Lindblad approach to nonlinear Jaynes-Cummings dynamics of a trapped ion

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The Lindblad approach to open quantum systems is introduced to study the dynamics of a single trapped ion prepared in nonclassical motional states and subjected to continuous measurement of its internal population. This results in an inhibition of the dynamics similar to the one occurring in the quantum Zeno effect. In particular, modifications to the Jaynes-Cummings collapses and revivals arising from an initial coherent state of motion in various regimes of interaction with the driving laser are dealt with in detail. [S1050-2947(97)04605-2]

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In recent years, several efforts have been made to clarify the role of the measurement process during the evolution of a quantum system. Besides the conceptual viewpoint, this also has a practical relevance in predicting the results of experiments and possibly pointing out some working conditions where genuine quantum phenomena are expected [1]. Despite the different languages, ranging from group algebraic and path-integral techniques [2,3] to stochastic evolution models [4,5], which have been employed to analyze the quantum measurement problem, all these approaches recognize that a measured system is *not* closed but interacts with the environment schematizing the meter. An equivalence loop connecting these different descriptions has been established in [6], resulting in a unified picture which incorporates the measurement process through a unique parameter proportional to the strenght of the system-meter interaction. In the present paper, we apply this scheme for investigating the influence of a continuous measurement process on the dynamics of a trapped and laser-irradiated ion. This system has lately attracted growing interest, culminating in the experimental generation of nonclassical motional states [7,8]. The possibility to read out the vibrational state of the ion following the evolution of its internal levels under a Jaynes-Cummings type interaction has been at the heart of the experimental procedure. As we shall see, by including the measurement process an inhibition of the two-level transitions dynamics emerges, which could be made observable via the quenching of collapses and revivals in a coherent-state prepared ion.

Our physical system is a single trapped ion interacting with a classical laser field and undergoing a continuous measurement of its internal population. In practice, one is dealing with a collection of independent evolutions of single ions starting from the same initial state and *averaged* results are actually registered as outcomes of the experiment. According to the general discussion of [6], we can either model each single history as a realization of a pure-state stochastic evolution, then averaging over the associated *selective* measurement results, or we can consider a deterministic ensemble evolution which directly leads to averaged *nonselective* predictions. Our choice of the latter method relies on the exact solvability of the model. The analysis is based on the master equation for the reduced density operator $\hat{\rho}(t)$ of the system, which results from tracing out the variables of the external environment from the overall system plus environment density operator and is written in the Lindblad form [2,6]

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}[\hat{H}(t),\hat{\rho}(t)] - \frac{\kappa}{2}[\hat{A},[\hat{A},\hat{\rho}(t)]], \tag{1}$$

where $\hat{H}(t)=\hat{H}_0+\hat{H}_{int}(t)$ describes the dynamics of the unmeasured closed system and the Lindblad operator, representing, in general, the influence of the environment on the system, is related in this case to the measured observable $\hat{A}$. The parameter κ expresses the strength of the coupling of the measured system to the meter [6], hereafter assumed to be time independent. The Hamiltonian $\hat{H}_0$ describes the one-dimensional harmonic oscillator associated to the center-of-mass degree of freedom and the two internal states of the ion

$$\hat{H}_0 = \hat{H}^{cm} + \hat{H}^{el} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\omega_{21}\hat{\sigma}_z, \tag{2}$$

ω being the vibrational frequency, ω_{21} the two-level transition frequency, and $\hat{a}$ and $\hat{\sigma}_z$ the boson annihilation operator and the pseudospin $\hat{z}$ operator, respectively. The trap frequency is supposed large enough to neglect the atomic spontaneous emission (strong confinement limit [7]). The interaction $\hat{H}_{int}(t)$ with the laser field is modeled by a nonlinear multiquantum Jaynes-Cummings model (JCM) which, in the rotating-wave approximation, is written as [9]

$$\hat{H}^{int}(t) = \frac{\hbar\Omega_0}{2} e^{i\omega_L t}\cos[\eta(\hat{a}+\hat{a}^\dagger)+\varphi]\hat{\sigma}_- + \text{H.c.}, \tag{3}$$

where ω_L is the laser frequency, Ω_0 the fundamental Rabi frequency, $\hat{\sigma}_-$ the lowering operator, and $\eta=\omega_L\Delta x_{SQL}/c$ the Lamb-Dicke parameter related to the position standard quantum limit of the ion. In Eq. (3) a standing-wave laser field is considered, the phase φ fixing the position of the trap potential with respect to the wave. The measured observable $\hat{A}$ depends upon the specific experiment. If the population of the internal ground state is measured, for instance by collect-

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ing the fluorescence emitted after stimulated transition to a third auxiliary level as done by Meekhof *et al.* [7], the operator $\hat{A}$ is the occupancy of level $\downarrow$, *i.e.*, $\hat{A}=\hat{\sigma}_-\hat{\sigma}_+$ in our formalism.

It is convenient to introduce the representation defined by the eigenkets of $\hat{H}_0$, $|S,n\rangle$, $S=\downarrow,\uparrow,n=0,\ldots,\infty$, with matrix elements $\rho_{Sn,S'm}(t)=\langle S,n|\hat{\rho}(t)|S',m\rangle$. Moreover, let us assume that the ion is in the low excitation regime ($\omega\gg\Omega_0$) and the laser is tuned to the kth vibrational sideband, i.e., $\omega_L=\omega_{21}+k\omega$, $k\in Z$, two conditions usually fulfilled in laboratory [7]. This implies that transitions between states $|\downarrow,n\rangle$ and $|\uparrow,n+k\rangle$, involving the exchange of k vibrational quanta, are resonantly enhanced, whereas all the off-resonant couplings are rapidly oscillating with frequency ω and can be disregarded (JCM approximation [9]). As a notable case, when $\varphi=\pm\pi/2$ and $\eta\ll 1$ (the Lamb-Dicke limit), the well-known linear one-quantum JCM or anti-JCM operators $\eta(\hat{a}\hat{\sigma}_+ +\hat{a}^\dagger\hat{\sigma}_-)$, $\eta(\hat{a}^\dagger\hat{\sigma}_+ +\hat{a}\hat{\sigma}_-)$ are recovered by tuning to the first red or blue sideband, $k=-1$ or $k=1$, respectively, [10]. In what follows we focus on the properties of the internal dynamics of the ion, which can be extracted from the reduced atomic density matrix $\sigma_{SS'}(t)=\Sigma_{n=0}^{\infty}\rho_{Sn,S'n}(t)$. We can, therefore, restrict ourselves to the subset of the density matrix $\rho_{Sn,S'm}$ having $n=m$. According to the master equation (1) and for a generic k-quantum resonance, the time development is ruled by

$$
\begin{aligned}
\dot{\rho}_{\downarrow n\downarrow n}(t)&=-i\Omega_{nn+k}/2[\rho_{\uparrow n+k\downarrow n}(t)e^{i\omega_L t}-\rho_{\downarrow n\uparrow n+k}(t)e^{-i\omega_L t}],\\
\dot{\rho}_{\uparrow n+k\uparrow n+k}(t)&=-i\Omega_{nn+k}/2[\rho_{\downarrow n\uparrow n+k}(t)e^{-i\omega_L t}-\rho_{\uparrow n+k\downarrow n}(t)e^{i\omega_L t}],\\
\dot{\rho}_{\downarrow n\uparrow n+k}(t)&=(i\omega_{21}-\kappa/2)\rho_{\downarrow n\uparrow n+k}(t)-i\Omega_{nn+k}/2[\rho_{\uparrow n+k\uparrow n+k}(t)-\rho_{\downarrow n\downarrow n}(t)]e^{i\omega_L t},\\
\dot{\rho}_{\uparrow n+k\downarrow n}(t)&=(-i\omega_{21}-\kappa/2)\rho_{\uparrow n+k\downarrow n}(t)-i\Omega_{nn+k}/2[\rho_{\downarrow n\downarrow n}(t)-\rho_{\uparrow n+k\uparrow n+k}(t)]e^{-i\omega_L t},
\end{aligned}\tag{4}
$$

where, as usual, the nonlinear k-quantum Rabi frequencies expressed in terms of the Laguerre polinomials $\mathcal{L}_n^k$

$$
\begin{aligned}
\Omega_{nn+k}&=\Omega_0\langle n|\cos[\eta(\hat{a}+\hat{a}^\dagger)+\varphi]|n+k\rangle\\
&=\Omega_0[e^{i\varphi}+(-1)^k e^{-i\varphi}]\\
&\quad\times(i\eta)^k\left(\frac{n!}{(n+k)!}\right)^{1/2}e^{-\eta^2/2}\mathcal{L}_n^k(\eta^2)
\end{aligned}\tag{5}
$$

have been introduced [9]. In the Lamb-Dicke regime ($\eta\ll 1$) they reduce to the linear ones, e.g., $\Omega_{nn+1}=\eta\Omega_0\sqrt{n+1}$ for one-quantum interaction. The following solution of Eq. (4) holds for $n\geq -k$

$$
\begin{aligned}
\rho_{\downarrow n\downarrow n}(t)&=\frac{1}{2}(\rho_{\uparrow n+k\uparrow n+k}+\rho_{\downarrow n\downarrow n})(0)-\frac{1}{2}e^{-\kappa t/4}\\
&\quad\times\Bigg\{(\rho_{\uparrow n+k\uparrow n+k}-\rho_{\downarrow n\downarrow n})(0)\Bigg[\cos(w_{nn+k}t)\\
&\quad+\frac{\kappa}{4w_{nn+k}}\sin(w_{nn+k}t)\Bigg]\\
&\quad+i\,\mathrm{Im}[\rho_{\downarrow n\uparrow n+k}(0)]\frac{2\Omega_{nn+k}}{w_{nn+k}}\sin(w_{nn+k}t)\Bigg\},\\
\rho_{\uparrow n+k\uparrow n+k}(t)&=(\rho_{\uparrow n+k\uparrow n+k}+\rho_{\downarrow n\downarrow n})(0)-\rho_{\downarrow n\downarrow n}(t),\\
\rho_{\downarrow n\uparrow n+k}(t)&=e^{ik\omega t-\kappa t/4}\Bigg\{\mathrm{Re}[\rho_{\downarrow n\uparrow n+k}(0)]e^{-\kappa t/4}\\
&\quad+\mathrm{Im}[\rho_{\downarrow n\uparrow n+k}(0)]\Bigg[\cos(w_{nn+k}t)\\
&\quad-\frac{\kappa}{4w_{nn+k}}\sin(w_{nn+k}t)\Bigg]\\
&\quad-i(\rho_{\uparrow n+k\uparrow n+k}-\rho_{\downarrow n\downarrow n})(0)\\
&\quad\times\frac{\Omega_{nn+k}}{2w_{nn+k}}\sin(w_{nn+k}t)\Bigg\},
\end{aligned}\tag{6}
$$

where

$$
w_{nn+k}=\left(\Omega^2_{nn+k}-\frac{\kappa^2}{16}\right)^{1/2}\tag{7}
$$

and $\rho_{\uparrow n+k\downarrow n}(t)=\rho^*_{\downarrow n\uparrow n+k}(t)$. Equations (6), holding within every *fixed* Jaynes-Cummings manifold $|\downarrow,n\rangle$, $|\uparrow,n+k\rangle$, are formally equivalent to Eqs. (84)–(85) of Ref. [6], describing a two-level system subjected to a continuous occupancy measurement. When $\kappa=0$ the effect of the measurement disappears and oscillations with angular frequency Ω_{nn+k} result for fixed n. In the opposite limit of strong measurement coupling the frequency (7) becomes imaginary and an overdamped regime occurs in which transitions are inhibited, the so-called quantum Zeno effect. The borderline between the two regimes is ruled by the threshold $w_{nn+k}=0$, corresponding to a critical measurement coupling $\kappa^{crit}_{nn+k}=4\Omega_{nn+k}$. In a general situation where different vibrational subspaces are involved, the n dependence of this critical value globally results in a smoother transition. The measured occupancy $P_\downarrow(t)=\sigma_{\downarrow\downarrow}(t)$ is straightforwardly evaluated from Eqs. (6) once the initial conditions are specified. We assume here, according to [7], that no entanglement between the internal and motional degrees of freedom occurs and only the ground internal level is populated, yielding

$$P_\downarrow(t)=\frac{1}{2}\left\{1+e^{-\kappa t/4}\sum_{n=0}^{\infty}\rho_{nn}^{cm}(0)\left[\cos(w_{nn+k}t)+\frac{\kappa}{4w_{nn+k}}\sin(w_{nn+k}t)\right]\right\}, \tag{8}$$

the matrix elements ρ_{nn}^{cm} characterizing the number state distribution of the vibrational motion. The presence of entanglement, crucial to extend the description to Schrödinger-cat states [8], can be included as well. Equation (8) displays the effect of the measurement for an arbitrary vibronic distribution, namely, an overall damping of the signal amplitude and, for each vibrational component, both a frequency shift (7) and a κ-weighted term.

Let us now specialize the internal state evolution (8) to an initial coherent distribution, $\rho_{nn}^{cm}(0)=\overline{n}^n e^{-\overline{n}}/n!$. In the limit in which no measurement is present ($\kappa=0$) and a linear Jaynes-Cummings interaction is realized by a sufficiently small Lamb-Dicke parameter, $P_\downarrow(t)$ undergoes collapses and revivals, a well-known quantum phenomenon due to dephasing and rephasing between the various Rabi oscillations, as firstly predicted by Eberly, Narozhny, and Sanchez-Mondragon [11] and recently observed in cavity QED by using Rydberg atoms [12]. Figure 1(a) shows the collapses and revivals occurring in a quantized trap, the center of mass being in a coherent state of average quantum number $\overline{n}$. The modifications intervening when the same system is coupled to the meter are depicted in Fig. 1(b) for a strength $\kappa=10^{-1}\kappa_{01}^{crit}$, in units of a reference value $\kappa_{01}^{crit}=4\Omega_{01}$. Provided $\kappa\ll\kappa_{\tilde{n}\tilde{n}+k}^{crit}$, $\tilde{n}=\mathrm{Int}(\overline{n})$, an approximate evaluation of the series in Eq. (8) leads to

$$P_\downarrow(t)\simeq 1+\cos(\eta\Omega_0\sqrt{\overline{n}+1}\,t)\exp\left(-\frac{\kappa}{4}t-\frac{\eta^2\Omega_0^2}{8}\frac{\overline{n}}{\overline{n}+1}t^2\right), \tag{9}$$

which holds for a time scale $t\ll 2\sqrt{\overline{n}}/\eta\Omega_0$, still including all the collapse evolution, and generalizes the result already quoted in [11] for $\kappa=0$. Thus, even in this weak-coupling regime, the measurement has a leading influence at short times due to its linear dependence in the exponent of the envelope in Eq. (9). Similar considerations can be repeated for the nonlinear multiquantum Jaynes-Cummings interaction, which is restored outside the Lamb-Dicke limit and has been extensively studied by Vogel and de Matos-Filho in case $\kappa=0$ [9]. Two representative examples, corresponding to one-quantum and two-quantum couplings are shown in Figs. 2 and 3, respectively. The parameters have been chosen to reproduce, in the closed-system limit, the experimental results reported by [7] for a nonlinear one-quantum interaction and the same reference ratio $\kappa/\kappa_{0k}^{crit}=10^{-1}$, $k=1,2$, has been maintained as before. In all the cases the measurement tends to wash out collapses and revivals making it more difficult to infer the quantized nature of the vibrational motion in the trap. For larger values of $\kappa/\kappa_{0k}^{crit}$ the suppression is even more pronounced, while for $\kappa/\kappa_{0k}^{crit}>1$ a complete freezing of the population dynamics to the initial occupation is achieved, regardless of the initial vibrational state. A qualitatively similar quenching of the collapse-revival phenomena for the linear one-quantum Jaynes-Cummings dynamics of a single two-level atom in an optical cavity is described in [13] by including both spontaneous emission and cavity damping mechanisms. The relative insensitivity to spontaneous emission damping reported there further enforces the possibility of neglecting such effect in our description.

Trapped ions seem particularly adequate to investigate the vanishing of revivals in coherent states induced by the measurement of their occupancy probability. Relaxation mechanisms due to spontaneous decay of the electronic transition or mechanical dissipation of the ion vibrational motion are indeed negligible [14,15]. A technical source of decoherence has been instead identified in [7] and ascribed to laser and frequency trap fluctuations, contributing to a finite linewidth for the electronic transition depending upon the vibrational state, $\gamma_n=\gamma_0(n+1)^{0.7}$, $\gamma_0=11.94$ kHz. This yields an average decay time $\tau_{\overline{n}}=\gamma_{\overline{n}}^{-1}=31.2\mu$s for $\overline{n}=3.1$. On the other hand, the data collected in the quantum Zeno experiment [14], exploiting the same auxiliary transition ${}^2S_{1/2}\rightarrow{}^2P_{3/2}$ of ${}^9Be^+$ also used in [7], are fitted through Eq. (1) with $\kappa\approx 4.9\times 10^4$ s^{-1} [6]. The corresponding decay time is $\tau=4\kappa^{-1}\simeq 816$ μs, one order of magnitude larger than the

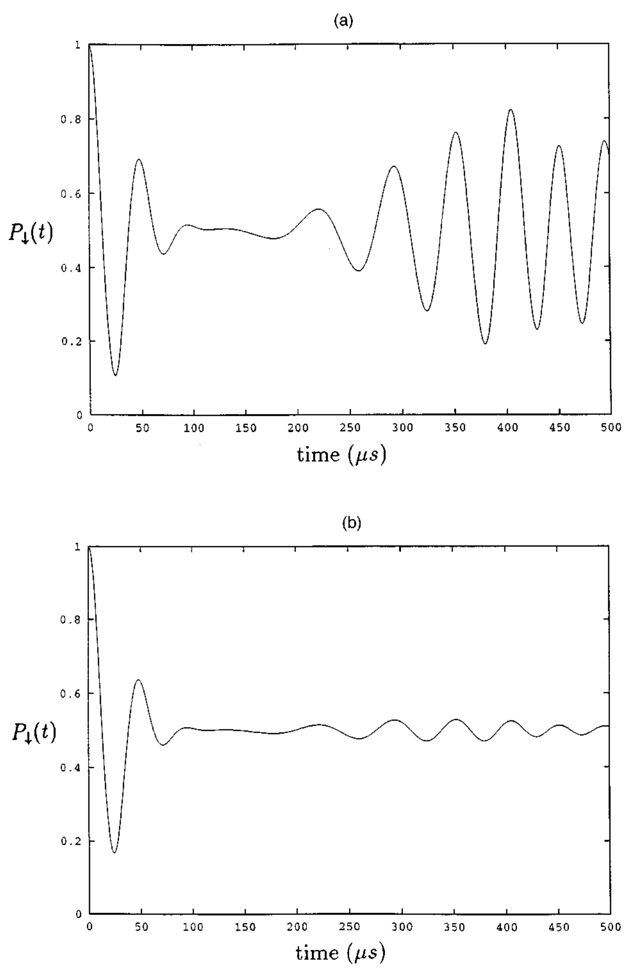

FIG. 1. (a) Ground-state occupation vs time for a linear one-quantum resonance and an initial coherent-state distribution, showing collapses and revivals. The parameters $\overline{n}=3.1$, $\eta=10^{-2}$, and $\Omega_{01}/2\pi=4.75$ KHz have been used and $\kappa=0$. (b) Same as in (a) except $\kappa/\kappa_{01}^{crit}=10^{-1}$.

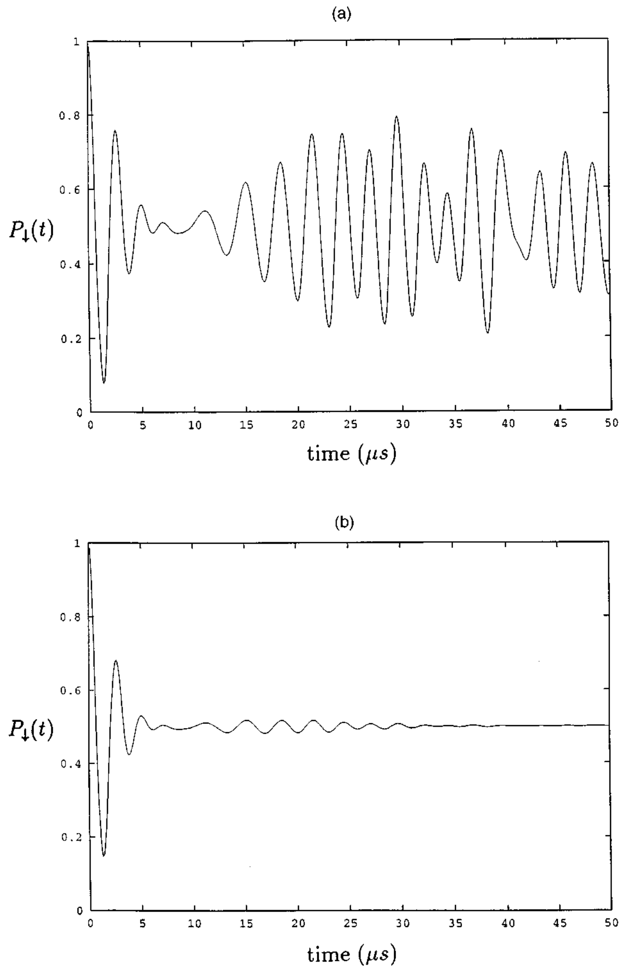

FIG. 2. (a) Same as in Fig. 1(a) but for a one-quantum nonlinear Jaynes-Cummings model, $\Omega_{01}/2\pi = 94$ KHz and $\eta = 0.202$. (b) $\kappa/\kappa_{01}^{crit} = 10^{-1}$.

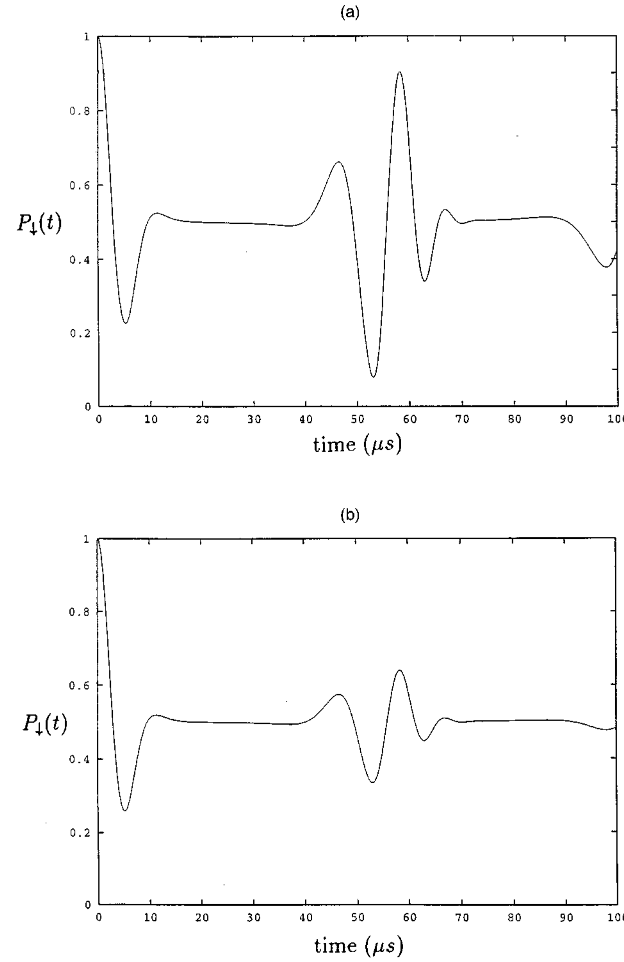

FIG. 3. (a) Same as in Fig. 2(a) but for a two-quantum nonlinear interaction, $\Omega_{02}/2\pi$ =13.4 KHz. (b) $\kappa/\kappa_{02}^{crit} = 10^{-1}$.

technical decoherence decay constant, suggesting that an improvement in the stability performances of the experimental setup by roughly a factor 10^2 should imply a measurement-dominated decoherence. The previous numerical example corresponds to a ratio $\kappa/\kappa_{\tilde{n}\tilde{n}+k}^{crit} = 1.1 \times 10^{-2}$, i.e., a weak-coupling regime. An increase of this ratio by two orders of magnitude, for instance, attainable by using the same measurement configuration but a smaller Rabi frequency Ω_0, allows one to test the region where neither a closed system evolution nor a von Neumann instantaneous projection of the state are suitable for reproducing the experimental results. A description in terms of the Lindblad equation (1) is instead mandatory.

The predicted effect does not exhaust the influence of the measurement process on the ion dynamics. As investigated in [16] in a different context, the measurement also *indirectly* affects the average vibrational motion of the trapped ion, a quantum damping without classical analog whose features will be described in [17]. Furthermore, analogous decoherence effects can be studied within the same framework as in other physical systems, atomic Bose-Einstein condensates being particularly appealing in connection with a recent analysis of their collapses and revivals dynamics [18]. From a general viewpoint, experimental investigations of the effect of the measurement process as the one proposed here will allow for a quantitative description of this ultimate source of decoherence, a topic which, besides its established importance in defining the boundary between quantum and classical worlds [19], is becoming crucial for the issue of quantum computation [20].

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