|nlm\rangle$ state in calculating the cross section of the one-photon excitation of atomic hydrogen by electron impact with fixed momentum transfer. In the second one, Cionga, Florescu, Maquet, and Taïeb [18] evaluate the cross section of the laser-assisted photoeffect at moderate intensities by employing the velocity-gauge LRV's associated to the ground state and to the Coulomb scattering waves.

The present paper is intended as a comprehensive, but by no means exhaustive treatment of the LRV's, based on the consistent use of the CGF, via the already employed generating function $\mathcal{F}$ of the linear response [19]. The notations of I and, to a large extent, those introduced in Refs. [5,8] are preserved. In Sec. II we revisit the LRV's in terms of which the first-order perturbed wave functions are expressed both in the length and velocity gauges. In Secs. III and IV, we describe our parallel three-step derivations of the LRV's associated in coordinate space to spherical and parabolic stationary states, respectively, whose common starting point is the generating function $\mathcal{F}$. The LRV's are then evaluated also in momentum space as Fourier transforms of their integral representations in coordinate space. The momentum-space LRV's are first written as closed-form contour integrals and then explicitly in terms of generalized hypergeometric functions with several parameters and variables. Sections V and VI are devoted to the evaluation of the momentum-space reduced LRV's, by Fourier transformation, for both spherical and parabolic energy eigenstates. In Sec. VII the general formulas we have obtained are specialized to the atomic ground state. Moreover, we point out the low-frequency behavior of the linear response from the ground state in the length gauge, both in coordinate and momentum representations. The velocity gauge is chosen in Sec. VIII to discuss the high-frequency limit of the linear response in coordinate space for any energy eigenstate. We afterwards summarize the results and conclude by emphasizing briefly their relevance. Appendixes A and B collect some formulas needed for deriving or expressing the linear response in coordinate and momentum representation, respectively.

II. LINEAR-RESPONSE WAVE FUNCTIONS REVISITED

In the EDA the laser field acts on the atom as a uniform harmonic electric field that is in general elliptically polarized:

$$\boldsymbol{E}(t)=\frac{1}{2}\mathcal{E}_0\{\exp(-i\omega t)\hat{\boldsymbol{\epsilon}}+\exp(i\omega t)\hat{\boldsymbol{\epsilon}}^*\} \quad (\mathcal{E}_0>0). \tag{2.1}$$

The action of the magnetic field is systematically ignored [20]:

$$\boldsymbol{B}=\boldsymbol{0}. \tag{2.2}$$

As shown by Eq. (2.1), $\mathcal{E}_0$ is the amplitude of the laser field if and only if its polarization is linear: $\hat{\boldsymbol{\epsilon}}^*=\hat{\boldsymbol{\epsilon}}$. However, in all cases, $\frac{1}{2}\mathcal{E}_0^2$ is the time average of the squared electric field strength.

There are two usual choices of the electromagnetic potentials from which the fields (2.1) and (2.2) derive. They are well known as [21] (a) the length (Göppert-Mayer) gauge,

$$\Phi'(\boldsymbol{r},t)=-\frac{1}{2}\mathcal{E}_0[\exp(-i\omega t)(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{r})+\exp(i\omega t)(\hat{\boldsymbol{\epsilon}}^*\cdot\boldsymbol{r})],$$

$$\boldsymbol{A}'=\boldsymbol{0}, \tag{2.3a}$$

and (b) the velocity gauge,

$$\Phi=0,\ \boldsymbol{A}(t)=\frac{c}{2i\omega}\mathcal{E}_0[\exp(-i\omega t)\hat{\boldsymbol{\epsilon}}-\exp(i\omega t)\hat{\boldsymbol{\epsilon}}^*]. \tag{2.3b}$$

The gauge transformation from Eq. (2.3b) to Eq. (2.3a) has the generating function

$$\chi_0(\boldsymbol{r},t)=-\frac{c}{2i\omega}\mathcal{E}_0[\exp(-i\omega t)(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{r})-\exp(i\omega t)(\hat{\boldsymbol{\epsilon}}^*\cdot\boldsymbol{r})]. \tag{2.4}$$

The field-atom interaction Hamiltonian in the length gauge is

$$H_L^{(1)}=e\boldsymbol{E}(t)\cdot\boldsymbol{r}, \tag{2.5a}$$

while in the velocity gauge, after removing the $\boldsymbol{A}^2$ term, it reads

$$H_V^{(1)}=\frac{e}{m_e c}\boldsymbol{A}(t)\cdot\boldsymbol{P}. \tag{2.5b}$$

Recall that in the EDA the radiation gauge goes into the velocity gauge, while the Poincaré and any multipolar gauge reduces to the length gauge [22].

The electric field (2.1) is supposed to increase adiabatically in the time interval $(-\infty,0)$. In the remote past $(t\to-\infty)$, the state of the electron is a stationary one:

$$\Psi_N^{(0)}(\boldsymbol{r},t)=\exp\left(-\frac{i}{\hbar}E_n t\right)u_N(\boldsymbol{r}). \tag{2.6}$$

The energy eigenfunction $u_N(\boldsymbol{r})$, associated to the nth Bohr level E_n, describes in coordinate space either an angular-momentum eigenstate $|N\rangle\equiv|nlm\rangle$ or a Stark state $|N\rangle\equiv|nn_e m\rangle$. In the latter case, n_e denotes the electric quantum number, defined as the difference

$$n_e\equiv n_\xi-n_\eta, \tag{2.7}$$

where n_ξ and n_η are the parabolic quantum numbers fulfilling the condition

$$n_\xi+n_\eta+|m|+1=n. \tag{2.8}$$

According to Dirac's perturbation theory, the first-order corrections to the wave function (2.6) in the length and velocity gauge read for $t \geqslant 0$

$$\Psi^{(1)}_{(L)\ N}(\omega;\boldsymbol{r},t) = -\frac{1}{2} e\mathcal{E}_0 \exp\left(-\frac{i}{\hbar}E_n t\right) \times [\exp(-i\omega t)\hat{\boldsymbol{\epsilon}}\cdot \boldsymbol{v}_N(\Omega_1;\boldsymbol{r}) + \exp(i\omega t)\hat{\boldsymbol{\epsilon}}^*\cdot \boldsymbol{v}_N(\Omega_2;\boldsymbol{r})] \quad (2.9a)$$

and, respectively,

$$\Psi^{(1)}_{(V)\ N}(\omega;\boldsymbol{r},t) = -\frac{e}{2im_e\omega}\mathcal{E}_0 \exp\left(-\frac{i}{\hbar}E_n t\right) \times [\exp(-i\omega t)\hat{\boldsymbol{\epsilon}}\cdot \boldsymbol{w}_N(\Omega_1;\boldsymbol{r}) - \exp(i\omega t)\hat{\boldsymbol{\epsilon}}^*\cdot \boldsymbol{w}_N(\Omega_2;\boldsymbol{r})]. \quad (2.9b)$$

The vector functions occurring in Eqs. (2.9), which we call *linear-response vectors*, are defined by means of the Schrödinger CGF in position representation [23] as

$$\boldsymbol{v}_N(\Omega;\boldsymbol{r}) \equiv -\int d^3x'\, G(\Omega;\boldsymbol{r},\boldsymbol{r}')\mathbf{r}' u_N(\boldsymbol{r}') \quad (2.10a)$$

and

$$\boldsymbol{w}_N(\Omega;\boldsymbol{r}) \equiv -\int d^3x'\, G(\Omega;\boldsymbol{r},\boldsymbol{r}')\boldsymbol{P}' u_N(\boldsymbol{r}'). \quad (2.10b)$$

The parameters Ω_1 and Ω_2 are associated to the nth Bohr level:

$$\Omega_1 = E_n + \hbar\omega + i0, \quad \Omega_2 = E_n - \hbar\omega. \quad (2.11)$$

We write out the gauge transformation connecting the corrections (2.9a) and (2.9b) which is induced by the generating function (2.4):

$$\Psi^{(1)}_{(L)\ N}(\omega;\boldsymbol{r},t) = \Psi^{(1)}_{(V)\ N}(\omega;\boldsymbol{r},t) - \frac{e\mathcal{E}_0}{2\hbar\omega}[\exp(-i\omega t)(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{r}) - \exp(i\omega t)(\hat{\boldsymbol{\epsilon}}^*\cdot\boldsymbol{r})]\Psi^{(0)}_N(\boldsymbol{r},t). \quad (2.12)$$

Owing to Eq. (A1) of I, the LRV's (2.10) satisfy the following inhomogeneous differential equations:

$$(H^{(0)} - \Omega)\boldsymbol{v}_N(\Omega;\boldsymbol{r}) = \boldsymbol{r}u_N(\boldsymbol{r}) \quad (2.13a)$$

and

$$(H^{(0)} - \Omega)\boldsymbol{w}_N(\Omega;\boldsymbol{r}) = \boldsymbol{P}u_N(\boldsymbol{r}), \quad (2.13b)$$

where $H^{(0)}$ is the unperturbed Coulomb Hamiltonian. Obviously, the LRV's in momentum representation are the Fourier transforms of those in the coordinate space:

$$\tilde{\boldsymbol{v}}_N(\Omega;\boldsymbol{p}) \equiv (2\pi\hbar)^{-3/2}\int d^3x \exp\left(-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}\right)\boldsymbol{v}_N(\Omega;\boldsymbol{r}) \quad (2.14a)$$

and

$$\tilde{\boldsymbol{w}}_N(\Omega;\boldsymbol{p}) \equiv (2\pi\hbar)^{-3/2}\int d^3x \exp\left(-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}\right)\boldsymbol{w}_N(\Omega;\boldsymbol{r}). \quad (2.14b)$$

Further, we introduce the kets whose representatives in coordinate space are the LRV's (2.10) and in momentum space the LRV's (2.14), namely,

$$|\boldsymbol{v}_N(\Omega)\rangle = -G(\Omega)\boldsymbol{R}|N\rangle \quad (2.15a)$$

and

$$|\boldsymbol{w}_N(\Omega)\rangle = -G(\Omega)\boldsymbol{P}|N\rangle. \quad (2.15b)$$

$G(\Omega)$ is thereupon the resolvent of the Coulomb Hamiltonian,

$$G(\Omega) = (\Omega I - H^{(0)})^{-1}, \quad (2.16)$$

while $\boldsymbol{R}$ and $\boldsymbol{P}$ denote the position vector and momentum operators of the electron. The resolvent (2.16), with I the unit operator, is defined for any complex value Ω other than an energy eigenvalue. It has the spectral resolution

$$G(\Omega) = \sum_{N'} \frac{|N'\rangle\langle N'|}{\Omega - E_{n'}}, \quad (2.17)$$

in terms of a complete orthonormal set of energy eigenvectors $|N'\rangle$ [24]. We recall that the LRV's (2.15) are connected by the identity [25]

$$\frac{\hbar}{im_e}|\boldsymbol{w}_N(\Omega)\rangle = \boldsymbol{R}|N\rangle + (\Omega - E_n)|\boldsymbol{v}_N(\Omega)\rangle, \quad (2.18)$$

which results in the gauge transformation (2.12).

The vector $|\boldsymbol{v}_N(\Omega)\rangle$ has a singular part when the parameter Ω approaches the Bohr level E_n:

$$P^{(n)}|\boldsymbol{v}_N(\Omega)\rangle = -\frac{1}{\Omega - E_n}P^{(n)}\boldsymbol{R}|N\rangle. \quad (2.19)$$

In Eq. (2.19), $P^{(n)}$ denotes the orthogonal projection operator onto the nth energy eigensubspace $\mathcal{U}_n$,

$$P^{(n)} = \sum_{N';(n'=n)} |N'\rangle\langle N'|. \quad (2.20)$$

Note that the only stationary state for which the secular term (2.19) vanishes is the atomic ground state $|100\rangle$. For any other state $|N\rangle$ the LRV (2.15a) includes a distinct regular part,

$$|\boldsymbol{v}'_N(\Omega)\rangle \equiv (I - P^{(n)})|\boldsymbol{v}_N(\Omega)\rangle, \quad (2.21)$$

belonging to the orthogonal complement $\mathcal{U}_n^{\perp}$ of the eigensubspace $\mathcal{U}_n$. We call its limit for $\Omega = E_n$ a *reduced* LRV:

$$|\boldsymbol{v}'_N(E_n)\rangle = \lim_{\Omega\to E_n}(I-P^{(n)})|\boldsymbol{v}_N(\Omega)\rangle. \qquad (2.22)$$

From Eq. (2.18) written in the form

$$\frac{\hbar}{im_e}|\boldsymbol{w}_N(\Omega)\rangle = (I-P^{(n)})\boldsymbol{R}|N\rangle + (\Omega-E_n)|\boldsymbol{v}'_N(\Omega)\rangle, \qquad (2.23)$$

we get an alternative formula of a reduced LRV:

$$|\boldsymbol{v}'_N(E_n)\rangle = \frac{\hbar}{im_e}\frac{\partial}{\partial\Omega}|\boldsymbol{w}_N(\Omega)\rangle|_{\Omega=E_n}. \qquad (2.24)$$

To write the *nondivergent* part $\Psi^{(1)\,\prime}_{(L)\,N}(\omega;\boldsymbol{r},t)$ of the length-gauge correction $\Psi^{(1)}_{(L)\,N}(\omega;\boldsymbol{r},t)$, one should only replace in Eq. (2.9a) the LRV $\boldsymbol{v}_N(\Omega;\boldsymbol{r})$ by its regular term $\boldsymbol{v}'_N(\Omega;\boldsymbol{r})$ [26]. Just notice that the reduced LRV's in coordinate space can be expressed in terms of the corresponding reduced CGF [27] as

$$\boldsymbol{v}'_N(E_n;\boldsymbol{r}) = -\int d^3x'\,G^{(n)}(E_n;\boldsymbol{r},\boldsymbol{r}')\boldsymbol{r}'u_N(\boldsymbol{r}'). \quad (2.25)$$

The reduced LRV's (2.25) have been evaluated for spherical states in Ref. [5] and for parabolic ones in Ref. [8]. We complement our previous results by deriving here their counterparts in momentum space:

$$\tilde{\boldsymbol{v}}'_N(E_n;\boldsymbol{p}) \equiv (2\pi\hbar)^{-3/2}\int d^3x \exp\left(-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}\right)\boldsymbol{v}'_N(E_n;\boldsymbol{r}). \qquad (2.26)$$

III. LINEAR-RESPONSE VECTORS FOR SPHERICAL STATIONARY STATES

A. Position representation

We present our three-step method of deriving the LRV's $\boldsymbol{v}_{nlm}(\Omega;\boldsymbol{r})$ and $\boldsymbol{w}_{nlm}(\Omega;\boldsymbol{r})$ associated to an arbitrary angular-momentum eigenstate $|nlm\rangle$. The only prerequisite analytic tool is the generating function of the linear response, $\mathcal{F}(\Omega,\boldsymbol{q},\lambda;\boldsymbol{r})$, defined by Eqs. (A9) and (A10) of I.

The first step consists of evaluating the integral

$$\mathcal{F}_{lm}(\Omega,\lambda;r,\theta,\varphi) \equiv -\int d^3x'\,G(\Omega;\boldsymbol{r},\boldsymbol{r}')\mathcal{U}(\boldsymbol{0},\lambda;\boldsymbol{r}')\times(r')^l Y_{lm}(\theta',\varphi'), \qquad (3.1)$$

built up with the homogeneous harmonic polynomial

$$r^l Y_{lm}(\theta,\varphi) = (4\pi)^{-1/2}C_{lm,\ j_1\ldots j_l}x_{j_1}\cdots x_{j_l} \quad (l=0,1,2,3,\ldots;\ \ m=-l,-l+1,\ldots,l). \qquad (3.2)$$

The coefficients $C_{lm,\ j_1\ldots j_l}$ in Eq. (3.2) are totally symmetric with respect to the Cartesian indices and have vanishing traces. From Eqs. (A10) of I and (3.2) it follows that

$$\mathcal{U}(\boldsymbol{0},\lambda;\boldsymbol{r})r^l Y_{lm}(\theta,\varphi) = \left(\frac{\hbar}{i}\right)^l(4\pi)^{-1/2}C_{lm,\ j_1\ldots j_l}\left.\frac{\partial^l\mathcal{U}(\boldsymbol{q},\lambda;\boldsymbol{r})}{\partial q_{j_1}\cdots\partial q_{j_l}}\right|_{\boldsymbol{q}=\boldsymbol{0}}. \qquad (3.3)$$

Taking note of the definitions (A9) of I and (3.1), we get the parallel identity

$$\mathcal{F}_{lm}(\Omega,\lambda;r,\theta,\varphi) = \left(\frac{\hbar}{i}\right)^l(4\pi)^{-1/2}C_{lm,j_1\ldots j_l}\left.\frac{\partial^l\mathcal{F}(\Omega,\boldsymbol{q},\lambda;\boldsymbol{r})}{\partial q_{j_1}\cdots\partial q_{j_l}}\right|_{\boldsymbol{q}=\boldsymbol{0}}. \qquad (3.4)$$

Combined with Eqs. (A14)–(A16) of I, Eq. (3.4) yields the integral representation

$$\mathcal{F}_{lm}(\Omega,\lambda;r,\theta,\varphi) = \frac{\tau}{Ze^2}r^l\,Y_{lm}(\theta,\varphi)\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\times\int_1^{(0+)}d\rho\rho^{-1-\tau}[f(X,0,\lambda;\rho)]^{l+1}\times\exp\left[-g(X,0,\lambda;\rho)\frac{1}{\tau}\frac{r}{a}\right]. \qquad (3.5)$$

The second step aims to employ Eq. (3.5) for calculating the function

$$\mathcal{G}^{(s)}_{nlm}(\Omega;r,\theta,\varphi) \equiv -\int d^3x'\,G(\Omega;\boldsymbol{r},\boldsymbol{r}')\frac{1}{r'}\exp(-\kappa_n r')(2\kappa_n r')^l \times {}_1F_1(l+1-n+s;2l+2;2\kappa_n r')Y_{lm}(\theta',\varphi') \quad (s\ \text{integer},\quad n-l-s-1\geqslant 0), \qquad (3.6)$$

with

$$\kappa_n \equiv \frac{1}{na}. \qquad (3.7)$$

In Eq. (3.7) a is the scaled Bohr radius, Eq. (2.10) of I. In fact, the Kummer function ${}_1F_1$ in Eq. (3.6) is proportional to a Laguerre polynomial. By using the integral representation (A5) of the Kummer hypergeometric function and the definition (3.1), we get the expression

$$\mathcal{G}^{(s)}_{nlm}(\Omega;r,\theta,\varphi) = (2\kappa_n)^l\left(-\frac{1}{2\pi i}\right)\frac{(2l+1)!(n-l-s)!}{(n+l-s)!}\times\int_1^{0+}du(-u)^{l-n+s}(1-u)^{l+n-s}\times\mathcal{F}_{lm}(\Omega,(1-2u)\hbar\kappa_n;r,\theta,\varphi). \qquad (3.8)$$

Taking into account Eq. (3.5), we notice that Eq. (3.8) is a double contour integral that can be reduced to a single one by performing the integral (A6):

$$\mathcal{G}^{(s)}_{nlm}(\Omega;r,\theta,\varphi)=\frac{4n^2\tau}{Ze^2}Y_{lm}(\theta,\varphi)\frac{ie^{i\pi a}}{2\sin(\pi a)}\times\int_1^{(0+)}d\rho\rho^{-\tau}\exp\left(-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{1}{\tau}\frac{r}{a}\right)\times\mathcal{N}^{n-1-s}_{n,-\tau}\mathcal{N}^{-n-1+s}_{n,\tau}(2\kappa_{n,\tau}r)^l\times{}_1F_1(l+1-n+s;2l+2;2\kappa_{n,\tau}r),\quad(3.9)$$

with

$$\kappa_{n,\tau}\equiv\frac{4n^2\rho}{\mathcal{N}_{n,-\tau}\mathcal{N}_{n,\tau}}\kappa_n.\quad(3.10)$$

The contour integral in Eq. (3.9) can be written in terms of the generalized hypergeometric function ${}_1\Phi_H$ with five parameters and four variables [28]:

$$\mathcal{G}^{(s)}_{nlm}(\Omega;r,\theta,\varphi)=\frac{\tau}{Ze^2}\left(\frac{n-\tau}{2n}\right)^{n-l-1-s}\left(\frac{n+\tau}{2n}\right)^{-n+s-\tau}\exp\left(-\frac{1}{\tau}\frac{r}{a}\right)(2\kappa_n r)^l\,Y_{lm}(\theta,\varphi)\frac{1}{l+1-\tau}{}_1\Phi_H\left(l+1-\tau;-n+1+s-\tau,l+1-n+s,2l+2;l+2-\tau;\frac{n-\tau}{2n},-\frac{(n+\tau)^2}{2n(n-\tau)},\frac{n-\tau}{\tau}\kappa_n r,\frac{4n}{n-\tau}\kappa_n r\right)\quad(n-l-1-s\geqslant 0).\quad(3.11)$$

According to Eq. (A8), the function ${}_1\Phi_H$ occurring in Eq. (3.11) reduces to a finite double sum of Humbert functions Φ_1 [29].

As a third step, the finite expansions of the vectors $\boldsymbol{r}u_{nlm}(r,\theta,\varphi)$ and $\boldsymbol{P}u_{nlm}(r,\theta,\varphi)$ associated to any $|nlm\rangle$ eigenstate [30] enable us to express the spherical components (A3) of the LRV's in both gauges in terms of functions (3.6):

$$v_{nlm;}{}^{\mu}(\Omega;r,\theta,\varphi)=\left(\frac{2}{\kappa_n}\right)^{1/2}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\sum_{q=1,-1}(-q)\times\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\frac{\langle l+q\;\;m-\mu,1\mu|l+q1,lm\rangle}{[2(l+q)+1]!}\times\sum_{s=-2}^{2}c^{(q,s)}_{n,l}\mathcal{G}^{(s)}_{n,l+q,m-\mu}(\Omega;r,\theta,\varphi)\quad(\mu=-1,0,1)\quad(3.12a)$$

and

$$w_{nlm;}{}^{\mu}(\Omega;r,\theta,\varphi)=\frac{i}{\hbar}m_e|2E_n|\left(\frac{2}{\kappa_n}\right)^{1/2}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\sum_{q=1,-1}(-q)\times\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\frac{\langle l+q\;\;m-\mu,1\mu|l+q1,lm\rangle}{[2(l+q)+1]!}\times\sum_{s=-1,1}d^{(q,s)}_{n,l}\mathcal{G}^{(s)}_{n,l+q,m-\mu}(\Omega;r,\theta,\varphi)\quad(\mu=-1,0,1).\quad(3.12b)$$

The symbol $\lambda_{l+q,l}$ takes on the values

$$\lambda_{l+q,l}=\begin{cases}-(l+1), & q=1\\ l, & q=-1\end{cases}\quad(3.13)$$

while the coefficients $c^{(q,s)}_{n,l}$ and $d^{(q,s)}_{n,l}$ are listed in Ref. [8], in Tables I and VI, respectively.

In order to simplify the presentation of complicated similar formulas written in the length and velocity gauges, we adopt in the remainder of the paper the following convention: after writing explicitly an equation in the length gauge, we specify only the relative differences of the corresponding expression in the velocity gauge and replace by an ellipsis the identical factors and terms. With this convention, Eq. (3.12b) is abbreviated to

$$w_{nlm;}{}^{\mu}(\Omega;r,\theta,\varphi)=\frac{i}{\hbar}m_e|2E_n|\cdots\sum_{s=-1,1}d^{(q,s)}_{n,l}\cdots.$$

By insertion of Eq. (3.9) into Eqs. (3.12), we recover the decomposition formula of the LRV's $\boldsymbol{v}_{nlm}$ and $\boldsymbol{w}_{nlm}$ in terms of two vector spherical harmonics (A4):

$$\boldsymbol{v}_{nlm}(\Omega;\boldsymbol{r})=\sum_{q=1,-1}(-q)\times\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\mathcal{A}_{n\,l\,l+q}(\Omega;r)\boldsymbol{V}_{l+q\;l\;m}(\hat{\boldsymbol{r}})\quad(3.14a)$$

and

$$\boldsymbol{w}_{nlm}(\Omega;\boldsymbol{r})=\frac{i}{\hbar}m_e\sum_{q=1,-1}(-q)\times\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\mathcal{B}_{n\,l\,l+q}(\Omega;r)\boldsymbol{V}_{l+q\;l\;m}(\hat{\boldsymbol{r}}).\quad(3.14b)$$

Notice that, by virtue of the identity

$$\boldsymbol{V}_{100}(\hat{\boldsymbol{r}})=-(4\pi)^{-1/2}\hat{\boldsymbol{r}},\quad(3.15)$$

in the special case $l=0$ Eqs. (3.14) are simply

$$\boldsymbol{v}_{n00}(\Omega;\boldsymbol{r})=(4\pi)^{-1/2}\mathcal{A}_{n01}(\Omega;r)\hat{\boldsymbol{r}} \tag{3.16a}$$

and

$$\boldsymbol{w}_{n00}(\Omega;\boldsymbol{r})=\frac{i}{\hbar}m_e(4\pi)^{-1/2}\mathcal{B}_{n01}(\Omega;r)\hat{\boldsymbol{r}}. \tag{3.16b}$$

The scalar radial functions in Eqs. (3.14) and (3.16) are found as closed-form contour integrals:

$$\begin{aligned}\mathcal{A}_{nll+q}(\Omega;r)&=\frac{1}{2|E_n|}(2\kappa_n)^{1/2}\frac{4n\tau}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\\&\quad\times\sum_{s=-2}^{2}c_{n,l}^{(q,s)}\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\rho^{-\tau}\\&\quad\times\exp\left(-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{1}{\tau}\frac{r}{a}\right)\mathcal{N}_{n,-\tau}^{n-1-s}\mathcal{N}_{n,\tau}^{-n-1+s}(2\kappa_{n,\tau}r)^{l+q}\\&\quad\times{}_1F_1(l+q+1-n+s;2(l+q)+2;2\kappa_{n,\tau}r)\end{aligned} \tag{3.17a}$$

and

$$\mathcal{B}_{nll+q}(\Omega;r)=2|E_n|\cdots\sum_{s=-1,1}d_{n,l}^{(q,s)}\cdots. \tag{3.17b}$$

Obviously, in Eqs. (3.17), $q=1$ for $l=0$ and $q=1,-1$ for $l>0$. On the other hand, substituting Eq. (3.11) into Eqs. (3.12), we find the explicit formulas of the scalar radial factors $\mathcal{A}_{nll+q}(\Omega;r)$ and $\mathcal{B}_{nll+q}(\Omega;r)$ which have been reported before [31]. These include several functions ${}_1\Phi_H$, each of them expressible as a finite double sum of Humbert functions Φ_1. Owing to the well-known identity

$$\hat{\boldsymbol{r}}Y_{lm}(\hat{\boldsymbol{r}})=\sum_{q=1,-1}(-q)\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\boldsymbol{V}_{l+qlm}(\hat{\boldsymbol{r}}), \tag{3.18}$$

Eq. (2.18), when written for a $|nlm\rangle$ eigenstate, reduces to the relationship

$$\mathcal{B}_{nll+q}(\Omega;r)=rR_{nl}(r)+(\Omega-E_n)\mathcal{A}_{nll+q}(\Omega;r)$$
$$(q=1\ \text{ for }\ l=0,\ \ q=1,-1\ \text{ for }\ l>0). \tag{3.19}$$

Note that Eqs. (3.19) have been checked by a direct method which employs the integral representations (3.17). By using some basic properties of the Kummer function [32], both integrands have been written in a convenient form in terms of the function ${}_1F_1(l+1-n;2l+2;2\kappa_{n,\tau}r)$. A tedious calculation involving several integrations by parts has led us to Eqs. (3.19).

B. Momentum representation

We evaluate the Fourier transforms (2.14) for an angular-momentum eigenstate $|N\rangle=|nlm\rangle$. Combining the expansions (3.14) and (A4) with the spherical-wave expansion of a scalar plane wave [33], we get the following formulas:

$$\tilde{\boldsymbol{v}}_{nlm}(\Omega;\boldsymbol{p})=\sum_{q=1,-1}(-q)\times\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\tilde{\mathcal{A}}_{n\,l\,l+q}(\Omega;p)\boldsymbol{V}_{l+q\,l\,m}(\hat{\boldsymbol{p}}) \tag{3.20a}$$

and

$$\tilde{\boldsymbol{w}}_{nlm}(\Omega;\boldsymbol{p})=\frac{i}{\hbar}m_e\sum_{q=1,-1}(-q)\times\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\tilde{\mathcal{B}}_{n\,l\,l+q}(\Omega;p)\boldsymbol{V}_{l+q\,l\,m}(\hat{\boldsymbol{p}}). \tag{3.20b}$$

For $l=0$, Eqs. (3.20) read

$$\tilde{\boldsymbol{v}}_{n00}(\Omega;\boldsymbol{p})=(4\pi)^{-1/2}\tilde{\mathcal{A}}_{n01}(\Omega;p)\hat{\boldsymbol{p}} \tag{3.21a}$$

and

$$\tilde{\boldsymbol{w}}_{n00}(\Omega;\boldsymbol{p})=\frac{i}{\hbar}m_e(4\pi)^{-1/2}\,\tilde{\mathcal{B}}_{n01}(\Omega;p)\hat{\boldsymbol{p}}. \tag{3.21b}$$

The scalar factors in Eqs. (3.20) and (3.21) include the radial integrals

$$\tilde{\mathcal{A}}_{nll+q}(\Omega;p)=(-i)^{l+q}\frac{4\pi}{(2\pi\hbar)^{3/2}}\int_0^\infty dr\ r^2\times j_{l+q}\left(\frac{p}{\hbar}r\right)\mathcal{A}_{n\,l\,l+q}(\Omega;r) \tag{3.22a}$$

and

$$\tilde{\mathcal{B}}_{nll+q}(\Omega;p)=(-i)^{l+q}\frac{4\pi}{(2\pi\hbar)^{3/2}}\int_0^\infty dr\ r^2\times j_{l+q}\left(\frac{p}{\hbar}r\right)\mathcal{B}_{n\,l\,l+q}(\Omega;r). \tag{3.22b}$$

After substituting in Eqs. (3.22) relation (B1) and the integral representations (3.17), we carry out the radial integration making use of Eqs. (B5)–(B7). It is convenient to introduce the dimensionless vector

$$\boldsymbol{\zeta}\equiv\frac{\boldsymbol{p}}{\hbar\kappa_1}. \tag{3.23}$$

The scalar functions (3.22) are finally obtained as contour integrals:

$$\widetilde{\mathcal{A}}_{nll+q}(\Omega;p)=\frac{1}{2|E_n|}\left(\frac{1}{\pi n}\right)^{1/2}\left(\frac{1}{\hbar\kappa_1}\right)^{3/2}\frac{1}{4\kappa_1}\frac{\tau}{\zeta^2}\frac{(l+q+1)!}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\left[\frac{-i2^4n\tau^2\zeta}{(1+\tau^2\zeta^2)(n^2-\tau^2)}\right]^{l+q+1}$$
$$\times\sum_{s=-2}^{2}\sum_{\tilde{s}=-1,1}\tilde{s}\,c_{n,l}^{(q,s)}\left(\frac{n-\tau}{n+\tau}\right)^{n-s}\left(\frac{1-i\tau\zeta}{1+i\tau\zeta}\right)^{-\tilde{s}}\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\rho^{l+q-\tau}$$
$$\times[1-x_1(n,\tau,\zeta)\rho]^{-(n-s-\tilde{s})}[1-x_2(n,-\tau,\zeta)\rho]^{-(l+q+1+s-n)}[1-x_2(n,\tau,\zeta)\rho]^{-(l+q+1+\tilde{s})}$$
$$\times{}_2F_1\left(l+q+1+s-n,l+q+1+\tilde{s};2(l+q)+2;\frac{i2^4n\tau^2\zeta}{(1+\tau^2\zeta^2)(n^2-\tau^2)}\right.$$
$$\left.\times\frac{\rho}{[1-x_2(n,-\tau,\zeta)\rho][1-x_2(n,\tau,\zeta)\rho]}\right) \tag{3.24a}$$

and

$$\widetilde{\mathcal{B}}_{nll+q}(\Omega;p)=2|E_n|\cdots\sum_{s=-1,1}d_{n,l}^{(q,s)}\cdots. \tag{3.24b}$$

In Eqs. (3.24) we have introduced the dimensionless variables

$$x_1(n,\tau,\zeta)\equiv\frac{(1-i\tau\zeta)(n-\tau)}{(1+i\tau\zeta)(n+\tau)},\quad x_2(n,\tau,\zeta)\equiv\frac{(1+i\tau\zeta)(n-\tau)}{(1-i\tau\zeta)(n+\tau)}. \tag{3.25}$$

The scalar functions (3.24) can be expressed in terms of a previously introduced [34] generalized hypergeometric function ${}_1F_E$ with six parameters and four variables, as follows:

$$\widetilde{\mathcal{A}}_{nll+q}(\Omega;p)=\frac{1}{2|E_n|}\left(\frac{1}{\pi n}\right)^{1/2}\left(\frac{1}{\hbar\kappa_1}\right)^{3/2}\frac{1}{4\kappa_1}\frac{\tau}{\zeta^2}\frac{(l+q+1)!}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\left[\frac{-i2^4n\tau^2\zeta}{(1+\tau^2\zeta^2)(n^2-\tau^2)}\right]^{l+q+1}$$
$$\times\sum_{s=-2}^{2}\sum_{\tilde{s}=-1,1}\tilde{s}\,c_{n,l}^{(q,s)}\left(\frac{n-\tau}{n+\tau}\right)^{n-s}\left(\frac{1-i\tau\zeta}{1+i\tau\zeta}\right)^{-\tilde{s}}\frac{1}{l+q+1-\tau}{}_1F_E\left(l+q+1-\tau;n-s-\tilde{s},l+q+1+s-n,\right.$$
$$\left.l+q+1+\tilde{s},2(l+q)+2;l+q+2-\tau;x_1(n,\tau,\zeta),x_2(n,-\tau,\zeta),x_2(n,\tau,\zeta),\frac{i2^4n\tau^2\zeta}{(1+\tau^2\zeta^2)(n^2-\tau^2)}\right) \tag{3.26a}$$

and

$$\widetilde{\mathcal{B}}_{nll+q}(\Omega;p)=2|E_n|\cdots\sum_{s=-1,1}d_{n,l}^{(q,s)}\cdots. \tag{3.26b}$$

Notice that in Eqs. (3.26) the parameters $l+q+1+s-n$ are nonpositive integers, so that, according to Eq. (B13), each function ${}_1F_E$ reduces to a finite double sum of Appell functions F_1.

IV. LINEAR-RESPONSE VECTORS FOR PARABOLIC STATIONARY STATES

A. Position representation

The derivation of the LRV's $\boldsymbol{v}_{nn_em}(\Omega;\boldsymbol{r})$ and $\boldsymbol{w}_{nn_em}(\Omega;\boldsymbol{r})$ associated to a parabolic state $|nn_em\rangle$ parallels that presented in Sec. III for the case of a spherical state $|nlm\rangle$. As a first step, we calculate the integral

$$\mathcal{F}_m(\Omega,\beta,\lambda;\xi,\eta,\varphi)\equiv-\int d^3x'\,G(\Omega;\boldsymbol{r},\boldsymbol{r}')\mathcal{U}(i\beta\hat{\boldsymbol{z}}',\lambda;\boldsymbol{r}')(\xi'\eta')^{|m|/2}\Phi_m(\varphi')\quad(m\ \text{integer}), \tag{4.1}$$

where we have denoted

$$\Phi_m(\varphi)\equiv(2\pi)^{-1/2}\exp(im\varphi). \tag{4.2}$$

The integrand in Eq. (4.1) includes a homogeneous harmonic polynomial in the Cartesian variables $x_1=x$ and $x_2=y$,

$$(\xi\eta)^{|m|/2}\Phi_m(\varphi)=(2\pi)^{-1/2}D_{m,j_1\ldots j_{|m|}}x_{j_1}\cdots x_{j_{|m|}}. \tag{4.3}$$

The coefficients D in Eq. (4.3) are totally symmetric with respect to the Cartesian indices and their contractions are zero. In addition, they vanish if at least one of the Cartesian indices is equal to three. Upon use of the function (A10) of I, Eq. (4.2) is equivalent to the identity

$$\mathcal{U}(i\beta\hat{\mathbf{z}},\lambda;\mathbf{r})(\xi\eta)^{|m|/2}\Phi_m(\varphi) = \left(\frac{\hbar}{i}\right)^{|m|}(2\pi)^{-1/2}D_{m,j_1\ldots j_{|m|}}\frac{\partial^{|m|}\mathcal{U}(\mathbf{q},\lambda;\mathbf{r})}{\partial q_{j_1}\cdots\partial q_{j_{|m|}}}\bigg|_{\mathbf{q}=i\beta\hat{\mathbf{z}}}. \tag{4.4}$$

Taking note of the definitions (A9) of I and (4.1), we write down the similar relation

$$\mathcal{F}_m(\Omega,\beta,\lambda;\xi,\eta,\varphi) = \left(\frac{\hbar}{i}\right)^{|m|}(2\pi)^{-1/2}\times D_{m,j_1\ldots j_{|m|}}\frac{\partial^{|m|}\mathcal{F}(\Omega,\mathbf{q},\lambda;\mathbf{r})}{\partial q_{j_1}\cdots\partial q_{j_{|m|}}}\bigg|_{\mathbf{q}=i\beta\hat{\mathbf{z}}}. \tag{4.5}$$

After substituting the integral (A14) of I into Eq. (4.5), we exploit two properties of the functions (A15) and (A16) of I,

$$f(X,-\beta^2,\lambda;\rho)=[f(X,0,\lambda+\beta;\rho)f(X,0,\lambda-\beta;\rho)]^{1/2} \tag{4.6}$$

and

$$\beta z f(X,-\beta^2,\lambda;\rho)+Xrg(X,-\beta^2,\lambda;\rho) = \frac{1}{2}X[\xi\, g(X,0,\lambda+\beta;\rho)+\eta\, g(X,0,\lambda-\beta;\rho)], \tag{4.7}$$

in order to factor the integrand of the resulting integral:

$$\mathcal{F}_m(\Omega,\beta,\lambda;\xi,\eta,\varphi)=\frac{\tau}{Ze^2}(\xi\eta)^{|m|/2}\Phi_m(\varphi)\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\rho^{-\tau}[f(X,0,\lambda+\beta;\rho)]^{(|m|+1)/2}\exp\left[-g(X,0,\lambda+\beta;\rho)\frac{1}{2\tau}\frac{\xi}{a}\right] \times[f(X,0,\lambda-\beta;\rho)]^{(|m|+1)/2}\exp\left[-g(X,0,\lambda-\beta;\rho)\frac{1}{2\tau}\frac{\eta}{a}\right]. \tag{4.8}$$

Second, Eq. (4.8) is employed to evaluate the integral

$$\mathcal{G}^{(s_1,s_2)}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)\equiv-\int d^3x'\,G(\Omega;\mathbf{r},\mathbf{r}')\frac{1}{r'}\exp(-\kappa_n r')(\kappa_n{}^2\xi'\eta')^{|m|/2}{}_1F_1(-n_\xi+s_1;|m|+1;\kappa_n\xi') \times{}_1F_1(-n_\eta+s_2;|m|+1;\kappa_n\eta')\Phi_m(\varphi')\quad (s_1,s_2 \text{ integers},\quad n_\xi-s_1\geqslant 0,\quad n_\eta-s_2\geqslant 0). \tag{4.9}$$

Both Kummer hypergeometric functions in the integrand of Eq. (4.9) are proportional to Laguerre polynomials. Applying for each of them the integral representation (A5), we find

$$\mathcal{G}^{(s_1,s_2)}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)=(\kappa_n)^{|m|}\frac{(|m|)!(n_\xi+1-s_1)!}{(|m|+n_\xi-s_1)!}\;\frac{(|m|)!(n_\eta+1-s_2)!}{(|m|+n_\eta-s_2)!}\left(-\frac{1}{2\pi i}\right)^2\int_1^{(0+)}dt(-t)^{-n_\xi-1+s_1}(1-t)^{|m|+n_\xi-s_1} \times\int_1^{(0+)}du(-u)^{-n_\eta-1+s_2}(1-u)^{|m|+n_\eta-s_2}\mathcal{F}_m[\Omega,(u-t)\hbar\kappa_n,(1-t-u)\hbar\kappa_n;\xi,\eta,\varphi]. \tag{4.10}$$

The structure (4.8) of the function $\mathcal{F}_m$ allows us to factor in the triple contour integral (4.10) two integrals of the type (A6) leading to the formula

$$\mathcal{G}^{(s_1,s_2)}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)=\frac{4n^2\tau}{Ze^2}\Phi_m(\varphi)\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\rho^{-\tau}\exp\left(-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{1}{\tau}\frac{\xi+\eta}{2a}\right)\mathcal{N}_{n,-\tau}^{n-1-s_1-s_2}\mathcal{N}_{n,\tau}^{-n-1+s_1+s_2}(\kappa^2_{n,\tau}\xi\eta)^{|m|/2} \times{}_1F_1(-n_\xi+s_1;|m|+1;\kappa_{n,\tau}\xi)\,{}_1F_1(-n_\eta+s_2;|m|+1;\kappa_{n,\tau}\eta). \tag{4.11}$$

We may express now the contour integral in Eq. (4.11) in terms of a generalized hypergeometric function ${}_2\Phi_H$ with seven parameters and five variables [35]:

$$\mathcal{G}_{n_\xi n_\eta m}^{(s_1,s_2)}(\Omega;\xi,\eta,\varphi)=\frac{\tau}{Ze^2}\left(\frac{n-\tau}{2n}\right)^{n-|m|-1-s_1-s_2}\left(\frac{n+\tau}{2n}\right)^{-n+s_1+s_2-\tau}\exp\left(-\frac{1}{\tau}\frac{\xi+\eta}{2a}\right)(\kappa_n^2\xi\eta)^{|m|/2}\Phi_m(\varphi)\frac{1}{|m|+1-\tau}$$
$$\times{}_2\Phi_H\left(|m|+1-\tau;-n+s_1+s_2+1-\tau,-n_\xi+s_1,-n_\eta+s_2,|m|+1,|m|+1;|m|+2-\tau;\frac{n-\tau}{2n},\right.$$
$$\left.-\frac{(n+\tau)^2}{2n(n-\tau)},\frac{n-\tau}{2\tau}\kappa_n(\xi+\eta),\frac{2n}{n-\tau}\kappa_n\xi,\frac{2n}{n-\tau}\kappa_n\eta\right)\quad(n_\xi-s_1\geqslant 0,\quad n_\eta-s_2\geqslant 0). \tag{4.12}$$

According to Eq. (A9), the function ${}_2\Phi_H$ occurring in Eq. (4.12) reduces to a finite triple sum of Humbert functions Φ_1.

The third step consists of employing the finite expansions of the vectors $\boldsymbol{r}u_{nn_em}(\xi,\eta,\varphi)$ and $\boldsymbol{P}u_{nn_em}(\xi,\eta,\varphi)$ associated to any Stark state $|nn_em\rangle$ [36]. They are used to express the spherical components (A3) of the corresponding LRV's in both gauges in terms of the functions (4.9):

$$v_{nn_em;}{}^{\mu}(\Omega;\boldsymbol{r})=(-1)^{(\mu+|\mu|)/2}2^{-|\mu|/2}\left(\frac{2}{\kappa_n}\right)^{1/2}\frac{1}{[(|m-\mu|)!]^2}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n}\right]^{1/2}\sum_{\widetilde{M}=0,M}\sum_{s=-2}^{2}\left(1-\frac{1}{2}\delta_{M-\widetilde{M}+s,\widetilde{M}}\right)$$
$$\times[c_{n_\xi,n_\eta,|m|}^{(M,\widetilde{M},s)}\mathcal{G}_{n_\xi-M,n_\eta,m-\mu}^{(s-\widetilde{M},\widetilde{M})}(\Omega;\xi,\eta,\varphi)-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)]\quad(\mu=-1,0,1) \tag{4.13a}$$

and

$$w_{nn_em;}{}^{\mu}(\Omega;\boldsymbol{r})=\frac{i}{\hbar}m_e2|E_n|\cdots\sum_{s=-1,1}[d_{n_\xi,n_\eta,|m|}^{(M,\widetilde{M},s)}\cdots]. \tag{4.13b}$$

In Eqs. (4.13) we have used the parameter

$$\widetilde{M}\equiv|m-\mu|-|m|. \tag{4.14}$$

The coefficients $c_{n_\xi,n_\eta,|m|}^{(M,\widetilde{M},s)}$ and $d_{n_\xi,n_\eta,|m|}^{(M,\widetilde{M},s)}$ are listed in Ref. [8], in Tables II and III, respectively. The symbol $(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)$ stands for the preceding expression inside the same brackets with the quoted quantities interchanged. Substitution of the integral representation (4.11) into Eqs. (4.13) yields the LRV's associated to a Stark state as compact contour integrals:

$$v_{nn_em;}{}^{\mu}(\Omega;\boldsymbol{r})=(-1)^{(\mu+|\mu|)/2}2^{-|\mu|/2}\frac{(2\kappa_n)^{1/2}}{2|E_n|}\frac{4n\tau}{[(|m-\mu|)!]^2}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n}\right]^{1/2}\Phi_{m-\mu}(\varphi)$$
$$\times\sum_{\widetilde{M}=0,M}\sum_{s=-2}^{2}\left(1-\frac{1}{2}\delta_{M-\widetilde{M}+s,\widetilde{M}}\right)\left[c_{n_\xi,n_\eta,|m|}^{(M,\widetilde{M},s)}\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\rho^{-\tau}\right.$$
$$\times\exp\left(-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{1}{\tau}\frac{\xi+\eta}{2a}\right)\mathcal{N}_{n,-\tau}^{n-1-s}\mathcal{N}_{n,\tau}^{-n-1+s}(\kappa_{n,\tau}^2\xi\eta)^{(|m|-\mu)/2}{}_1F_1(-n_\xi+M-\widetilde{M}+s;|m-\mu|+1;\kappa_{n,\tau}\xi)$$
$$\left.\times{}_1F_1(-n_\eta+\widetilde{M};|m-\mu|+1;\kappa_{n,\tau}\eta)-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)\right]\quad(\mu=-1,0,1) \tag{4.15a}$$

and

$$w_{nn_em;}{}^{\mu}(\Omega;\boldsymbol{r})=\frac{i}{\hbar}2|E_n|\cdots\sum_{s=-1,1}[d_{n_\xi,n_\eta,|m|}^{(M,\widetilde{M},s)}\cdots]. \tag{4.15b}$$

Introducing finally Eq. (4.12) into Eqs. (4.13), we find the explicit expressions of the LRV's $\mathbf{v}_{nn_em}(\Omega;\xi,\eta,\varphi)$ and $\mathbf{w}_{nn_em}(\Omega;\xi,\eta,\varphi)$ [37]. For instance, the velocity-gauge LRV is

$$\boldsymbol{w}_{nn_em}(\Omega;\xi,\eta,\varphi)=\frac{i}{\hbar}m_e(2\kappa_n)^{1/2}\frac{\tau}{n}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n}\right]^{1/2}\exp\left(-\frac{1}{\tau}\frac{\xi+\eta}{2a}\right)$$
$$\times\sum_{\mu=-1}^{1}\left\{(-1)^{(\mu+|\mu|)/2}2^{-|\mu|/2}\frac{1}{[(|m-\mu|)!]^2}(\kappa_n^2\xi\eta)^{|m-\mu|/2}\Phi_{m-\mu}(\varphi)\sum_{\widetilde{M}=0,M}\sum_{s=-1,1}\left(1-\frac{1}{2}\delta_{M-\widetilde{M}+s,\widetilde{M}}\right)\right.$$
$$\times\left(\frac{n-\tau}{2n}\right)^{n-|m-\mu|-1-s}\left(\frac{n+\tau}{2n}\right)^{-n+s-\tau}\frac{1}{|m-\mu|+1-\tau}\left[d^{(M,\widetilde{M},s)}_{n_\xi,n_\eta,|m|}\ {}_2\Phi_H\left(|m-\mu|+1-\tau;-n+s+1-\tau,\right.\right.$$
$$-n_\xi+M-\widetilde{M}+s,-n_\eta+\widetilde{M},|m-\mu|+1,|m-\mu|+1;|m-\mu|+2-\tau;\frac{n-\tau}{2n},$$
$$\left.\left.\left.-\frac{(n+\tau)^2}{2n(n-\tau)},\frac{n-\tau}{2\tau}\kappa_n(\xi+\eta),\frac{2n}{n-\tau}\kappa_n\xi,\frac{2n}{n-\tau}\kappa_n\eta\right)-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)\right]\right\}\hat{\boldsymbol{\chi}}_\mu. \tag{4.16}$$

Note that the LRV (4.16) includes several functions ${}_2\Phi_H$, each of them expressible as a finite triple sum (A9) of Humbert functions Φ_1.

B. Momentum representation

We have to evaluate the Fourier transforms (2.14) for a Stark eigenstate $|N\rangle=|nn_em\rangle$. We make use of the integral representations (4.15), performing the space integration in parabolic coordinates. The cylindrical-wave expansion (B9) of a plane wave allows one to carry out the integration over the polar angle φ. This is followed by an integration over the coordinate η by applying termwise the integral (B8) to the finite ascending power expansion of the Kummer hypergeometric function with argument $\kappa_{n,\tau}\eta$ in Eq. (4.15a). The integral (B7) is then employed to carry out the integration over the third parabolic coordinate ξ. To evaluate the remaining finite sum mentioned above, we use the integral representation (B10) of the resulting Gauss function ${}_2F_1$, which allows one to perform a binomial sum under the second contour integral. Making use once again of Eq. (B10), we are left with a single contour integral. Two recurrence relations between contiguous Gauss hypergeometric functions [38] finally lead to the following result:

$$\widetilde{v}_{nn_em;}{}^{\mu}(\Omega;\boldsymbol{p})=(-i)^{m-\mu}(-1)^{(1/2)[|m-\mu|-(m+|\mu|)]}2^{-|\mu|/2}\frac{(2\kappa_n)^{1/2}}{2|E_n|}\frac{8\pi}{(2\pi\hbar)^{3/2}}\left(\frac{\hbar}{X}\right)^3\frac{4n\tau}{(|m-\mu|)!}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n}\right]^{1/2}$$
$$\times(8n\tau^2\zeta_\rho)^{|m-\mu|}\Phi_{m-\mu}(\widetilde{\varphi})\sum_{\widetilde{M}=0,M}\sum_{s=-2}^{2}\left(1-\frac{1}{2}\delta_{M-\widetilde{M}+s,\widetilde{M}}\right)\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\rho^{|m-\mu|-\tau}$$
$$\times\left[\left(c^{(M,\widetilde{M},s)}_{n_\xi,n_\eta,|m|}\sum_{\nu_\xi=0}^{1}\sum_{\nu_\eta=0}^{1}\gamma_{\nu_\xi\nu_\eta}\{[1-(\nu_\xi+\nu_\eta)]\mathcal{N}_{\tau,n}\mathcal{N}_{n,\tau}+(\nu_\xi+\nu_\eta)(n^2-\tau^2)(1-\rho^2)\}\right.\right.$$
$$\times[(\mathcal{N}_{\tau,n})^2+(\tau\zeta)^2(\mathcal{N}_{n,\tau})^2]^{\nu_\xi+\nu_\eta-(n-s+1)}[\mathcal{N}_{-\tau,n}\mathcal{N}_{\tau,n}+(\tau\zeta)^2\mathcal{N}_{n,\tau}\mathcal{N}_{n,-\tau}+i8n\tau^2\rho\zeta_z]^{n_\xi-(M-\widetilde{M}+s)-\nu_\xi}$$
$$\times[\mathcal{N}_{-\tau,n}\mathcal{N}_{\tau,n}+(\tau\zeta)^2\mathcal{N}_{n,\tau}\mathcal{N}_{n,-\tau}-i8n\tau^2\rho\zeta_z]^{n_\eta-\widetilde{M}-\nu_\eta}{}_2F_1(-n_\xi+(M-\widetilde{M}+s)+\nu_\xi,$$
$$\left.\left.-n_\eta+\widetilde{M}+\nu_\eta;|m-\mu|+1;y(\rho))\right)-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)\right], \tag{4.17a}$$

$$\widetilde{w}_{nn_em;}{}^{\mu}(\Omega;\boldsymbol{p})=\frac{i}{\hbar}m_e2|E_n|\cdots\left(\sum_{s=-1,1}d^{(M,\widetilde{M},s)}_{n_\xi,n_\eta,|m|}\cdots\right). \tag{4.17b}$$

In Eqs. (4.17), along with already known symbols we have used the dimensionless variable

$$y(\rho)\equiv\frac{(8n\tau^2\rho\zeta_\rho)^2}{[\mathcal{N}_{-\tau,n}\mathcal{N}_{\tau,n}+(\tau\zeta)^2\mathcal{N}_{n,-\tau}\mathcal{N}_{n,\tau}]^2+(8n\tau^2\rho\zeta_z)^2} \tag{4.18}$$

built up with the cylindrical components ζ_ρ and ζ_z of the vector (3.23), as well as the four coefficients $\gamma_{\nu_\xi\nu_\eta}$:

$$\gamma_{00}=n-s,\quad \gamma_{10}=-[n_\xi-(M-\widetilde{M}+s)],\quad \gamma_{01}=-(n_\eta-\widetilde{M}),\quad \gamma_{11}=0. \tag{4.19}$$

According to Eq. (B14), the vectors (4.17) can be written out in terms of the *new* generalized hypergeometric function F_F having the explicit expression (B15). We get

$$\widetilde{v}_{nn_e m;}{}^{\mu}(\Omega;\boldsymbol{p})=(-i)^{m-\mu}(-1)^{(1/2)[|m-\mu|-(m+|\mu|)]}2^{-|\mu|/2}\frac{(2\kappa_n)^{1/2}}{2|E_n|}\frac{8\pi}{(2\pi\hbar)^{3/2}}\left(\frac{\hbar}{X}\right)^3\frac{4n\tau}{(|m-\mu|)!}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n}\right]^{1/2}$$
$$\times\frac{[z(n,\tau,\boldsymbol{\zeta})]^{|m-\mu|}}{(n^2-\tau^2)[1+(\tau\boldsymbol{\zeta})^2]^2}\Phi_{m-\mu}(\widetilde{\varphi})\sum_{\widetilde{M}=0,M}\sum_{s=-2}^{2}\left(1-\frac{1}{2}\delta_{M-\widetilde{M}+s,\widetilde{M}}\right)$$
$$\times\left(\frac{n-\tau}{n+\tau}\right)^{n-s}\Bigg[\Bigg(c^{(M,\widetilde{M},s)}_{n_\xi,n_\eta,|m|}\sum_{\nu_\xi=0}^{1}\sum_{\nu_\eta=0}^{1}\gamma_{\nu_\xi\nu_\eta}\Bigg\{\frac{1}{|m-\mu|+1-\tau}F_F(|m-\mu|+1-\tau;n-s+1-(\nu_\xi+\nu_\eta);$$
$$-[n_\xi-(M-\widetilde{M}+s)-\nu_\xi],-(n_\eta-\widetilde{M}-\nu_\eta);|m-\mu|+1;|m-\mu|+2-\tau;$$
$$x_1(n,\tau,\zeta),x_2(n,\tau,\zeta),y_1(n,\tau,\boldsymbol{\zeta}),y_2(n,\tau,\boldsymbol{\zeta}),y_1(n,\tau,-\boldsymbol{\zeta}),y_2(n,\tau,-\boldsymbol{\zeta}),z(n,\tau,\boldsymbol{\zeta}))$$
$$-\left(\frac{n-\tau}{n+\tau}\right)^{2-2(\nu_\xi+\nu_\eta)}\frac{1}{|m-\mu|+3-\tau}F_F(|m-\mu|+3-\tau;n-s+1-(\nu_\xi+\nu_\eta);-[n_\xi-(M-\widetilde{M}+s)-\nu_\xi],$$
$$-(n_\eta-\widetilde{M}-\nu_\eta);|m-\mu|+1;|m-\mu|+4-\tau;x_1(n,\tau,\zeta),x_2(n,\tau,\zeta),y_1(n,\tau,\boldsymbol{\zeta}),y_2(n,\tau,\boldsymbol{\zeta}),$$
$$y_1(n,\tau,-\boldsymbol{\zeta}),y_2(n,\tau,-\boldsymbol{\zeta}),z(n,\tau,\boldsymbol{\zeta}))\Bigg\}\Bigg)-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)\Bigg]\qquad(\mu=-1,0,1)\tag{4.20a}$$

and

$$\widetilde{w}_{nn_e m;}{}^{\mu}(\Omega;\boldsymbol{p})=\frac{i}{\hbar}m_e2|E_n|\cdots\left(\sum_{s=-1,1}d^{(M,\widetilde{M},s)}_{n_\xi,n_\eta,|m|}\cdots\right).\tag{4.20b}$$

In Eqs. (4.20), besides using the variables (3.25), we have introduced the variables y, which are determined by the conditions

$$y_1(n,\tau,\boldsymbol{\zeta})+y_2(n,\tau,\boldsymbol{\zeta})=\frac{[(n-\tau)^2+(n+\tau)^2][1-(\tau\zeta)^2]-i8n\tau^2\zeta_z}{(n^2-\tau^2)[1+(\tau\zeta)^2]}\tag{4.21a}$$

and

$$y_1(n,\tau,\boldsymbol{\zeta})\ y_2(n,\tau,\boldsymbol{\zeta})=1.\tag{4.21b}$$

We have further denoted

$$z(n,\tau,\boldsymbol{\zeta})\equiv\frac{8n\tau}{n^2-\tau^2}\quad\frac{\tau\zeta_\rho}{1+(\tau\zeta)^2}.\tag{4.22}$$

As shown by Eq. (B16), any function F_F entering Eqs. (4.20) reduces to a finite quintuple sum of Appell functions F_1 of the variables (3.25).

V. REDUCED LINEAR-RESPONSE VECTORS FOR SPHERICAL STATIONARY STATES

A. Position representation

Owing to the identity (3.18), the divergent vector (2.19) written for a state $|nlm\rangle$ has the structure

$$P^{(n)}\boldsymbol{v}_{nlm}(\Omega;\boldsymbol{r})=\sum_{q=1,-1}(-q)\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\times D\mathcal{A}_{n\,l\,l+q}(\Omega;r)\boldsymbol{V}_{l+q\,l\,m}(\hat{\boldsymbol{r}}),\tag{5.1}$$

which for s states becomes

$$P^{(n)}\boldsymbol{v}_{n00}(\Omega;\boldsymbol{r})=(4\pi)^{-1/2}D\mathcal{A}_{n01}(\Omega;r)\hat{\boldsymbol{r}}.\tag{5.2}$$

In Eqs. (5.1) and (5.2), the following singular radial functions are involved:

$$D\mathcal{A}_{n\,l\,l+q}(\Omega;r)=\frac{3}{2\kappa_n}\frac{[n^2-(\lambda_{l+q,l})^2]^{1/2}}{\Omega-E_n}R_{n,l+q}(r)\qquad(q=1\ \text{for}\ \ l=0;\ \ q=1,-1\ \text{for}\ \ l>0).\tag{5.3}$$

Taking the limit $\Omega=E_n$ of the identity (3.19), we find the equation

$$\mathcal{B}_{n\,l\,l+q}(E_n;r)=rR_{nl}(r)+(\Omega-E_n)D\mathcal{A}_{n\,l\,l+q}(\Omega;r).\tag{5.4}$$

Substitution of Eqs. (A10) and (5.3) into Eq. (5.4) provides the formula

$$\mathcal{B}_{n\,l\,l+q}(E_n\,;r)=(2\kappa_n)^{1/2}\frac{1}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\exp(-\kappa_n r)(2\kappa_n r)^{l+q}$$
$$\times\sum_{s=-1}^{1}\left(1-\frac{3}{2}\delta_{s0}\right)e_{n,l}^{(q,s)}\,{}_1F_1(l+q+1-n+s;2(l+q)+2;2\kappa_n r)\quad(q=1\ \text{for}\ \ l=0;\ \ q=1,-1\ \text{for}\ \ l>0). \tag{5.5}$$

The coefficients $e_{n,l}^{(q,s)}$ are listed in Table I. The projection of the LRV (3.14a) onto the subspace $\mathcal{U}_n^{\perp}$ has a similar structure, written previously [39]:

$$\boldsymbol{v}'_{nlm}(\Omega;\boldsymbol{r})=\sum_{q=1,-1}(-q)\times\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\mathcal{A}'_{n\,l\,l+q}(\Omega;r)\boldsymbol{V}_{l+q\,l\,m}(\hat{\boldsymbol{r}}). \tag{5.6}$$

When $l=0$, Eq. (5.6) reads

$$\boldsymbol{v}'_{n00}(\Omega;\boldsymbol{r})=(4\pi)^{-1/2}\mathcal{A}'_{n01}(\Omega;r)\hat{\boldsymbol{r}}. \tag{5.7}$$

In Eqs. (5.6) and (5.7), any factor $\mathcal{A}'_{nll+q}(\Omega;r)$ is the regular part of the corresponding scalar function $\mathcal{A}_{nll+q}(\Omega;r)$,

$$\mathcal{A}'_{n\,l\,l+q}(\Omega;r)=\mathcal{A}_{n\,l\,l+q}(\Omega;r)-D\mathcal{A}_{n\,l\,l+q}(\Omega;r). \tag{5.8}$$

The reduced LRV for a spherical eigenstate is the limit $\Omega=E_n$ of Eq. (5.6),

$$\boldsymbol{v}'_{nlm}(E_n\,;\boldsymbol{r})=\sum_{q=1,-1}(-q)\times\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\mathcal{A}'_{n\,l\,l+q}(E_n\,;r)\boldsymbol{V}_{l+q\,l\,m}(\hat{\boldsymbol{r}}), \tag{5.9}$$

or, in the special case $l=0$, that of Eq. (5.7),

$$\boldsymbol{v}'_{n00}(E_n\,;\boldsymbol{r})=(4\pi)^{-1/2}\mathcal{A}'_{n01}(E_n\,;r)\hat{\boldsymbol{r}}. \tag{5.10}$$

TABLE I. The coefficients $e_{n,l}^{(q,s)}$ in the expansions (A10) of the function $rR_{nl}(r)$.

q	s	$e_{n,l}^{(q,s)}$
1	-1	$(n+l+1)(n+l+2)$
1	0	$-2(n+l+1)(n-l-1)$
1	1	$(n-l-1)(n-l-2)$
-1	-1	1
-1	0	-2
-1	1	1

The reduced scalar radial functions $\mathcal{A}'_{nll+q}(E_n\,;r)$ have been evaluated in two different ways by making use of the explicit form either of the functions $\mathcal{A}_{nll+q}(\Omega;r)$ or of the factors $\mathcal{B}_{nll+q}(\Omega;r)$ [31].

(i) The first way consists of writing the right-hand side of Eq. (5.8) explicitly and then selecting all nonvanishing terms of its limit $\tau=n$.

(ii) The second way exploits the identity

$$\mathcal{A}'_{n\,l\,l+q}(E_n\,;r)=\left.\frac{\partial\mathcal{B}_{nll+q}(\Omega;r)}{\partial\Omega}\right|_{\Omega=E_n}, \tag{5.11}$$

derived from Eqs. (3.19), (5.8), and (5.3). From the explicit expression of $\mathcal{B}_{nll+q}(E_n+\delta\Omega;r)$, we collect all terms proportional to $\delta\tau\equiv\tau-n$ owing to the relationship

$$\delta\Omega=|E_n|\left[1-\left(1+\frac{\delta\tau}{n}\right)^{-2}\right]. \tag{5.12}$$

Both methods give a result reported previously [40], that we shall apply below:

$$\mathcal{A}'_{n\,l\,l+q}(E_n\,;r)=\frac{(2\kappa_n)^{1/2}}{2|E_n|[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\times\exp(-\kappa_n r)(2\kappa_n r)^{l+q}\sum_{s=-2}^{2}C_{n,l}^{(0;q,s)}\times{}_1F_1(l+q+1-n+s;2(l+q)+2;2\kappa_n r)$$
$$(q=1\ \text{if}\ \ l=0;\quad q=1,-1\ \text{if}\ \ l>0). \tag{5.13}$$

TABLE II. The coefficients $C_{n,l}^{(0;q,s)}$ in the expansions (5.13) of the reduced linear-response radial functions $\mathcal{A}'_{nl\,l+q}(E_n\,;r)$.

q	s	$C_{n,l}^{(0;q,s)}$
1	-2	$-\frac{1}{2}(n+l+1)(n+l+2)(n+l+3)$
1	-1	$(n+l+1)(n+l+2)(n+l+3)$
1	0	$(n+l+1)(n-l-1)(2l+5)$
1	1	$-(n-l-1)(n-l-2)(n-l-3)$
1	2	$\frac{1}{2}(n-l-1)(n-l-2)(n-l-3)$
-1	-2	$-\frac{1}{2}(n+l+1)$
-1	-1	$n-l+2$
-1	0	$-(2l-3)$
-1	1	$-(n+l-2)$
-1	2	$\frac{1}{2}(n-l-1)$

The coefficients $C_{n,l}^{(0;q,s)}$ are listed in Table II. It is worth noting that Jhanwar and Meath [41] derived the z-axis component of the reduced LRV (5.9). They found it by solving the appropriate inhomogeneous radial differential equations.

B. Momentum representation

We employ the technique developed in Sec. III B to evaluate the Fourier transform of the coordinate-space projection (5.1) and find the expression

$$P^{(n)}\tilde{\boldsymbol{v}}_{nlm}(\Omega;\boldsymbol{p})=\sum_{q=1,-1}(-q)\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\times D\tilde{\mathcal{A}}_{n\,l\,l+q}(\Omega;p)\boldsymbol{V}_{l+q\,l\,m}(\hat{\boldsymbol{p}}), \tag{5.14}$$

which is simpler for an s state:

$$P^{(n)}\tilde{\boldsymbol{v}}_{n00}(\Omega;\boldsymbol{p})=(4\pi)^{-1/2}D\tilde{\mathcal{A}}_{n01}(\Omega;p)\hat{\boldsymbol{p}}. \tag{5.15}$$

The singular radial factors in Eqs. (5.14) and (5.15) are

$$D\tilde{\mathcal{A}}_{n\,l\,l+q}(\Omega;p)=(-i)^{l+q}\frac{3}{2\kappa_n}\frac{[n^2-(\lambda_{l+q,l})^2]^{1/2}}{\Omega-E_n}\frac{2^{2(l+q)+3}n}{\pi^{1/2}(\hbar\kappa_n)^{3/2}}\frac{(l+q)!}{[2(l+q)+1]!}\left[\frac{(n+l+q)!}{(n-l-q-1)!2n}\right]^{1/2}\frac{(n\zeta)^{l+q}}{[(n\zeta)^2+1]^{l+q+2}}\times{}_2F_1\left(l+q+1-n,l+q+1+n;l+q+\frac{3}{2};\frac{1}{(n\zeta)^2+1}\right). \tag{5.16}$$

Note that the secular term (5.16) vanishes for $l+q=n$.

By Fourier transforming the vector $\boldsymbol{w}_{nlm}(E_n;\boldsymbol{r})$ characterized by the scalar factors (5.5), we get a similar structure in momentum space, with the factors

$$\tilde{\mathcal{B}}_{n\,l\,l+q}(E_n;p)=(-i)^{l+q}\frac{2^{2(l+q)+3}n}{\pi^{1/2}(\hbar\kappa_n)^{3/2}\kappa_n}\frac{(l+q)!}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\frac{(n\zeta)^{l+q}}{[(n\zeta)^2+1]^{l+q+2}}\times\sum_{s=-1}^{1}\left(1-\frac{3}{2}\delta_{s0}\right)(n-s)e_{n,l}^{(q,s)}{}_2F_1\left(l+q+1-n+s,l+q+1+n-s;l+q+\frac{3}{2};\frac{1}{(n\zeta)^2+1}\right)$$

$$(q=1\quad\text{if}\quad l=0;\qquad q=1,-1\quad\text{if}\quad l>0). \tag{5.17}$$

When applying the same transformations to the reduced LRV $\boldsymbol{v}'_{nlm}$ in coordinate space, Eq. (5.9), we get its counterpart in momentum space:

$$\tilde{\boldsymbol{v}}'_{nlm}(E_n;\boldsymbol{p})=\sum_{q=1,-1}(-q)\left(\frac{|\lambda_{l+q,l}|}{2l+1}\right)^{1/2}\tilde{\mathcal{A}}'_{n\,l\,l+q}(E_n;p)\boldsymbol{V}_{l+q\,l\,m}(\hat{\boldsymbol{p}}). \tag{5.18}$$

For $l=0$, Eq. (5.18) reads

$$\tilde{\boldsymbol{v}}'_{n00}(E_n;\boldsymbol{p})=(4\pi)^{-1/2}\tilde{\mathcal{A}}'_{n01}(E_n;p)\hat{\boldsymbol{p}}. \tag{5.19}$$

The scalar functions $\tilde{\mathcal{A}}'_{nll+q}(E_n;p)$ in Eqs. (5.18) and (5.19) are

$$\tilde{\mathcal{A}}'_{n\,l\,l+q}(E_n;p)=(-i)^{l+q}\frac{1}{2|E_n|}\frac{2^{2(l+q)+1}}{\pi^{1/2}(\hbar\kappa_n)^{3/2}\kappa_n}\frac{(l+q)!}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\frac{(n\zeta)^{l+q}}{[(n\zeta)^2+1]^{l+q+2}}\times\sum_{s=-2}^{2}(n-s)C_{n,l}^{(0;q,s)}{}_2F_1\left(l+q+1-n+s;l+q+1+n-s;l+q+\frac{3}{2};\frac{1}{(n\zeta)^2+1}\right)$$

$$(q=1\quad\text{if}\quad l=0;\qquad q=1,-1\quad\text{if}\quad l>0). \tag{5.20}$$

Remark that, according to Eq. (B3), all the Gauss hypergeometric functions included in Eqs. (5.16), (5.17), and (5.20) are Gegenbauer polynomials of the dimensionless variable $q_0=[(n\zeta)^2-1]/[(n\zeta)^2+1]$. [See Eq. (6.11) below.]

VI. REDUCED LINEAR-RESPONSE VECTORS FOR PARABOLIC STATIONARY STATES

A. Position representation

The secular term (2.19) of the velocity-gauge vector for a Stark state $\boldsymbol{v}_{nn_e m}(\Omega;\boldsymbol{r})$ can be extracted from the explicit expression of the latter [37]:

$$P^{(n)}\boldsymbol{v}_{nn_e m}(\Omega;\boldsymbol{r})=\frac{(2\kappa_n)^{1/2}}{\Omega-E_n}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n}\right]^{1/2}\left(-\frac{1}{n}\right)\exp\left(-\frac{1}{2}\kappa_n(\xi+\eta)\right)\sum_{\mu=-1}^{1}(-1)^{(\mu+|\mu|)/2}2^{-|\mu|/2}$$
$$\times\frac{1}{[(|m-\mu|)!]^2}\Phi_{m-\mu}(\varphi)(\kappa_n^2\xi\eta)^{|m-\mu|/2}\left(1-\frac{1}{2}\delta_{M,0}\right)[c^{(M,0,0)}_{n_\xi,n_\eta,|m|}{}_1F_1(-n_\xi+M;|m-\mu|+1;\kappa_n\xi)$$
$$\times{}_1F_1(-n_\eta;|m-\mu|+1;\kappa_n\eta)-(-1)^\mu(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)]\hat{\boldsymbol{\chi}}_\mu. \tag{6.1}$$

Equation (6.1) has been reported before [42]. The reduced LRV (2.25) for a Stark state $\boldsymbol{v}'_{nn_e m}(E_n;\boldsymbol{r})$ has been evaluated in Ref. [8] in three independent ways [43]. We need here its explicit expression:

$$\boldsymbol{v}'_{nn_e m}(E_n;\boldsymbol{r})=\frac{(2\kappa_n)^{1/2}}{4|E_n|}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n}\right]^{1/2}\exp\left(-\frac{1}{2}\kappa_n(\xi+\eta)\right)\sum_{\mu=-1}^{1}(-1)^{(\mu+|\mu|)/2}2^{-|\mu|/2}\frac{1}{[(|m-\mu|)!]^2}\Phi_{m-\mu}(\varphi)$$
$$\times(\kappa_n^2\xi\eta)^{|m-\mu|/2}\Bigg\{\Bigg[-2\delta_{\mu,0}(|m|+1+n_\xi)n_\eta{}_1F_1(-n_\xi-1;|m|+1;\kappa_n\xi){}_1F_1(-n_\eta+1;|m|+1;\kappa_n\eta)$$
$$+\sum_{\widetilde{M}=0,M}\sum_{s=-2}^{2}\left(1-\frac{1}{2}\delta_{M-\widetilde{M}+s,\widetilde{M}}\right)C^{(0;M,\widetilde{M},s)}_{n_\xi,n_\eta,|m|}{}_1F_1(-n_\xi+M-\widetilde{M}+s;|m-\mu|+1;\kappa_n\xi)$$
$$\times{}_1F_1(-n_\eta+\widetilde{M};|m-\mu|+1;\kappa_n\eta)\Bigg]-(-1)^\mu(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)\Bigg\}\hat{\boldsymbol{\chi}}_\mu. \tag{6.2}$$

The coefficients $C^{(0;M,\widetilde{M},s)}_{n_\xi,n_\eta,|m|}$ are listed in Table V of Ref. [8]. Notice that each Kummer function in Eqs. (6.1) and (6.2) is proportional to a Laguerre polynomial.

B. Momentum representation

We carry out the Fourier transformations of the vectors (6.1) and (6.2) along the lines presented in Sec. IV B. We get the divergent projection onto $\mathcal{U}_n$,

$$P^{(n)}\tilde{\boldsymbol{v}}_{nn_e m}(\Omega;\boldsymbol{p})=\frac{1}{\Omega-E_n}\frac{-4}{\pi^{1/2}(\hbar\kappa_n)^{3/2}\kappa_n}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n}\right]^{1/2}\frac{1}{[(n\zeta)^2+1]^{n+1}}$$
$$\times\sum_{\mu=-1}^{1}(-1)^{[|m-\mu|-(m+\mu)]/2}(-i)^{m-\mu}2^{-|\mu|/2}\frac{1}{(|m-\mu|)!}\Phi_{m-\mu}(\widetilde{\varphi})(2n\zeta_\rho)^{|m-\mu|}\left(1-\frac{1}{2}\delta_{M,0}\right)$$
$$\times\Bigg\{\Bigg[c^{(M,0,0)}_{n_\xi,n_\eta,|m|}[(n\zeta)^2-1+2in\zeta_z]^{n_\xi-M}[(n\zeta)^2-1-2in\zeta_z]^{n_\eta}$$
$$\times{}_2F_1\left(-n_\xi+M,-n_\eta;|m-\mu|+1;-\frac{(2n\zeta_\rho)^2}{[(n\zeta)^2-1]^2+(2n\zeta_z)^2}\right)\Bigg]-(-1)^\mu[\zeta_\xi\leftrightarrow\zeta_\eta,n_\xi\leftrightarrow n_\eta]\Bigg\}\hat{\boldsymbol{\chi}}_\mu, \tag{6.3}$$

and the reduced LRV

$$\tilde{\boldsymbol{v}}'_{nn_e m}(E_n;\boldsymbol{p}) = \frac{1}{2|E_n|}\,\frac{2}{\pi^{1/2}(\hbar\kappa_n)^{3/2}\kappa_n}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n}\right]^{1/2}$$
$$\times\sum_{\mu=-1}^{1}(-1)^{(1/2)[|m-\mu|-(m+|\mu|)]}(-i)^{m-\mu}2^{-|\mu|/2}\frac{1}{(|m-\mu|)!}\Phi_{m-\mu}(\widetilde{\varphi})(2n\zeta_\rho)^{|m-\mu|}$$
$$\times\Bigg\{\Bigg[-2\delta_{\mu,0}(|m|+1+n_\xi)n_\eta\frac{n}{[(n\zeta)^2+1]^{n+1}}[(n\zeta)^2-1+2in\zeta_z]^{n_\xi+1}[(n\zeta)^2-1-2in\zeta_z]^{n_\eta-1}$$
$$\times{}_2F_1\left(-n_\xi-1,-n_\eta+1;|m|+1;-\frac{(2n\zeta_\rho)^2}{[(n\zeta)^2-1]^2+(2n\zeta_z)^2}\right)+\sum_{\widetilde{M}=0,M}\sum_{s=-2}^{2}\left(1-\frac{1}{2}\delta_{M-\widetilde{M}+s,\widetilde{M}}\right)$$
$$\times C^{(0;M,\widetilde{M},s)}_{n_\xi,n_\eta,|m|}\frac{n-s}{[(n\zeta)^2+1]^{n-s+1}}[(n\zeta)^2-1+2in\zeta_z]^{n_\xi-(M-\widetilde{M}+s)}[(n\zeta)^2-1-2in\zeta_z]^{n_\eta-\widetilde{M}}$$
$$\times{}_2F_1\left(-n_\xi+M-\widetilde{M}+s,-n_\eta+\widetilde{M};|m-\mu|+1;-\frac{(2n\zeta_\rho)^2}{[(n\zeta)^2-1]^2+(2n\zeta_z)^2}\right)\Bigg]$$
$$-(-1)^\mu[\zeta_\xi\leftrightarrow\zeta_\eta,n_\xi\leftrightarrow n_\eta]\Bigg\}\hat{\boldsymbol{\chi}}_\mu. \tag{6.4}$$

The vectors (6.3) and (6.4) can be expressed via Jacobi polynomials, Eq. (B2), in terms of matrix elements of the $SU(2)$ irreducible representations,

$$\mathcal{D}^{(j)}_{mn}(\alpha,\beta,\gamma)=\exp(-im\alpha)d^{(j)}_{mn}(\beta)\exp(-in\gamma), \tag{6.5}$$

where α,β,γ are the Euler angles [44]. These are introduced as follows:

$$\alpha=\widetilde{\psi}+\left(\widetilde{\varphi}-\frac{\pi}{2}\right),\qquad \beta=\arccos\left(1-\frac{8(n\zeta_\rho)^2}{[(n\zeta)^2+1]^2}\right),$$
$$\gamma=\widetilde{\psi}-\left(\widetilde{\varphi}-\frac{\pi}{2}\right). \tag{6.6}$$

The angle $\widetilde{\varphi}$ is already known to be the longitude in momentum space, while $\widetilde{\psi}$ denotes the principal value of the argument of a complex number occurring in Eqs. (6.3) and (6.4):

$$\widetilde{\psi}\equiv\arg[(n\zeta)^2-1+2in\zeta_z]. \tag{6.7}$$

Using the Euler angle parametrization of an $SU(2)$ transformation

$$U=U(\alpha,\beta,\gamma) \tag{6.8}$$

that is specified by Eqs. (6.6), we find the alternative expressions

$$P^{(n)}\tilde{\boldsymbol{v}}_{nn_e m}(\Omega;\boldsymbol{p})=\frac{1}{\Omega-E_n}\,\frac{-1}{\pi\kappa_n(\hbar\kappa_n)^{3/2}}\,\frac{1}{[(n\zeta)^2+1]^2}\sum_{\mu=-1}^{1}(-1)^{(\mu+|\mu|)/2}2^{-|\mu|/2+1/2}\left(1-\frac{1}{2}\delta_{M,0}\right)$$
$$\times\left[\left\{c^{(M,0,0)}_{n_\xi,n_\eta,|m|}\left(\frac{(n_\xi-M)!(n_\eta+|m|)!}{n_\xi!(n_\eta+|m|+M)!n}\right)^{1/2}\mathcal{D}^{((n-1)/2)}_{-(m-\mu+n_e-M)/2,(m-\mu-n_e+M)/2}(U)\right\}\right.$$
$$\left.-(-1)^\mu\{\zeta_\xi\leftrightarrow\zeta_\eta,n_\xi\leftrightarrow n_\eta\}\right]\hat{\boldsymbol{\chi}}_\mu \tag{6.9}$$

and

$$\tilde{\boldsymbol{v}}'_{nn_em}(E_n,\boldsymbol{p})=\frac{1}{2|E_n|}\frac{1}{\pi\kappa_n(\hbar\kappa_n)^{3/2}}\left(\frac{1}{4n}\right)^{1/2}\frac{1}{[(n\zeta)^2+1]^2}\sum_{\mu=-1}^{1}(-1)^{(\mu+|\mu|)/2}2^{-|\mu|/2+1/2}$$
$$\times\Bigg[\Bigg\{-2\,\delta_{\mu,0}n[(|m|+1+n_\xi)n_\xi(n_\xi+1)(n_\eta+|m|)]^{1/2}\mathcal{D}^{((n-1)/2)}_{-(m+n_e+2)/2,(m-n_e-2)/2}(U)$$
$$+\sum_{\tilde{M}=0,M}\sum_{s=-2}^{2}\left(1-\frac{1}{2}\delta_{M-\tilde{M}+s,\tilde{M}}\right)(n-s)C^{(0;M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}\left(\frac{(n_\xi+1)_{|m|}(n_\eta+1)_{|m|}}{(n_\xi-(M-\tilde{M}+s)+1)_{|m-\mu|}(n_\eta-\tilde{M}+1)_{|m-\mu|}}\right)^{1/2}$$
$$\times\mathcal{D}^{((n-s-1)/2)}_{-[m-\mu+n_e-(M-2\tilde{M}+s)]/2,[m-\mu-n_e+(M-2\tilde{M}+s)]/2}(U)\Bigg\}-(-1)^{\mu}\{\zeta_\xi\leftrightarrow\zeta_\eta,n_\xi\leftrightarrow n_\eta\}\Bigg]\hat{\boldsymbol{\chi}}_\mu. \tag{6.10}$$

Notice that the momentum $\boldsymbol{p}$ equally determines the Euler-Rodrigues parameters [45] of the $SU(2)$ operator (6.8):

$$q_0=\frac{(n\zeta)^2-1}{(n\zeta)^2+1},\quad \boldsymbol{q}=\frac{2n\zeta}{(n\zeta)^2+1}\hat{\boldsymbol{p}}. \tag{6.11}$$

Accordingly, we get the corresponding unit vector $\hat{\boldsymbol{u}}$ of the rotation axis as well as the rotation angle A [45]:

$$\hat{\boldsymbol{u}}=\hat{\boldsymbol{p}},\qquad \cot\frac{A}{4}=n\zeta. \tag{6.12}$$

VII. LINEAR RESPONSE FROM THE ATOMIC GROUND STATE

A. Position representation

In the special case of the ground state the LRV's are of the type (3.16)

$$\boldsymbol{v}_{100}(\Omega;\boldsymbol{r})=(4\pi)^{-1/2}\mathcal{A}_{101}(\Omega;r)\hat{\boldsymbol{r}} \tag{7.1a}$$

and

$$\boldsymbol{w}_{100}(\Omega;\boldsymbol{r})=\frac{i}{\hbar}m_e(4\pi)^{-1/2}\mathcal{B}_{101}(\Omega;r)\hat{\boldsymbol{r}}. \tag{7.1b}$$

According to Eqs. (3.17), the scalar radial functions have the following integral representations: in the length gauge,

$$\mathcal{A}_{101}(\Omega;r)=\frac{1}{2|E_1|}a^{-3/2}2\tau r\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\times\int_1^{(0+)}d\rho\rho^{1-\tau}\exp\left(-\frac{\mathcal{N}_{\tau,1}}{\mathcal{N}_{1,\tau}}\frac{1}{\tau}\frac{r}{a}\right)\times\left(\frac{2}{\mathcal{N}_{1,\tau}}\right)^5\left[2\tau(1-\rho)+\frac{2}{\mathcal{N}_{1,\tau}}\rho\frac{r}{a}\right] \tag{7.2a}$$

and in the velocity gauge,

$$\mathcal{B}_{101}(\Omega;r)=a^{-3/2}2\tau r\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\rho^{1-\tau}\times\exp\left(-\frac{\mathcal{N}_{\tau,1}}{\mathcal{N}_{1,\tau}}\frac{1}{\tau}\frac{r}{a}\right)\left(\frac{2}{\mathcal{N}_{1,\tau}}\right)^4. \tag{7.2b}$$

Consequently, they have simple explicit expressions [31] in terms of Humbert hypergeometric functions Φ_1:

$$\mathcal{A}_{101}(\Omega;r)=\frac{1}{2|E_1|}a^{-3/2}\left(\frac{2}{1+\tau}\right)^{3+\tau}2\tau r\exp\left(-\frac{1}{\tau}\frac{r}{a}\right)\frac{1}{3-\tau}\times\left[\frac{2\tau}{2-\tau}\Phi_1\left(2-\tau,-1-\tau,4-\tau;\frac{1-\tau}{2},\frac{1-\tau}{\tau}\frac{r}{a}\right)+\frac{r}{a}\Phi_1\left(3-\tau,-2-\tau,4-\tau;\frac{1-\tau}{2},\frac{1-\tau}{\tau}\frac{r}{a}\right)\right] \tag{7.3a}$$

and

$$\mathcal{B}_{101}(\Omega;r)=a^{-3/2}\left(\frac{2}{1+\tau}\right)^{2+\tau}2\tau r\exp\left(-\frac{1}{\tau}\frac{r}{a}\right)\frac{1}{2-\tau}\times\Phi_1\left(2-\tau,-1-\tau,3-\tau;\frac{1-\tau}{2},\frac{1-\tau}{\tau}\frac{r}{a}\right). \tag{7.3b}$$

It goes without saying that we find Eqs. (7.1) and (7.2) again when we specialize to the case of the ground state the integral representations (4.15) of the LRV's for an arbitrary Stark state. Similarly, Eqs. (7.1) and (7.3) are obtained by the same particularization from the explicit expressions of the vectors $\boldsymbol{v}_{nn_em}(\Omega;\boldsymbol{r})$ [37] and $\boldsymbol{w}_{nn_em}(\Omega;\boldsymbol{r})$, Eq. (4.16). Recall that the neglect of retardation amounts to replacing both Poincaré and multipolar gauges by the length gauge, and the radiation gauge by the velocity gauge [22]. As a consequence, Eqs. (6.10a) and (6.12a) from I, written with $L=1$, are equivalent to Eqs. (7.1a), (7.2a), and (7.3a), while Eqs.

(3.13)–(3.15) from I coincide with Eqs. (7.1b), (7.2b), and (7.3b). Note also that relation (3.18), when taken for the ground state,

$$\mathcal{B}_{101}(\Omega;r)=rR_{10}(r)+(\Omega-E_1)\mathcal{A}_{101}(\Omega;r), \tag{7.4}$$

is obtained directly after an integration by parts in Eq. (7.2a). However, it is also retrievable upon comparison between Eqs. (6.10a) and (6.10b) from I or between Eqs. (6.12a) and (6.12b) from I, when both pairs are written for $L=1$.

B. Momentum representation

The LRV's for the ground state are of the form (3.21):

$$\tilde{\boldsymbol{v}}_{100}(\Omega;\boldsymbol{p})=(4\pi)^{-1/2}\tilde{\mathcal{A}}_{101}(\Omega;p)\hat{\boldsymbol{p}} \tag{7.5a}$$

and

$$\tilde{\boldsymbol{w}}_{100}(\Omega;\boldsymbol{p})=\frac{i}{\hbar}m_e(4\pi)^{-1/2}\tilde{\mathcal{B}}_{101}(\Omega;p)\hat{\boldsymbol{p}}. \tag{7.5b}$$

The integral representations (3.24) of the scalar functions from the right-hand sides of Eqs. (7.5) are

$$\tilde{\mathcal{A}}_{101}(\Omega;p)=\frac{1}{2|E_1|}\frac{1}{(2\pi)^{1/2}(\hbar\kappa_1)^{3/2}\kappa_1}\frac{2^8\tau^5}{(1+\tau)^4[1+(\tau\zeta)^2]^2}\sum_{\tilde{s}=-1,1}\tilde{s}\left(\frac{1-i\tau\zeta}{1+i\tau\zeta}\right)^{-\tilde{s}}\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}$$
$$\times\int_1^{(0+)}d\rho\rho^{1-\tau}\Bigg\{\frac{1-\tau}{1+\tau}\left(1-\frac{1}{x_2}\rho\right)(1-x_1\rho)^{-(3-\tilde{s})}(1-x_2\rho)^{-(2+\tilde{s})}-(2+\tilde{s})\frac{4i\tau^2\zeta}{(1+\tau)^2[1+(\tau\zeta)^2]}$$
$$\times\rho(1-x_1\rho)^{-(3-\tilde{s})}(1-x_2\rho)^{-(3+\tilde{s})}-(1-x_1\rho)^{-(2-\tilde{s})}(1-x_2\rho)^{-(2+\tilde{s})}\Bigg\}, \tag{7.6a}$$

and

$$\tilde{\mathcal{B}}_{101}(\Omega;p)=-\frac{1}{(2\pi)^{1/2}(\hbar\kappa_1)^{3/2}\kappa_1}\frac{2^7\tau^5}{(1+\tau)^4[1+(\tau\zeta)^2]^2}\sum_{\tilde{s}=-1,1}\tilde{s}\left(\frac{1-i\tau\zeta}{1+i\tau\zeta}\right)^{-\tilde{s}}\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}$$
$$\times\int_1^{(0+)}d\rho\rho^{1-\tau}(1-x_1\rho)^{-(2-\tilde{s})}(1-x_2\rho)^{-(2+\tilde{s})}. \tag{7.6b}$$

The explicit expressions of the linear-response scalar functions (7.6) can be found either directly or by particularization of Eqs. (3.26). We get

$$\tilde{\mathcal{A}}_{101}(\Omega;p)=\frac{1}{2|E_1|}\frac{1}{(2\pi)^{1/2}(\hbar\kappa_1)^{3/2}\kappa_1}\frac{2^8\tau^5}{(1+\tau)^4[1+(\tau\zeta)^2]^2}\sum_{\tilde{s}=-1,1}\tilde{s}\left(\frac{1-i\tau\zeta}{1+i\tau\zeta}\right)^{-\tilde{s}}$$
$$\times\Bigg\{\frac{1-\tau}{1+\tau}\left[\frac{1}{2-\tau}F_1(2-\tau;3-\tilde{s},2+\tilde{s};3-\tau;x_1,x_2)-\frac{1}{x_2}\frac{1}{3-\tau}F_1(3-\tau;3-\tilde{s},2+\tilde{s};4-\tau;x_1,x_2)\right]$$
$$-\frac{4i\tau^2\zeta}{(1+\tau)^2[1+(\tau\zeta)^2]}\frac{2+\tilde{s}}{3-\tau}F_1(3-\tau;3-\tilde{s},3+\tilde{s};4-\tau;x_1,x_2)-\frac{1}{2-\tau}F_1(2-\tau;2-\tilde{s},2+\tilde{s};3-\tau;x_1,x_2)\Bigg\} \tag{7.7a}$$

and

$$\widetilde{\mathcal{B}}_{101}(\Omega;p) = -\frac{1}{(2\pi)^{1/2}(\hbar\kappa_1)^{3/2}\kappa_1}\frac{2^7\tau^5}{(1+\tau)^4[1+(\tau\zeta)^2]^2}\sum_{\tilde{s}=-1,1}\tilde{s}\left(\frac{1-i\tau\zeta}{1+i\tau\zeta}\right)^{-\tilde{s}}\frac{1}{2-\tau}F_1(2-\tau;2-\tilde{s},2+\tilde{s};3-\tau;x_1,x_2). \quad (7.7b)$$

In Eqs. (7.6) and (7.7), the variables x_1 and x_2 correspond to

$$x_1 = x_1(1,\tau,\zeta), \quad x_2 = x_2(1,\tau,\zeta). \quad (7.8)$$

Obviously, Eqs. (7.5) and (7.6) have been retrieved from the general formulas (4.17) which present the LRV's for a Stark state as contour integrals. Specialization to the ground state of the explicit formulas (4.20) of the LRV's $\tilde{\boldsymbol{v}}_{nn_em}(\Omega;\boldsymbol{p})$ and $\tilde{\boldsymbol{w}}_{nn_em}(\Omega;\boldsymbol{p})$ yields alternative explicit expressions of the scalar functions $\widetilde{\mathcal{A}}_{101}(\Omega;p)$ and $\widetilde{\mathcal{B}}_{101}(\Omega;p)$ which are equivalent to Eqs. (7.7). Here we only mention that they are written in terms of seven and, respectively, two Appell functions F_1, Eq. (B2) of I, with $b'=b$.

C. Low-frequency behavior: Position representation

The ground state is the only atomic stationary state for which the projection (2.19) vanishes,

$$P^{(1)}|\boldsymbol{v}_{100}\rangle = \boldsymbol{0}, \quad (7.9)$$

so that the LRV

$$|\boldsymbol{v}_{100}(\Omega)\rangle = |\boldsymbol{v}'_{100}(\Omega)\rangle \quad (7.10)$$

has a finite static limit (2.22), which in coordinate space reads

$$\boldsymbol{v}_{100}(E_1;\boldsymbol{r}) = (4\pi)^{-1/2}\mathcal{A}_{101}(E_1;r)\hat{\boldsymbol{r}}. \quad (7.11)$$

Consequently, the length-gauge linear-response correction (2.9a) to the ground-state wave function,

$$\Psi^{(1)}_{(L)\,100}(\omega;\boldsymbol{r},t) = -\frac{1}{2}(4\pi)^{-1/2}e\mathcal{E}_0\exp\left(-\frac{i}{\hbar}E_1t\right)\times[\exp(-i\omega t)(\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{r}})\mathcal{A}_{101}(E_1+\hbar\omega+i0;r) + \exp(i\omega t)(\hat{\boldsymbol{\epsilon}}^*\cdot\hat{\boldsymbol{r}})\mathcal{A}_{101}(E_1-\hbar\omega;r)] \quad (7.12)$$

is regular in the range of low frequencies. In Eq. (7.12) we replace the function $\mathcal{A}_{101}(E_1+\delta\Omega;r)$ by the first three terms in its Taylor series, which is a good approximation when $|\delta\Omega|\ll|E_1|$. We get these terms from Eq. (7.3) of I taken for $L=1$:

$$\mathcal{A}_{101}(E_1+\delta\Omega;r) = \frac{1}{2|E_1|}a^{-3/2}\exp\left(-\frac{r}{a}\right)2r\left\{1+\frac{1}{2}\frac{r}{a} + \frac{\delta\Omega}{2|E_1|}\left[\frac{11}{6}+\frac{11}{12}\frac{r}{a}+\frac{1}{6}\left(\frac{r}{a}\right)^2\right] + \left(\frac{\delta\Omega}{2|E_1|}\right)^2\left[\frac{287}{72}+\frac{287}{144}\frac{r}{a}+\frac{31}{72}\left(\frac{r}{a}\right)^2 + \frac{1}{24}\left(\frac{r}{a}\right)^3\right] + O\left[\left(\frac{\delta\Omega}{2|E_1|}\right)^3\right]\right\}. \quad (7.13)$$

Therefore, for low frequencies ($\hbar\omega\ll|E_1|$) and in the special case of linear polarization, Eq. (7.12) becomes

$$\Psi^{(1)}_{(L)\,100}(\omega;\boldsymbol{r},t) = -\frac{e\mathcal{E}_0}{2|E_1|}(\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{r}})\left(\left\{1+\frac{1}{2}\frac{r}{a}+\left(\frac{\hbar\omega}{2|E_1|}\right)^2\times\left[\frac{287}{72}+\frac{287}{144}\frac{r}{a}+\frac{31}{72}\left(\frac{r}{a}\right)^2+\frac{1}{24}\left(\frac{r}{a}\right)^3\right]\right\}\times\cos(\omega t) - i\frac{\hbar\omega}{2|E_1|}\left[\frac{11}{6}+\frac{11}{12}\frac{r}{a}+\frac{1}{6}\left(\frac{r}{a}\right)^2\right]\times\sin(\omega t) + O\left[\left(\frac{\hbar\omega}{2|E_1|}\right)^3\right]\right)\Psi^{(0)}_{100}(\boldsymbol{r},t). \quad (7.14)$$

According to Eq. (2.9a), the atomic linear response from the ground state to a *static uniform electric field* $\mathbf{E}=\mathcal{E}_0\hat{\boldsymbol{\epsilon}}$ is determined by the reduced LRV (7.11):

$$|\Psi^{(1)}_{(L)\,100}(0;t)\rangle = -\exp\left(-\frac{i}{\hbar}E_1t\right)e\mathbf{E}\cdot|\boldsymbol{v}_{100}(E_1)\rangle. \quad (7.15)$$

To find the static limit (7.15) of the length-gauge correction to the wave function we have to set $\omega=0$ in Eq. (7.14):

$$\Psi^{(1)}_{(L)\,100}(0;\boldsymbol{r},t) = -\frac{e}{2|E_1|}(\mathbf{E}\cdot\boldsymbol{r})\left(1+\frac{1}{2}\frac{r}{a}\right)\Psi^{(0)}_{100}(\boldsymbol{r},t). \quad (7.16)$$

As a side remark, evaluation of the correction (7.16), by solving Eq. (2.13a) in the special case $|N\rangle=|100\rangle$ and $\Omega=E_1$, has become a standard textbook application of conventional stationary perturbation theory [46–48]. Apparently, use is made of the ingenious method devised by Dalgarno and Lewis [49] that emphasizes a *general* sum rule permitting, in particular, the perturbative calculation of the long-range forces between a proton and a hydrogen atom in the ground state.

D. Low-frequency behavior: Momentum representation

The momentum-space counterpart of the linear-response correction (7.12) is its Fourier transform:

$$\widetilde{\Psi}^{(1)}_{(L)\,100}(\omega;\boldsymbol{p},t)=-\frac{1}{2}(4\pi)^{-1/2}e\mathcal{E}_0\exp\left(-\frac{i}{\hbar}E_1 t\right)\times[\exp(-i\omega t)(\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{p}})\widetilde{\mathcal{A}}_{101}(E_1+\hbar\omega+i0;p)+\exp(i\omega t)(\hat{\boldsymbol{\epsilon}}^*\cdot\hat{\boldsymbol{p}})\widetilde{\mathcal{A}}_{101}(E_1-\hbar\omega;p)]. \tag{7.17}$$

We substitute the expansion (7.13) into Eq. (3.22a) written for the ground state. By employing Eq. (B1), the integral (B4), and then applying a quadratic transformation and an analytic-continuation property of the Gauss hypergeometric function [50], we finally get the approximate formula

$$\widetilde{\mathcal{A}}_{101}(E_1+\delta\Omega;p)=-\frac{i}{2|E_1|}\frac{2^4a^{5/2}}{(2\pi)^{1/2}\hbar^{3/2}}\frac{\zeta}{(\zeta^2+1)^3}\left\{1+\frac{6}{\zeta^2+1}+\frac{\delta\Omega}{2|E_1|}\times\left[\frac{11}{6}+\frac{5}{\zeta^2+1}+\frac{16}{(\zeta^2+1)^2}\right]+\left(\frac{\delta\Omega}{2|E_1|}\right)^2\times\left[\frac{287}{72}+\frac{119}{12}\frac{1}{\zeta^2+1}+\frac{52}{3}\frac{1}{(\zeta^2+1)^2}+\frac{40}{(\zeta^2+1)^3}\right]+O\left[\left(\frac{\delta\Omega}{2|E_1|}\right)^3\right]\right\}. \tag{7.18}$$

Recall the ground-state energy eigenfunction that parallels Eq. (2.11) of I:

$$\widetilde{u}_{100}(\boldsymbol{p})=\frac{1}{\pi}\left(2\frac{a}{\hbar}\right)^{3/2}\frac{1}{(\zeta^2+1)^2}. \tag{7.19}$$

Accordingly, the momentum-space analog of Eq. (7.14) is

$$\widetilde{\Psi}^{(1)}_{(L)\,100}(\omega;\boldsymbol{p},t)=i\frac{e\mathcal{E}_0 a}{2|E_1|}\frac{1}{\zeta^2+1}(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{\zeta})\left(\left\{1+\frac{6}{\zeta^2+1}+\left(\frac{\hbar\omega}{2|E_1|}\right)^2\left[\frac{287}{72}+\frac{119}{12}\frac{1}{\zeta^2+1}+\frac{52}{3}\frac{1}{(\zeta^2+1)^2}+\frac{40}{(\zeta^2+1)^3}\right]\right\}\cos(\omega t)-i\frac{\hbar\omega}{2|E_1|}\left[\frac{11}{6}+\frac{5}{\zeta^2+1}+\frac{16}{(\zeta^2+1)^2}\right]\times\sin(\omega t)+O\left[\left(\frac{\hbar\omega}{2|E_1|}\right)^3\right]\right)\widetilde{\Psi}^{(0)}_{100}(\boldsymbol{p},t). \tag{7.20}$$

The first-order correction to the wave function in the static limit, Eq. (7.15), is built up with the reduced LRV

$$\tilde{\boldsymbol{v}}_{100}(E_1;\boldsymbol{p})=(4\pi)^{-1/2}\widetilde{\mathcal{A}}_{101}(E_1;p)\hat{\boldsymbol{p}} \tag{7.21}$$

and is obtained as the limit $\omega=0$ of Eq. (7.20):

$$\widetilde{\Psi}^{(1)}_{(L)\,100}(0;\boldsymbol{p},t)=i\frac{ea}{2|E_1|}(\mathbf{E}\cdot\boldsymbol{\zeta})\frac{2}{\zeta^2+1}\left(1+\frac{6}{\zeta^2+1}\right)\widetilde{\Psi}^{(0)}_{100}(\boldsymbol{p},t). \tag{7.22}$$

VIII. DISCUSSION AND SUMMARY

A. High-frequency limit

We consider the high-frequency limit of the velocity-gauge correction (2.9b) to the wave function. By use of Eq. (A3) of I or inspection of Eq. (2.13b), we find the behavior of the LRV (2.10b) at large values of the parameter $|\Omega|$:

$$\boldsymbol{w}_N(\Omega;\boldsymbol{r})\underset{|\Omega|\to\infty}{\sim}-\frac{1}{\Omega}\boldsymbol{P}u_N(\boldsymbol{r}). \tag{8.1}$$

Now, taking notice of Eqs. (2.1) and (2.11), the wave function (2.9b) behaves in the range of high frequencies ($\hbar\omega\gg|E_1|$) as

$$\Psi^{(1)}_{(V)N}(\omega;\boldsymbol{r},t)\underset{|\Omega|\to\infty}{\sim}-\frac{i}{\hbar}\frac{e}{m_e\omega^2}\mathbf{E}(t)\cdot\boldsymbol{P}\Psi^{(0)}_N(\boldsymbol{r},t). \tag{8.2}$$

We denote by $\boldsymbol{\alpha}(t)$ the radius vector of a classical electron in its quiver motion produced by the electric field (2.1):

$$\boldsymbol{\alpha}(t)\equiv\frac{e}{m_e\omega^2}\mathbf{E}(t). \tag{8.3}$$

The real amplitude of this motion when the electron is driven by a linearly polarized electric field is

$$\alpha_0=\frac{e\mathcal{E}_0}{m_e\omega^2}. \tag{8.4}$$

Consequently, a linear-response wave function in the velocity gauge has the behavior

$$\Psi^{(0)}_N(\boldsymbol{r},t)+\Psi^{(1)}_{(V)N}(\boldsymbol{r},t)\underset{|\Omega|\to\infty}{\sim}[I-\boldsymbol{\alpha}(t)\cdot\boldsymbol{\nabla}]\Psi^{(0)}_N(\boldsymbol{r},t). \tag{8.5}$$

Remark that, by using the asymptotic expression of a Humbert function Φ_1, Eq. (B12) of I, we have recovered the prediction (8.5) for an arbitrary angular-momentum eigenstate $|nlm\rangle$, as well as for any Stark state $|nn_e m\rangle$.

On the other hand, the *exact* high-frequency limit of Floquet theory in the oscillating (Kramers-Henneberger) reference frame is equivalent to an energy eigenvalue problem for the potential *dressed* by the radiation field [51]:

$$V(\alpha_0;\boldsymbol{r}')=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}dt V[\boldsymbol{r}'+\boldsymbol{\alpha}(t)]. \tag{8.6}$$

In Eq. (8.6), $\boldsymbol{r}'$ is the electron position vector in the oscillating reference frame, while the amplitude α_0 is supposed fixed. To get the linear-response approach, which is valid only at low values of the parameter α_0, one has to neglect in the exact theory all terms whose order in α_0 exceeds one.

Within this approximation, the atomic dressed potential (8.6) reduces to the pure Coulomb potential:

$$V(\alpha_0;r')=-\frac{Ze^2}{r'}+O(\alpha_0^2). \tag{8.7}$$

Accordingly, the energy eigenvalue problem in the oscillating frame becomes that of the field-free Hamiltonian in the laboratory reference frame. It follows that a wave function in the laboratory frame is approximately

$$\Psi_N(\boldsymbol{r},t)=\Psi_N^{(0)}[\boldsymbol{r}-\boldsymbol{\alpha}(t),t]+O(\alpha_0^2). \tag{8.8}$$

To lowest order in perturbation theory, one has to keep in the right-hand side of Eq. (8.8) just the first two terms in the Taylor expansion of $\Psi_N^{(0)}$: they are precisely those written as the linear-response wave function (8.5).

B. Summary

Within the semiclassical theory of the photon-atom interaction, in nonrelativistic treatment, the EDA plays the role of the simplest, yet accurate approach. We have complemented earlier work on hydrogenic atoms, Refs. [5,8], deriving in a unified way the LRV's in coordinate space. Our starting point has been their generating function $\mathcal{F}$, built up in I with the Schrödinger Coulomb Green function. We have treated the length and velocity gauges on an equal footing. In each of them, we have established closed-form integral representations of the LRV's associated to any spherical eigenstate $|nlm\rangle$, Eqs. (3.14) and (3.17), as well as to any parabolic one $|nn_em\rangle$, Eqs. (4.15). Both have been used to evaluate their momentum-space counterparts via Fourier transformation. We have obtained the latter first as contour integrals and then explicitly in terms of two generalized hypergeometric functions with several parameters and variables, ${}_1F_E$ and F_F, respectively. The linear response from an atomic state $|N\rangle$ to a uniform static electric field is determined by the reduced LRV $|\boldsymbol{v}_N'(E_n)\rangle$. The reduced LRV's in coordinate space, already evaluated in Refs. [5,8], have been used to find those in momentum space, as their Fourier transforms. The atomic ground state is very special not only because we may benefit from the results established in I: it also lies at the intersection of our two sets of formulas regarding the spherical and parabolic stationary states, respectively. We have pointed out the low-frequency behavior of the linear response from the ground state in the length gauge, both in coordinate and momentum representations. An analysis of the high-frequency limit of the linear response has been finally made in the coordinate space and velocity gauge.

The above-presented study of the linear response in the EDA is extensive albeit not complete. For instance, we have not dealt with the Sturmian expansions of the LRV's which have been established in Refs. [7,8] only for spherical states $|nlm\rangle$ in coordinate space. Neither have we derived the LRV's associated to the Coulomb scattering waves of which only the velocity-gauge vector has been reported in Ref. [6]. Nevertheless, both topics deserve special analysis.

We stress that Sturmian series of the LRV's for spherical states in momentum space, as well as for parabolic states in both the position and momentum representations, are still to be explored. In spite of their noncompact character, such expansions are expected to be efficient in numerical calculations. As a related matter, we mention two recent papers dealing with the series expansion of the first-order Dirac CGF in coordinate space in terms of relativistic Coulomb Sturmians [52,53].The obtained formulas are applied to evaluate relativistic effects on the static electric polarizability in the ground state [52] and on two-photon transitions from the ground state towards low-lying excited atomic states [53].

To conclude, the LRV's for one electron in a pure Coulomb field are basic analytic tools in atomic physics. As pointed out in the Introduction, their compact expressions as contour integrals considerably simplify the evaluation of the two-photon amplitudes [13–15]. In addition, they are useful in the study of laser-assisted processes like photon-atom interactions or electron-atom collisions.

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APPENDIX A: FORMULAS CONNECTED WITH THE LINEAR RESPONSE IN COORDINATE SPACE

We recall that any complex vector $\boldsymbol{v}$ can be decomposed with respect to the orthonormal basis

$$\hat{\boldsymbol{\chi}}_{-1}=2^{-1/2}(\hat{\boldsymbol{x}}-i\hat{\boldsymbol{y}}),\quad \hat{\boldsymbol{\chi}}=\hat{\boldsymbol{z}},\quad \hat{\boldsymbol{\chi}}_1=-2^{-1/2}(\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}}), \tag{A1}$$

as

$$\boldsymbol{v}=\sum_{\mu=-1}^{1} v^{\mu}\hat{\boldsymbol{\chi}}_{\mu}, \tag{A2}$$

where v^{μ} are its contravariant spherical components:

$$v^{-1}=2^{-1/2}(v_x+iv_y),\quad v^0=v_z,\quad v^1=-2^{-1/2}(v_x-iv_y). \tag{A3}$$

For instance, the decomposition (A2) of the vector spherical harmonics reads

$$\boldsymbol{V}_{ljm}(\hat{\boldsymbol{r}})=\sum_{\mu=-1}^{1}\langle l\ m-\mu\ ,1\mu|l1,\ jm\rangle Y_{l\ m-\mu}(\hat{\boldsymbol{r}})\hat{\boldsymbol{\chi}}_{\mu}. \tag{A4}$$

It is indispensable to write down two integral representations of the Kummer hypergeometric function ${}_1F_1$ that are valid only in the special case when it is proportional to a Laguerre polynomial:

$${}_1F_1(a;c;y)=-\frac{1}{2\pi i}\frac{\Gamma(c)\Gamma(1-a)}{\Gamma(c-a)}\times\int_1^{(0+)}du(-u)^{a-1}(1-u)^{c-a-1}e^{yu}$$

$$[\mathrm{Re}(c-a)>0, \quad a \text{ nonpositive integer}] \tag{A5}$$

and

$$
\begin{aligned}
{}_1F_1(a;c;y) &= (1-x)^{-a}\left(-\frac{1}{2\pi i}\right)\frac{\Gamma(c)\Gamma(1-a)}{\Gamma(c-a)} \\
&\times \int_1^{(0+)} dt(-t)^{a-1}(1-t)^{c-a-1} \\
&\times \left(1-\frac{x}{x-1}t\right)^{-c}\exp\left(\frac{yt}{1-x+xt}\right) \\
&\left(\mathrm{Re}(c-a)>0, \quad a \text{ nonpositive integer}, \right. \\
&\left. x\neq 1, \quad \arg\left|\frac{x}{x-1}\right|<\pi \right).
\end{aligned} \tag{A6}
$$

Equation (A6) is obtained from Eq. (A5) by a change of the variable of integration,

$$u=\frac{t}{1-x+xt} \qquad (x\neq 1), \tag{A7}$$

which introduces the redundant parameter x.

We mention the finite expansions of the generalized hypergeometric functions ${}_1\Phi_H$ and ${}_2\Phi_H$ in terms of Humbert functions Φ_1:

$$
\begin{aligned}
&{}_1\Phi_H(a;b,a_1,c_1;c;x,x',y,z) \\
&= \sum_{\nu'=0}^{\infty}\sum_{\nu_1=0}^{\infty}\frac{(a)_{\nu'+\nu_1}(a_1)_{\nu'+\nu_1}}{(c)_{\nu'+\nu_1}(c_1)_{\nu_1}}\frac{(x')^{\nu'}z^{\nu_1}}{\nu'!\,\nu_1!} \\
&\times \Phi_1(a+\nu'+\nu_1,b+\nu',c+\nu'+\nu_1;x,y)
\end{aligned} \tag{A8}
$$

and

$$
\begin{aligned}
&{}_2\Phi_H(a;b,a_1,a_2,c_1,c_2;c;x,x',y,z_1,z_2) \\
&= \sum_{\nu'=0}^{\infty}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\frac{(a)_{\nu'+\nu_1+\nu_2}(a_1)_{\nu_1}(a_2)_{\nu_2}}{(c)_{\nu'+\nu_1+\nu_2}(c_1)_{\nu_1}(c_2)_{\nu_2}} \\
&\times (a_1+a_2+\nu_1+\nu_2)_{\nu'}\frac{(x')^{\nu'}z_1^{\nu_1}z_2^{\nu_2}}{\nu'!\,\nu_1!\,\nu_2!} \\
&\times \Phi_1(a+\nu'+\nu_1+\nu_2,b+\nu',c+\nu'+\nu_1+\nu_2;x,y).
\end{aligned} \tag{A9}
$$

In Eqs. (A8) and (A9), a_1 and a_2 are nonpositive integers.

A pair of finite expansions which is alternative to that written as Eq. (A6) in Ref. [8] can be found by combining two recurrence relations between contiguous Kummer hypergeometric functions [Ref. [32], p. 254, Eqs. (4) and (5)]:

$$
\begin{aligned}
rR_{nl}(r) &= (2\kappa_n)^{1/2}\frac{1}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l+1)!2n}\right]^{1/2} \\
&\times \exp(-\kappa_n r)(2\kappa_n r)^{l+q}\sum_{s=-1}^{1} e_{n,l}^{(q,s)} \\
&\times {}_1F_1(l+q+1+s-n;2(l+q)+2;2\kappa_n r) \\
&(q=1, \text{ if } l=0, \text{ and } q=1,-1 \text{ if } l>0).
\end{aligned} \tag{A10}
$$

The coefficients $e_{n,l}^{(q,s)}$ are listed in Table I.

APPENDIX B: FORMULAS CONNECTED WITH THE LINEAR RESPONSE IN MOMENTUM SPACE

We recall the expressions in terms of hypergeometric functions of (1) a spherical Bessel function,

$$
\begin{aligned}
j_l(z) &= \frac{l!}{(2l+1)!}e^{iz}(2z)^l\ {}_1F_1(l+1;2l+2;-2iz) \\
&(l=0,1,2,3,\ldots),
\end{aligned} \tag{B1}
$$

(2) a Jacobi polynomial,

$$P_l^{(a,b)}(w)=\binom{l+a}{a}\left(\frac{1+w}{2}\right)^l {}_2F_1\left(-l,-l-b;a+1;\frac{w-1}{w+1}\right), \tag{B2}$$

and (3) a Gegenbauer polynomial [54],

$$C_n^{\lambda}(w)=\frac{(2\lambda)_n}{n!}\ {}_2F_1\left(-n,n+2\lambda;\lambda+\frac{1}{2};\frac{1}{2}(1-w)\right). \tag{B3}$$

The following infinite integrals proved to be necessary. First [55],

$$
\begin{aligned}
&\int_0^{\infty} dy e^{-\lambda y}y^{\nu}{}_1F_1(a;c;\mu y)=\frac{\Gamma(\nu+1)}{\lambda^{\nu+1}}\ {}_2F_1\left(a,\nu+1;c;\frac{\mu}{\lambda}\right) \\
&[\mathrm{Re}(\lambda)>|\mathrm{Re}(\mu)|, \quad \mathrm{Re}(\nu)>-1, \\
&\mathrm{Re}(c-a)>0, \quad a\neq 1,2,3,\ldots].
\end{aligned} \tag{B4}
$$

Second, consider

$$
\begin{aligned}
&J_{c-1}^{(\sigma,0)}(\lambda;a',a;\mu',\mu) \\
&\equiv \int_0^{\infty} dy e^{-\lambda y}y^{c-1+\sigma}{}_1F_1(a';c;\mu' y)\ {}_1F_1(a;c;\mu y) \\
&[\sigma=0,1; \quad \mathrm{Re}(c)>0, \quad \mathrm{Re}(c-a)>0].
\end{aligned} \tag{B5}
$$

In Eq. (B5), a' and/or a are nonpositive integers: $\mathrm{Re}(\lambda)>0$ if a' and a are both integers, $\mathrm{Re}(\lambda-\mu)>0$ if only a' is an integer, and $\mathrm{Re}(\lambda-\mu')>0$ if only a is an integer. We note the expression

$$J_{c-1}^{(0,0)}(\lambda;a',a;\mu',\mu) = \Gamma(c)\lambda^{a+a'-c}(\lambda-\mu')^{-a'}(\lambda-\mu)^{-a} \times {}_2F_1\left(a',a;c;\frac{\mu'\mu}{(\lambda-\mu')(\lambda-\mu)}\right). \quad \text{(B6)}$$

Then, a recurrence relation between contiguous Kummer hypergeometric functions [Ref. [32], p. 254, Eq. (2)] yields the identity

$$J_{c-1}^{(1,0)}(\lambda;a',a;\mu',\mu) = \frac{1}{\mu}[-(c-a)J_{c-1}^{(0,0)}(\lambda;a',a-1;\mu',\mu) + (c-2a)J_{c-1}^{(0,0)}(\lambda;a',a;\mu',\mu) + aJ_{c-1}^{(0,0)}(\lambda;a',a+1;\mu',\mu)]. \quad \text{(B7)}$$

Third, Hankel's formula [56] leads to the integral [57]

$$\int_0^\infty dy e^{-\lambda y} y^{(c-1)/2+n} J_{c-1}[2(\mu y)^{1/2}] = (c)_n \mu^{(c-1)/2} \lambda^{-(c+n)} \exp\left(-\frac{\mu}{\lambda}\right) {}_1F_1\left(-n;c;\frac{\mu}{\lambda}\right)$$
$$[\mathrm{Re}(\lambda)>0, \quad \mathrm{Re}(\mu)>0, \quad \mathrm{Re}(c)>0, \quad n=0,1,2,3,\ldots]. \quad \text{(B8)}$$

In Eq. (B8), J_ν is a Bessel function of the first kind.

A prerequisite for evaluating the Fourier transform of an energy eigenfunction or of a LRV in parabolic coordinates is the expansion of a plane wave into cylindrical waves:

$$\exp\left(\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}\right) = 2\pi \exp\left(\frac{i}{\hbar}p_z z\right) \times \sum_{m=-\infty}^{\infty} i^m J_m\left(\frac{1}{\hbar}p_\rho \rho\right)\Phi_m(\tilde{\varphi})\Phi_m^*(\varphi). \quad \text{(B9)}$$

In Eq. (B9), ρ,φ,z, and $p_\rho,\tilde{\varphi},p_z$ are the cylindrical coordinates in position and momentum space, respectively. $\Phi_m(\varphi)$ is a Fourier function (4.2). Equation (B9) originates in the Laurent series of the generating function of the Bessel coefficients [58].

Remark that Eq. (A5) can be retrieved by confluence from an appropriate integral representation of a Gauss hypergeometric function [59]:

$${}_2F_1(a,b;c;x) = -\frac{1}{2\pi i}\frac{\Gamma(c)\Gamma(1-a)}{\Gamma(c-a)} \times \int_1^{(0+)} du(-u)^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}$$
$$[\mathrm{Re}(c-a)>0, \quad a \text{ nonpositive integer}]. \quad \text{(B10)}$$

We also recall the definition of a Lauricella hypergeometric function F_D with n variables [60],

$$F_D(a;b_1,\ldots,b_n;c;x_1,\ldots,x_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\frac{-ie^{-i\pi a}}{2\sin(\pi a)}\int_1^{(0+)} \times d\rho\rho^{a-1}(1-\rho)^{c-a-1}(1-x_1\rho)^{-b_1}\cdots(1-x_n\rho)^{-b_n}$$
$$[\mathrm{Re}(c-a)>0; \quad a\neq 1,2,3,\ldots; \quad |\arg(-x_p)|\leqslant\pi,\ p=1,\ldots,n], \quad \text{(B11)}$$

as well as its multiple ascending power series,

$$F_D(a;b_1,\ldots,b_n;c;x_1,\ldots,x_n) = \sum_{\nu_1=0}^\infty \cdots \sum_{\nu_n=0}^\infty \frac{(a)_{\nu_1+\cdots+\nu_n}}{(c)_{\nu_1+\cdots+\nu_n}} \times \frac{(b_1)_{\nu_1}\cdots(b_n)_{\nu_n}}{\nu_1!\cdots\nu_n!} x_1^{\nu_1}\cdots x_n^{\nu_n}$$
$$(|x_p|<1, \quad p=1,\ \ldots\ ,n). \quad \text{(B12)}$$

Notice that in the particular cases $n=1$ and $n=2$ the Lauricella function F_D reduces to the Gauss function ${}_2F_1$ and the Appell function F_1, respectively.

It is worth writing down the finite expansion of the generalized hypergeometric function ${}_1F_E$ [34] in terms of Appell functions F_1:

$${}_1F_E(a;b,a_1,b_1,c_1;c;x,x',x'',z) = \sum_{\nu'=0}^\infty \sum_{\nu_1=0}^\infty \frac{(a)_{\nu'+\nu_1}(a_1)_{\nu'+\nu_1}(b_1)_{\nu_1}}{(c)_{\nu'+\nu_1}(c_1)_{\nu_1}}\frac{(x')^{\nu'}z^{\nu_1}}{(\nu')!\,\nu_1!}F_1(a+\nu'+\nu_1;b,b+\nu_1;c+\nu'+\nu_1;x,x'')$$
$$(a_1 \text{ nonpositive integer}, \quad x'\neq x'', \quad x'\neq 1, \quad x''\neq 1). \quad \text{(B13)}$$

We finally introduce a new generalized hypergeometric function F_F, with six parameters and seven variables, defined as a contour integral:

$$F_F(a;b;a',b';c';c;x_1,x_2,y_1,y_2,y_1',y_2',z) \equiv \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\frac{-ie^{-i\pi a}}{2\sin(\pi a)}\int_1^{(0+)} d\rho\rho^{a-1}(1-\rho)^{c-a-1}[(1-x_1\rho)(1-x_2\rho)]^{-b}[(1-y_1\rho)(1-y_2\rho)]^{-a'} \times[(1-y_1'\rho)(1-y_2'\rho)]^{-b'}{}_2F_1\left(a',b';c';\frac{-z^2\rho^2}{(1-y_1\rho)(1-y_2\rho)(1-y_1'\rho)(1-y_2'\rho)}\right)$$
$$[\mathrm{Re}(c-a)>0;\quad a\neq 1,2,3,\ldots;\quad a',\ b'\ \text{nonpositive integers}]. \tag{B14}$$

Making use of the power series expansion of the Gauss function in the integrand on the right-hand side of Eq. (B14), we find the finite expansion of the function F_F in terms of Lauricella functions F_D of six variables:

$$F_F(a;b;a',b';c';c;x_1,x_2,y_1,y_2,y_1',y_2',z) = \sum_{\mu=0}^{\infty}\frac{(a)_{2\mu}}{(c)_{2\mu}}\frac{(a')_\mu(b')_\mu}{(c')_\mu\ \mu!}(-z^2)^\mu F_D(a+2\mu;b,b,a'+\mu,a'+\mu,b'+\mu,b'+\mu;c+2\mu;x_1,x_2,y_1,y_2,y_1',y_2'). \tag{B15}$$

In this paper we contemplate only the case in which a' and b' are nonpositive integers. According to Eqs. (B15) and (B12), such a function F_F is a finite quintuple sum of Appell functions F_1:

$$F_F(a;b;a',b';c';c;x_1,x_2,y_1,y_2,y_1',y_2',z) = \sum_{\mu=0}^{\infty}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{\nu_1'=0}^{\infty}\sum_{\nu_2'=0}^{\infty}\frac{(a)_{2\mu+\nu_1+\nu_2+\nu_1'+\nu_2'}}{(c)_{2\mu+\nu_1+\nu_2+\nu_1'+\nu_2'}}\frac{(a')_{\mu+\nu_1}(a')_{\mu+\nu_2}}{(a')_\mu\nu_1!\,\nu_2!}\frac{(b')_{\mu+\nu_1'}(b')_{\mu+\nu_2'}}{(b')_\mu\ \nu'!\,\nu''!} \times\frac{1}{(c')_\mu\ \mu!}y_1^{\nu_1}y_2^{\nu_2}(y_1')^{\nu_1'}(y_2')^{\nu_2'}(-z^2)^\mu F_1(a+2\mu+\nu_1+\nu_2+\nu_1'+\nu_2';b,b;c+2\mu+\nu_1+\nu_2+\nu_1'+\nu_2';x_1,x_2). \tag{B16}$$

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[21] In Ref. [8], the length gauge has been designated as the ϕ gauge, while the velocity gauge has been called the $\boldsymbol{A}$ gauge.
[22] This can be seen upon inspection of the linear parts of the

interaction Hamiltonian in the Poincaré gauge, Eq. (2.7) of I subject to the gauge transformation (4.7) of I, and of the truncated one in the multipolar gauge, Eqs. (5.8)–(5.10) of I. Along the same lines, the generating function (2.4) is the $\boldsymbol{\kappa}=\mathbf{0}$ limit of the function (4.8) of I, alone and with the term (4.15) of I added. Indeed, these are the generating functions of the gauge transformations from the radiation gauge to the Poincaré and the multipolar gauge, respectively, within the second-order approximation of retardation.

[23] See Eq. (A2) of I.

[24] Equation (A2) of I is the coordinate-space version of Eq. (2.17).

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[29] See I, Appendix B, Eqs. (B5)–(B10).

[30] Reference [8]. See Eqs. (A8) and (A9) along with Tables I and VI, respectively.

[31] The explicit compact expansion of $\mathcal{A}_{nll+q}(\Omega;r)$ has been presented in Ref. [8] as Eq. (28), while that of $\mathcal{B}_{nll+q}(\Omega;r)$ has been reported as Eq. (28) of Ref. [5].

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[35] The hypergeometric function ${}_2\Phi_H$ has been defined as a contour integral in Ref. [8], Eq. (38). Its series expansion, Eq. (A9), has been given in the same paper as Eq. (39).

[36] Reference [8]. See Eqs. (A12) and (A13) along with Tables II and III, respectively. In Ref. [8] the Stark states are designated as $|n_\xi\, n_\eta\, m\rangle$, unlike the present work where they are denoted by $|nn_e m\rangle$.

[37] The expression of the vector $\boldsymbol{v}_{nn_em}(\Omega;\xi,\eta,\varphi)$ has been written as Eq. (40) of Ref. [8].

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