Exact solution of the Jaynes-Cummings model with cavity damping

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Operating in Laplace language and making use of a representation based on photon-number states, we find the exact solution for the density operator that belongs to the Jaynes-Cummings model with cavity damping. The detuning parameter is set equal to zero and the optical resonator does not contain any thermal photons. It is shown that the master equation for the density operator can be replaced by two algebraic recursion relations for vectors of dimension 2 and 4. These vectors are built up from suitably chosen matrix elements of the density operator. By performing an iterative procedure, the exact solution for each matrix element is found in the form of an infinite series. We demonstrate that all series are convergent and discuss how they can be truncated when carrying out numerical work. With the help of techniques from function theory, it is proved that our solutions respect the following conditions on the density operator: conservation of trace, Hermiticity, convergence to the initial state for small times, and convergence to the ground state for large times. We compute some matrix elements of the density operator for the case of weak damping and find that their analytic structure becomes much simpler. Finally, it is shown that the exact atomic density matrix converges to the state of maximum von Neumann entropy if the time, the square of the initial electromagnetic energy density, and the inverse of the cavity-damping parameter tend to infinity equally fast. The initial condition for the atom can be chosen freely, whereas the field may start from either a coherent or a photon-number state. [S1050-2947(97)00108-X]

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I. INTRODUCTION

Since the early 1980s, we have been witnessing a growing interest [1–8] in dissipative variants of the Jaynes-Cummings model [9], one of the principal assets in quantum optics [10]. Most workers have introduced dissipation by means of a coupling to one or more Markovian reservoirs. In doing so, one is led to a fully quantum-mechanical master equation. The theoretical efforts have been initiated primarily by experimental successes. Major progress has been made in the experimental realization of the interaction between a two-level atom and a single mode of the quantized electromagnetic (em) radiation field [11]. In short, it has become possible to carry out precise tests on theoretical predictions that are obtained by solving those extended versions of the Jaynes-Cummings model that include all of the relevant damping mechanisms.

Besides the experimental drive, there exists also a theoretical motivation to add damping to the Jaynes-Cummings model. If energy leaks away from the system of atom and field mode, its dynamics becomes much more interesting. For short times one still encounters the famous collapses and revivals in the Rabi oscillations of the atomic inversion [12], but for large times one now observes an exponential decay of atom and field mode toward their ground states. In other words, we can study within a fully quantum-mechanical context a time evolution that exhibits a crossover from a regime of quasireversible character to a regime that is truly irreversible. Such a program requires that exact analytical solutions be available. It is our aim to derive these for the case that depletion occurs in the field mode.

The reason for making the above choice of damping mechanism is a practical one. In an experimental setup energy losses stem from photon escape through the cavity mirrors, as well as from spontaneous emission of photons by the atom. If the cavity contains many photons, the first mechanism causes by far the greatest losses during a fixed time interval [13]. This statement can be made plausible by a simple argument. The atom can absorb only one photon at a time, whereas the cavity mirrors can let through any number of photons at a time. Hence, for the Jaynes-Cummings model spontaneous emission gives rise to a damping mechanism of sequential nature, which acts much slower than the mechanism originating from the finite transparency of the cavity mirrors. Of course, if the excited state of the atom becomes very short lived, then our reasoning is no longer valid.

In extracting exact results from dissipative counterparts of the Jaynes-Cummings model, a variety of strategies can be adopted. The most recent one relies upon the use of damping bases [14] and has the advantage that master equations of a high complexity can be handled [15,16]. In particular, the temperature inside the cavity may differ from zero. Then the method yields a set of recursion relations, the solution of which can be represented with the help of matrix continued fractions [17]. The latter have been evaluated numerically [18]. For the case of zero temperature the damped Jaynes-Cummings model can be solved analytically by employing the method of damping bases [14].

The other approaches have a more traditional character. For instance, the mathematical problem at hand can be formulated in terms of partial differential equations for a set of classical distribution functions [19]. These have been solved for the case that the cavity does not contain any thermal photons [20]. Analytic expressions for the diagonals of the atomic density matrix and the mean photon number have been derived [21].

Finally, the most familiar strategy [6,13,22–24] is based on representations of the full density operator that employ direct products between atomic states and photon-number states. The master equation is replaced by a set of ordinary

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differential equations for the matrix elements of the density operator. It has been suggested in the literature [6,13,16,22,24,25] that in solving such a system, one is obliged to treat at least N^2 equations simultaneously, where N linearly depends on some truncation parameter for photon number. We shall demonstrate that this obligation does not exist at all. For the case of cavity damping at zero temperature, the complete set of matrix elements will be evaluated via a simple recipe. Therefore, we shall be able to present the solution for the full atomic density matrix. The latter gives access to the von Neumann entropy of the atom.

This paper is organized as follows. In Sec. II we lay the mathematical foundations for our method, making use of several assumptions. The matrix elements of the density operator are evaluated in Sec. III and are shown to fulfill a number of important requirements. As discussed in Sec. IV, our solutions are consistent with all of the assumptions made earlier. In Sec. V we examine the behavior of the atomic density matrix for weak damping and large times. An interesting asymptotic limit is put forth. A summary of all results is given in Sec. VI.

In order to fix notations and make our paper self-contained, we close this introduction with a brief review of the model that will be solved. It describes the quantum-mechanical interaction between a motionless two-level atom, which is enclosed in an optical resonator, and a single mode of the em radiation field. The atom, mode, and cavity are in perfect resonance with each other. The mirrors of the cavity are slightly nonideal, so a cavity mode can lose energy to the surroundings. This is not the case for the atom, as it is supposed not to interact with em modes other than the privileged one. We assume that the cavity does not contain any thermal photons.

The density operator for the system of the atom (A) and field (F) is denoted by $\rho(t)$ and acts on Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_F$. We choose $\mathcal{H}_A=\mathrm{C}^2$, with the excited state and ground state of the atom represented by vectors $(1,0)^T$ and $(0,1)^T$, respectively. We define matrices

$$i_+=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad i_-=\begin{pmatrix}0&0\\0&1\end{pmatrix},$$

$$\sigma_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \sigma_-=\begin{pmatrix}0&0\\1&0\end{pmatrix} \tag{1}$$

and observe that each atomic operator can be represented by a linear combination of these. The ladder operators of the cavity mode are called a and $a^\dagger$, so the orthonormal photon-number states $\{|n\rangle\}_{n=0}^{\infty}$, which span Hilbert space $\mathcal{H}_F$, are given by

$$|n\rangle=(n!)^{-1/2}(a^\dagger)^n|0\rangle. \tag{2}$$

The commutator $[a,a^\dagger]$ equals unity and the state $a|0\rangle$ is identical to the zero element of $\mathcal{H}_F$.

We assume that in the interaction picture the time evolution of the density operator is governed by the master equation [26]

$$d\rho(t)/dt=\mathcal{L}_1[\rho(t)]+\kappa\mathcal{L}_2[\rho(t)], \tag{3}$$

where the new operators are defined as

$$\mathcal{L}_1[\rho]=-i[\sigma_+\otimes a+\sigma_-\otimes a^\dagger,\rho], \tag{4}$$

$$\mathcal{L}_2[\rho]=2(\mathbb{1}_2\otimes a)\rho(\mathbb{1}_2\otimes a^\dagger)-(\mathbb{1}_2\otimes a^\dagger a)\rho-\rho(\mathbb{1}_2\otimes a^\dagger a). \tag{5}$$

In the Jaynes-Cummings contribution (4) the rotating-wave and electric-dipole approximations have been employed. Contribution (5) brings about damping and is of Markovian nature. The master equation has been divided by the coupling constant of the Jaynes-Cummings term, so the time t and damping constant κ are dimensionless. For technical reasons κ will be limited to the interval $[0,2)$. We mention that contribution (5) preserves the trace, self-adjointness, and positivity of the density operator [27]. On the other hand, it provides a realistic description of photon loss through the cavity mirrors only for rather low values of κ [28].

II. METHOD OF SOLUTION

Our first step consists of transforming Eq. (3) into a set of c-number equations. To that end, we propose the decomposition

$$\rho(t)=i_+\otimes\rho_1(t)+\sigma_-\otimes\rho_2(t)+\sigma_+\otimes\rho_3(t)+i_-\otimes\rho_4(t), \tag{6}$$

with operators $\{\rho_j(t)\}$ acting on space $\mathcal{H}_F$. By making use of the linear independence of matrices (1), as well as the properties of ladder operators a and $a^\dagger$, one can derive equations of motion for the matrix elements

$$\rho_j(t)_{m,n}=\langle m|\rho_j(t)|n\rangle, \tag{7}$$

where on the right-hand side photon-number states (2) figure.

The result can be cast into a remarkable form, given by

$$d\mathbf{v}(t;m,n)/dt=\mathsf{A}(m,n)\ \mathbf{v}(t;m,n)+2\kappa\ \mathsf{S}(m,n)\ \mathbf{v}(t;m+1,n+1), \tag{8}$$

$$d\mathbf{w}(t;0,n)/dt=\mathsf{B}(n)\mathbf{w}(t;0,n)+2\kappa\mathsf{T}(n)\mathbf{w}(t;1,n+1), \tag{9}$$

$$d\rho_4(t)_{0,0}/dt=2\kappa\rho_4(t)_{1,1}. \tag{10}$$

Integers m and n run from zero to infinity. We have introduced vectors

$$\mathbf{v}(t;m,n)=[\rho_1(t)_{m,n},\rho_2(t)_{m+1,n},\rho_3(t)_{m,n+1},\rho_4(t)_{m+1,n+1}]^T,$$

$$\mathbf{w}(t;m,n)=[\rho_2(t)_{m,n},\rho_4(t)_{m,n+1}]^T \tag{11}$$

and matrices

$$\mathsf{A}(m,n)=\mathbb{1}_2\otimes\mathsf{B}(m)^\dagger+\mathsf{B}(n)\otimes\mathbb{1}_2, \tag{12}$$

$$\mathsf{S}(m,n)=\mathsf{T}(n)\otimes\mathsf{T}(m),$$

$$\mathsf{B}(m)=\begin{pmatrix}-\kappa m & i(m+1)^{1/2}\\ i(m+1)^{1/2} & -\kappa(m+1)\end{pmatrix},$$

$$\mathsf{T}(m)=\begin{pmatrix}(m+1)^{1/2} & 0\\ 0 & (m+2)^{1/2}\end{pmatrix}.$$

For the direct product of two matrices the standard definition has been employed [29]. Use of the self-adjointness of $\rho(t)$ in decomposition (6) leads to the symmetry relations

$$\rho_j(t)^*_{m,n}=\rho_j(t)_{n,m},\quad \rho_2(t)^*_{m,n}=\rho_3(t)_{n,m}, \tag{13}$$

for $j=1,4$. Therefore, vectors $\mathbf{v}$ and $\mathbf{w}$ deliver us all matrix elements (7), except for the element with $j=4$ and $m=n=0$.

The infinite set (8) of ordinary differential equations calls for the employment of Laplace transformation. Defining the transform of a function $f(t)$ as

$$\hat{f}(z)=-i\int_0^\infty dte^{izt}f(t), \tag{14}$$

with Imz positive, one obtains from Eq. (8) the set of algebraic equations

$$\hat{\mathbf{v}}(z;m,n)=[z\mathbb{1}_4-i\mathsf{A}(m,n)]^{-1}\{\mathbf{v}(t=0;m,n)+2i\kappa\,\mathsf{S}(m,n)\hat{\mathbf{v}}(z;m+1,n+1)\}. \tag{15}$$

As shown in Sec. IV, all imaginary parts of the eigenvalues of matrix $i\mathsf{A}(m,n)$ are smaller than $-\kappa$, so the inverse matrix in Eq. (15) exists.

We iterate the recursion relation (15) a finite number of times, carry out inverse Laplace transformation, and employ Jordan's lemma [30]. This brings us to

$$\mathbf{v}(t;m,n)=\mathbf{v}(t;N;m,n)+\mathbf{r}(t;N;m,n), \tag{16}$$

$$\mathbf{v}(t;N;m,n)=\sum_{k=0}^{N}\frac{(2i\kappa)^k}{2\pi i}\oint_{\Gamma(k;m,n)}dz\,e^{-izt}\mathsf{G}(z;k;m,n)\times\mathbf{v}(t=0;m+k,n+k), \tag{17}$$

$$\mathbf{r}(t;N;m,n)=\frac{i(2i\kappa)^{N+1}}{2\pi}\int_C dz\,e^{-izt}\mathsf{G}(z;N;m,n)\times\mathsf{S}(m+N,n+N)\times\hat{\mathbf{v}}(z;m+N+1,n+N+1), \tag{18}$$

where N may equal any positive integer and t must be chosen positive. Contour C runs above and parallel to the real axis, whereas the closed contour $\Gamma(k;m,n)$ encircles all poles of matrix $\mathsf{G}(z;k;m,n)$ counterclockwise. The latter must be constructed as

$$\begin{aligned}\mathsf{G}(z;k;m,n)=&[z\mathbb{1}_4-i\mathsf{A}(m,n)]^{-1}\\&\times\mathsf{S}(m,n)[z\mathbb{1}_4-i\mathsf{A}(m+1,n+1)]^{-1}\\&\times\mathsf{S}(m+1,n+1)\cdots\\&\times[z\mathbb{1}_4-i\mathsf{A}(m+k-1,n+k-1)]^{-1}\\&\times\mathsf{S}(m+k-1,n+k-1)\\&\times[z\mathbb{1}_4-i\mathsf{A}(m+k,n+k)]^{-1}.\end{aligned} \tag{19}$$

For $k=0$ the right-hand side (rhs) must be set equal to the matrix $[z\mathbb{1}_4-i\mathsf{A}(m,n)]^{-1}$.

In order that series (17) be convergent, the remainder $\mathbf{r}(t;N;m,n)$ should vanish as N tends to infinity. Going for an estimate of expression (18), we assume that the Euclidean norm of the vector $\mathbf{v}(t;m,n)$ satisfies the inequality

$$\|\mathbf{v}(t;m,n)\|\leqslant g_1(m!n!)^{-1/2}(1+m^{1/2})(1+n^{1/2})h^{m+n}\times e^{-\kappa t(m+n+1)}, \tag{20}$$

where g_1 and h are independent of indices m,n and time. Now each component of the vector $\hat{\mathbf{v}}(z;m,n)$ makes up a function that is analytic on the half space Im$z>-\kappa$. Since the same holds true for each element of the matrix $\mathsf{G}(z;N;m,n)$, we may shift in Eq. (18) the contour C below the real axis, toward the line Im$z=-\kappa/2$, for instance.

Employing a result of Sec. IV, viz.,

$$\|[z\mathbb{1}_4-i\mathsf{A}(m,n)]^{-1}\|\leqslant g_2|\mathrm{Im}z+\kappa(m+n+1)|^{-1}, \tag{21}$$

where g_2 is independent of m, n, and z, one can propose the following bound on the sup norm of the matrix G:

$$\begin{aligned}\|\mathsf{G}(z;N;m,n)\|\leqslant&\|[z\mathbb{1}_4-i\mathsf{A}(m,n)]^{-1}\|\\&\times\|[z\mathbb{1}_4-i\mathsf{A}(m+1,n+1)]^{-1}\|\\&\times\frac{1}{N!}\left(\frac{g_2}{2\kappa}\right)^{N-1}\\&\times\left[\frac{(m+N+1)!(n+N+1)!}{(m+1)!(n+1)!}\right]^{1/2},\end{aligned} \tag{22}$$

with Im$z>-\kappa$. Use has been made of the equality

$$\|S(m,n)\|=(m+2)^{1/2}(n+2)^{1/2}. \tag{23}$$

A bound on norm $\|\hat{\mathbf{v}}(z;m,n)\|$ can be inferred from definition (14) and assumption (20). The combination of all inequalities leads to the estimate we are after. It reads

$$\|\mathbf{r}(t;N;m,n)\|\leqslant 16\kappa g_1 e^{-\kappa t/2}\frac{(g_2h^2)^{N-1}}{\pi(N-1)!}\times\frac{|I(m,n)|h^{m+n+4}}{[(m+1)!(n+1)!]^{1/2}}, \tag{24}$$

where N has been taken larger than $\max(m+2,n+2)$ and the inequality $1+n^{1/2}\leqslant 2n^{1/2}$, valid for $n\geqslant 1$, has been employed. Furthermore, we have defined an integral

$$I(m,n)=\int_C dz\|[z\mathbb{1}_4-i\mathsf{A}(m,n)]^{-1}\|\times\|[z\mathbb{1}_4-i\mathsf{A}(m+1,n+1)]^{-1}\|. \tag{25}$$

It is convergent because the integrand behaves as $|z|^{-2}$ for $|z|$ large.

The identity (16) and estimate (24) enable us to write

$$\lim_{N\to\infty}\sup_{0<t<\infty}\|\mathbf{v}(t;m,n)-\mathbf{v}(t;N;m,n)\|=0. \tag{26}$$

In other words, for $t>0$ the series $\mathbf{v}(t;\infty;m,n)$ exists and represents the solution of the differential equation (8). Moreover, each component of $\mathbf{v}(t;\infty;m,n)$ is continuous on the positive time axis.

Of course, we have to verify *a posteriori* that the solution $\mathbf{v}(t;\infty;m,n)$ indeed obeys condition (20). This technical job will be taken care of in Sec. IV. There we also show that the series $\mathbf{v}(t=0;\infty;m,n)$ is convergent. Hence limit (26) implies that

$$\lim_{t\to 0^+}\|\mathbf{v}(t;\infty;m,n)-\mathbf{v}(t=0;\infty;m,n)\|=0. \qquad (27)$$

The second contribution inside the norm can be evaluated by resorting to standard techniques [31].

In Eq. (17) the contour $\Gamma(k;m,n)$ may be blown up to become a circle $|z|=R$. The radius R is chosen such that for each matrix $\mathsf{A}(k,l)$, figuring in Eq. (19), the inequality $\|\mathsf{A}(k,l)\|/R<1$ is true. Now all resolvents can be expanded into Neumann series. Since these converge uniformly on the circle $|z|=R$, we may integrate in Eq. (17) term by term. Owing to the identity $\oint_\Gamma dz\, z^{-n}=2\pi i\,\delta_{1,n}$, with n an integer, one ends up with the simple result

$$\frac{1}{2\pi i}\oint_{\Gamma(k;m,n)} dz\,\mathsf{G}(z;k;m,n)=\mathbb{1}_4\delta_{0,k}. \qquad (28)$$

The combination of Eqs. (17), (27), and (28) yields the satisfactory statement

$$\lim_{t\to 0^+}\|\mathbf{v}(t;\infty;m,n)-\mathbf{v}(t=0;m,n)\|=0, \qquad (29)$$

which tells us that our solution of the differential equation (8) respects the accessory initial condition.

The differential equation (9) can also be solved with the help of Laplace transformation. Employing the convolution theorem, one finds for each positive time

$$\mathbf{w}(t;0,n)=\mathsf{D}(t;n)\mathbf{w}(t=0;0,n)+2\kappa\int_0^t dt'\,\mathsf{D}(t-t';n)\times\mathsf{T}(n)\mathbf{w}(t';1,n+1). \qquad (30)$$

The new matrix is given by

$$\mathsf{D}(t;n)=\frac{1}{2\pi i}\oint_{\Delta(n)} dz\,e^{-izt}[z\mathbb{1}_2-i\mathsf{B}(n)]^{-1}, \qquad (31)$$

where the closed contour $\Delta(n)$ encircles all poles of the integrand on the right-hand side. By repeating the argument presented above (28), one proves that solution (30) converges to the initial vector $\mathbf{w}(t=0;0,n)$ for small times.

Upon computing matrix (31) we recognize that

$$\|\mathsf{D}(t;n)\|\leq g_3 e^{-\kappa t(n+1/2)}, \qquad (32)$$

where again g_3 denotes a positive constant, independent of the index n and time. With the aid of inequalities (20) and (32) we deduce from Eq. (30) the estimate

$$\|\mathbf{w}(t;0,n)\|\leq g_4(n!)^{-1/2}(1+n^{1/2})h^n e^{-\kappa t(n+1/2)}. \qquad (33)$$

The constant g_4 shares its properties with g_3. We have assumed that the initial vector $\mathbf{w}(t=0;0,n)$ satisfies Eq. (33).

In this section a mathematical framework has been assembled that allows us to solve the Jaynes-Cummings model with cavity damping exactly. From Eqs. (20) and (33) one derives the following inequality for both the sup norm and the trace norm of $\rho(t)$:

$$\|\rho(t)\|\leq\sum_{j=1}^{4}\sum_{m,n=0}^{\infty}|\rho_j(t)_{m,n}|<\infty. \qquad (34)$$

It demonstrates the existence of the density operator that is generated by our method. Therefore, we can commence evaluating matrix elements $\rho_j(t)_{m,n}$.

III. CALCULATION OF MATRIX ELEMENTS

We calculate the resolvent $[z\mathbb{1}_4-i\mathsf{A}(m,n)]^{-1}$ for $\kappa\leq 2$ and find that it gives rise to the poles

$$z=i\mu_{\eta_1,\eta_2}(m,n),$$

$$\mu_{\eta_1,\eta_2}(m,n)=-\kappa(m+n+1)-i\eta_1 u_m-i\eta_2 u_n, \qquad (35)$$

with the square root

$$u_n=(n+1-\kappa^2/4)^{1/2} \qquad (36)$$

and the prescription

$$(\eta_1,\eta_2)=(+1,+1),(+1,-1),(-1,+1),(-1,-1),\ m\neq n$$

$$(\eta_1,\eta_2)=(+1,+1),(+1,-1),(-1,-1),\quad m=n. \qquad (37)$$

Thus, in elaborating Eq. (17), one has to treat the cases $m\neq n$ and $m=n$ separately.

The reader can check that the following results are obtained:

$$\mathbf{v}(t;\infty;m,n)=\frac{1}{8}\sum_{\eta_1,\eta_2}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(2i\kappa)^k\exp[\mu_{\eta_1,\eta_2}(m+l,n+l)t]}{u_{m+l}u_{n+l}(\eta_2 u_{m+l}+\eta_1 u_{n+l})}\times\frac{\mathsf{V}[i\mu_{\eta_1,\eta_2}(m+l,n+l);k;m,n]\mathbf{v}(t=0;m+k,n+k)}{\prod_{\eta_3,\eta_4}\prod_{p=0}^{k}{}'\,[i\mu_{\eta_1,\eta_2}(m+l,n+l)-i\mu_{\eta_3,\eta_4}(m+p,n+p)]}, \qquad (38)$$

$$\mathbf{v}(t;\infty;n,n)=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-1)^l\exp[\mu_{+1,-1}(n+l,n+l)t]}{l!(k-l)!}\frac{\mathsf{W}(i\mu_{+1,-1}(n+l,n+l);k;n)\,\mathbf{v}(t=0;n+k,n+k)}{\prod_{\eta_1}\prod_{p=0}^{k}[i\mu_{+1,-1}(n+l,n+l)-i\mu_{\eta_1,\eta_1}(n+p,n+p)]}$$
$$+\frac{1}{8}\sum_{\eta_1}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(2i\kappa)^k\exp[\mu_{\eta_1,\eta_1}(n+l,n+l)t]}{u_{n+l}u_{n+l}}\frac{\mathsf{W}[i\mu_{\eta_1,\eta_1}(n+l,n+l);k;n]\mathbf{v}(t=0;n+k,n+k)}{\prod'_{\eta_2,\eta_3}\prod'^{k}_{p=0}[i\mu_{\eta_1,\eta_1}(n+l,n+l)-i\mu_{\eta_2,\eta_3}(n+p,n+p)]},\tag{39}$$

where m and n are non-negative integers. Each index η_i belonging to a sum or product runs through the values -1 and $+1$. The prime in the product over η_2 and η_3 indicates that the choice $(\eta_2,\eta_3)=(-1,+1)$ should not be made. In a primed product over p the choice $p=l$ must be excluded. In the case that k and l are zero, the latter product equals unity.

The matrix V, figuring in Eq. (38), must be constructed according to the recipe

$$\begin{aligned}\mathsf{V}(z;k;m,n)=&\mathsf{R}(z;m,n)\mathsf{S}(m,n)\mathsf{R}(z;m+1,n+1)\\&\times\mathsf{S}(m+1,n+1)\cdots\mathsf{R}(z;m+k-1,n+k-1)\\&\times\mathsf{S}(m+k-1,n+k-1)\mathsf{R}(z;m+k,n+k),\end{aligned}\tag{40}$$

where k stands for a positive integer and z may be any complex number. For $k=0$ the choice $\mathsf{V}(z;0;m,n)=\mathsf{R}(z;m,n)$ must be made. The matrix R must be obtained from

$$\mathsf{R}(z;m,n)=\det[z\mathbb{1}_4-i\,\mathsf{A}(m,n)][z\mathbb{1}_4-i\mathsf{A}(m,n)]^{-1}.\tag{41}$$

The computation of the resolvent provides us with the explicit results

$$\mathsf{R}(z;m,n)_{11}=a_{m,n}(z)[a_{m,n}(z)^2+i\kappa a_{m,n}(z)-m-n-2],$$

$$\mathsf{R}(z;m,n)_{13}=(n+1)^{1/2}[-a_{m,n}(z)^2-i\kappa a_{m,n}(z)-m+n],$$

$$\mathsf{R}(z;m,n)_{14}=\mathsf{R}(z;m,n)_{23}=-2(m+1)^{1/2}(n+1)^{1/2}a_{m,n}(z),$$

$$\begin{aligned}\mathsf{R}(z;m,n)_{33}=&a_{m,n}(z)[a_{m,n}(z)^2+\kappa^2]\\&-(m+1)[a_{m,n}(z)+i\kappa]\\&-(n+1)[a_{m,n}(z)-i\kappa],\end{aligned}\tag{42}$$

where we have defined

$$a_{m,n}(z)=z+i\kappa(m+n+1).\tag{43}$$

The remaining elements of R are determined by the symmetry relations

$$\mathsf{R}(z;m,n)_{kl}=\mathsf{R}(z;m,n)_{lk},$$

$$\begin{aligned}\mathsf{R}[a^*_{m,n}(z^*);m,n]_{kl}&=-\mathsf{R}[a^*_{m,n}(-z^*);n,m]_{P(k)P(l)}\\&=-\mathsf{R}[a^*_{m,n}(-z);n,m]^*_{Q(k)Q(l)}.\end{aligned}\tag{44}$$

Characters P and Q denote the permutations $(1234)\rightarrow(4231)$ and $(1234)\rightarrow(1324)$, respectively.

The matrix W, figuring in Eq. (39), is created from

$$\mathsf{W}(z;k;n)=\mathsf{V}(z;k;n,n)\prod_{l=0}^{k}a^{-1}_{n+l,n+l}(z).\tag{45}$$

For the sake of clarity we mention that the matrix $\mathsf{V}(z;k;n,n)$ contains the product $\Pi_{l=0}^{k}a_{n+l,n+l}(z)$ as a factor. Relations (40)–(45) completely specify how solutions (38) and (39) depend on the damping parameter κ, time t, and integers m,n.

The result (38) furnishes the solution for the vector $\mathbf{w}(t;1,n+1)$, on account of definitions (11). We substitute this solution into the rhs of Eq. (30) and interchange the integration over time with all summations. The last step is permitted because, by Eq. (20), the series for vector $\mathbf{w}(t;1,n+1)$ converges uniformly on the positive time axis. The integration can now be done, and for each non-negative integer n one arrives at

$$\begin{pmatrix}\rho_2(t)_{0,n}\\\rho_4(t)_{0,n+1}\end{pmatrix}=\frac{1}{2u_n}\sum_{\eta_1}\exp[\mu_{\eta_1,\eta_1}(n,n)t/2]\times\begin{pmatrix}u_n+i\eta_1\kappa/2&-\eta_1(n+1)^{1/2}\\-\eta_1(n+1)^{1/2}&u_n-i\eta_1\kappa/2\end{pmatrix}\times\begin{pmatrix}\rho_2(t=0)_{0,n}+\mathbf{s}_{\eta_1}(t;n)_2\\\rho_4(t=0)_{0,n+1}+\mathbf{s}_{\eta_1}(t;n)_4\end{pmatrix}.\tag{46}$$

The rhs contains the second and fourth components of a new vector, given by

$$\mathbf{s}_{\eta_1}(t;n)=\frac{\kappa}{4\sqrt{2}}\sum_{\eta_2,\eta_3}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(2i\kappa)^k\{1-\exp[-\mu_{\eta_1,\eta_1}(n,n)t/2+\mu_{\eta_2,\eta_3}(l,n+l+1)t]\}}{u_l u_{n+l+1}(\eta_3 u_l+\eta_2 u_{n+l+1})[\mu_{\eta_1,\eta_1}(n,n)/2-\mu_{\eta_2,\eta_3}(l,n+l+1)]}$$
$$\times\frac{\mathsf{S}(0,n)\mathsf{V}[i\mu_{\eta_2,\eta_3}(l,n+l+1);k;0,n+1]\mathbf{v}(t=0;k,n+k+1)}{\prod_{\eta_4,\eta_5}\prod_{p=0}^{k}{}'\,[i\mu_{\eta_2,\eta_3}(l,n+l+1)-i\mu_{\eta_4,\eta_5}(p,n+p+1)]}. \tag{47}$$

For summations and products the same conventions have been used as in Eqs. (38) and (39).

Evaluation of the last matrix element $\rho_4(t)_{0,0}$ can happen by integrating Eq. (10) over time. Its rhs follows from Eq. (39), with n set equal to zero. Operating in a similar vein as above, one is led to

$$\rho_4(t)_{0,0}=\rho_4(t=0)_{0,0}+\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-1)^l\{1-\exp[\mu_{+1,-1}(l,l)t]\}\{\mathsf{W}[i\mu_{+1,-1}(l,l);k;0]\mathbf{v}(t=0;k,k)\}_4}{(l+1/2)l!(k-l)!\prod_{\eta_1}\prod_{p=0}^{k}[i\mu_{+1,-1}(l,l)-i\mu_{\eta_1,\eta_1}(p,p)]}$$
$$+\frac{1}{4}\kappa\sum_{\eta_1}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(2i\kappa)^k\{\exp[\mu_{\eta_1,\eta_1}(l,l)t]-1\}\{\mathsf{W}[i\mu_{\eta_1,\eta_1}(l,l);k;0]\mathbf{v}(t=0;k,k)\}_4}{u_l u_l\mu_{\eta_1,\eta_1}(l,l)\prod_{\eta_2,\eta_3}{}'\prod_{p=0}^{k}{}'\,[i\mu_{\eta_1,\eta_1}(l,l)-i\mu_{\eta_2,\eta_3}(p,p)]}. \tag{48}$$

From the results (38), (39), (46), and (48), supplemented with symmetry relations (13), all matrix elements $\rho_j(t)_{m,n}$ can be found. As an alternative, one can first compute the complete set of matrix elements and subsequently check that the constraints (13) are indeed fulfilled. The symmetry properties (44) of the matrix $\mathsf{R}(z;m,n)$ should be used. A further check on our solutions is carried out by letting the damping parameter κ go to zero. Then one indeed recovers the matrix elements that represent the density operator of the undamped Jaynes-Cummings model.

In Sec. II we have proved that each matrix element converges to its initial value if time t tends to zero. As to the asymptotic behavior of our solutions, a physical argument can be brought into play: If one waits sufficiently long, all em energy will leak away from the cavity and the atom will stay in its ground state. We thus expect that the following limit is valid:

$$\lim_{t\to\infty}\|\rho(t)-i_-\otimes|0\rangle\langle 0|\|=0. \tag{49}$$

Since for both the sup norm and trace norm one has $\||m\rangle\langle n|\|=1$, the validity of Eq. (49) is implied by the statements

$$\lim_{t\to\infty}\sum_{j=1}^{4}\sum_{m,n=0}^{\infty}(1-\delta_{4,j}\delta_{0,m}\delta_{0,n})|\rho_j(t)_{m,n}|=0, \tag{50a}$$

$$\lim_{t\to\infty}\rho_4(t)_{0,0}=1. \tag{50b}$$

The series $\Sigma_{n=0}^{\infty}(n!)^{-1/2}h^n$ is convergent, so inequalities (20) and (33) tell us that Eq. (50a) is true.

For completion of the proof we return to Eq. (10). Evaluation of the limit (50b) requires that matrix element $\mathbf{v}(t;0,0)_4$ be integrated over the complete positive time axis. By Eq. (24) this is equivalent to first integrating the function $\mathbf{v}(t;N;0,0)_4$ and subsequently taking the integer N to infinity. In doing so, we extract from Eq. (10) the equality

$$\lim_{t\to\infty}\rho_4(t)_{0,0}=\rho_4(t=0)_{0,0}-\frac{\kappa}{\pi}\sum_{k=0}^{\infty}(2i\kappa)^k \times\oint_{\Gamma(k;0,0)}dz\,z^{-1}[\mathsf{G}(z;k;0,0)\times\mathbf{v}(t=0;k,k)]_4. \tag{51}$$

In virtue of an argument similar to the one used near Eq. (28), the above integral vanishes if $\Gamma(k;0,0)$ is replaced by a contour encircling all poles of matrix $\mathsf{G}(z;k;0,0)$, as well as the origin. Thus the integral is equal to the matrix $-2\pi i\mathsf{G}(z=0;k;0,0)$. The $4j$ element of the latter is given by $-\pi\kappa^{-1}(2i\kappa)^{-k}(\delta_{1,j}+\delta_{4,j})$, a result that can be established by means of induction. Substitution of our findings into Eq. (51) leads to the desired limit.

In finishing the foregoing proof, we have made use of the fact that for $t=0$ the trace of the density operator equals unity. Since this property is conserved in time, it should be made sure that our solutions respect the condition

$$\mathrm{Tr}_A\,\mathrm{Tr}_F\rho(t)=\sum_{j=1,4}\sum_{n=0}^{\infty}\mathbf{v}(t;\infty;n,n)_j+\rho_4(t)_{0,0}=1 \tag{52}$$

for all $t\geqslant 0$.

We repeat the derivation of Eq. (51), but refrain from taking t to infinity. For the matrix element $\rho_4(t=0)_{0,0}$ we insert the result that is found by uniting Eqs. (50b) and (51). The ensuing identity enables us to eliminate the matrix element $\rho_4(t)_{0,0}$ in Eq. (52). Upon eliminating the matrix element $\mathbf{v}(t;\infty;n,n)_j$ with the help of Eq. (17) and choosing new summation indices, we see that Eq. (52) is equivalent to the condition

$$\sum_{j=1}^{4}\sum_{k=0}^{\infty}\oint_{\Gamma_k} dz\, z^{-1}e^{-izt}\psi(z;k)_j\,\mathbf{v}(t=0;k,k)_j=0,$$

$$\psi(z;k)_j=z\sum_{p=1,4}\sum_{l=0}^{k}(2i\kappa)^l\mathsf{G}(z;l;k-l,k-l)_{pj}+(2i\kappa)^{k+1}\mathsf{G}(z;k;0,0)_{4j}, \tag{53}$$

where the contour Γ_k lies below the real axis and encircles all poles of the matrices $\{\mathsf{G}(z;l;k-l,k-l)\}_{l=0}^{k}$. By means of induction the identity $\psi(z;k)_j=\delta_{1,j}+\delta_{4,j}$ can be proved. Hence the integral (53) is indeed identical to zero.

In this section we have obtained explicit expressions for all matrix elements of the density operator. They have been shown to satisfy a number of important physical requirements. However, we still have to check whether our results are consistent with inequalities (20) and (21).

IV. CONVERGENCE

The material that is presented below demonstrates the mathematical soundness of our treatment. Besides that, it is of great use in performing a numerical evaluation of our solutions. We shall be able to tell how many terms of a series for a matrix element must be computed in order to achieve a given accuracy.

The initial density matrix $\rho(t=0)$ is assumed to satisfy the constraint

$$|\rho_j(t=0)_{m,n}|\leqslant g_5(m!n!)^{-1/2}h^{m+n}, \tag{54}$$

with g_5 independent of the integers m,n and the parameter h defined below Eq. (20). The above inequality implies that Eqs. (20) and (33) hold true at time zero, as required.

A possible choice for $\rho(t=0)$ is given by $\rho_A\otimes|\alpha\rangle\langle\alpha|$, where the coherent state is defined as $|\alpha\rangle=\exp(-|\alpha|^2/2+\alpha a^\dagger)|0\rangle$, α being a complex number. The character ρ_A stands for a fixed atomic density matrix. Constants h and g_5 come out as $|\alpha|$ and $\exp(-|\alpha|^2)$, respectively. The above initial condition gives rise to rich dynamics inside the optical cavity. As an example, one may mention the famous collapses and revivals in the Rabi oscillations of the atomic inversion [12]. A second choice that obeys Eq. (54) is given by $\rho_A\otimes|n\rangle\langle n|$.

Our first aim is the derivation of a bound on the sup norm of the non-Hermitian matrix $\mathsf{R}(z;m,n)$, defined in Eq. (41). Consider a complex block matrix M of dimension d; it satisfies the eigenvalue equation $\mathsf{M}\mathbf{v}^{(j)}=\lambda_j\mathbf{v}^{(j)}$, with $\{\mathbf{v}^{(j)}\}_{j=1}^{d}$ a set of normalized and linearly independent eigenvectors. The set of eigenvalues $\{\lambda_j\}_{j=1}^{d}$ may contain degeneracies. Now an arbitrary vector can be decomposed according to $\mathbf{v}=\Sigma_{j=1}^{d}c_j(\mathbf{v})\mathbf{v}^{(j)}$, where the objects $\{c_j(\mathbf{v})\}_{j=1}^{d}$ are complex coefficients. They can be expressed as $c_j(\mathbf{v})=\langle\Sigma_{k=1}^{d}c_j^*(\mathbf{w}^{(k)})\,\mathbf{w}^{(k)},\mathbf{v}\rangle$ if the set $\{\mathbf{w}^{(j)}\}_{j=1}^{d}$ is orthonormal.

On the basis of the last remark the inequality $|c_j(\mathbf{v})|\leqslant\|\mathbf{v}\|[\Sigma_{k=1}^{d}|c_j(\mathbf{w}^{(k)})|^2]^{1/2}$ can be proved. It enables us to propose for the norm of the matrix M the estimate

$$\|\mathsf{M}\|=\sup{}'\|\mathsf{M}\mathbf{v}\|/\|\mathbf{v}\|\leqslant\sum_{j=1}^{d}|\lambda_j|\left(\sum_{k=1}^{d}|c_j(\mathbf{w}^{(k)})|^2\right)^{1/2}, \tag{55}$$

where the prime reminds us of the condition $\mathbf{v}\neq 0$. The coefficients on the rhs are determined by the constraints

$$\sum_{k=1}^{d}\langle\mathbf{w}^{(i)},\mathbf{v}^{(k)}\rangle c_k(\mathbf{w}^{(j)})=\delta_{i,j}, \tag{56}$$

with $i,j=1,2,3,\ldots,d$.

The above result can be directly applied to the matrix at hand. From representation (12) we derive the eigenvalue equation

$$\mathsf{A}(m,n)\mathbf{q}_{\eta_1,\eta_2}(m,n;\kappa)=\mu_{\eta_1,\eta_2}(m,n)\mathbf{q}_{\eta_1,\eta_2}(m,n;\kappa). \tag{57}$$

Eigenvalues $\mu_{\eta_1,\eta_2}(m,n)$ can be found in Eq. (35). The normalized and linearly independent eigenvectors are given by

$$\mathbf{q}_{\eta_1,\eta_2}(m,n;\kappa)=\tfrac{1}{2}[1,b_{\eta_2}(n)]^T\otimes[1,-b_{\eta_1}(m)]^T,$$
$$b_{\eta_1}(m)=(m+1)^{-1/2}(-\eta_1u_m+i\kappa/2). \tag{58}$$

They are not orthogonal as long as the parameter κ differs from zero.

Obviously, the set $\{\mathbf{q}_{\eta_1,\eta_2}(m,n;\kappa=0)\}$ should serve as orthonormal frame $\{\mathbf{w}^{(j)}\}$. Then inversion of the matrix $\langle\mathbf{w}^{(i)},\mathbf{v}^{(k)}\rangle$does not demand great algebraic efforts because we merely need to deal with a direct product of two 2×2 matrices. The solution of Eq. (56) is now a straightforward affair and takes us to the identity

$$\left(\sum_{k=1}^{4}|c_j(\mathbf{w}^{(k)})|^2\right)^{1/2}=(m+1)^{1/2}(n+1)^{1/2}(u_mu_n)^{-1}. \tag{59}$$

The index j does not occur on the right. Hence, from estimate (55) we can deduce the statement

$$\|[z\mathbb{1}_4-i\mathsf{A}(m,n)]^{-1}\|\leqslant\frac{(m+1)^{1/2}(n+1)^{1/2}}{u_mu_n}\times\sum_{\eta_1,\eta_2}|z-i\mu_{\eta_1,\eta_2}(m,n)|^{-1}. \tag{60}$$

Multiplication of both sides by a factor

$$|\det[z\mathbb{1}_4-i\mathsf{A}(m,n)]|=\prod_{\eta_1,\eta_2}|z-i\mu_{\eta_1,\eta_2}(m,n)| \tag{61}$$

yields the desired bound on the norm of matrix $\mathsf{R}(z;m,n)$. Each denominator of Eq. (60) cancels out against a factor of Eq. (61), so for all complex z the bound is finite.

Statement (21) is a direct consequence of (60), in view of the inequalities

$$|z-i\mu_{\eta_1,\eta_2}(m,n)|\geqslant|\mathrm{Im}z+\kappa(m+n+1)|, \tag{62}$$

$$(m+k+1)^{1/2}/u_{m+k} \leqslant (k+1)^{1/2}/u_k, \tag{63}$$

with k, m, and n non-negative integers. The constant g_2 is found as $16/(4-\kappa^2)$, so indeed κ must be chosen inside the interval $[0,2)$.

The proof of statement (20) is more delicate. Assumption (54) allows us to control the norm of initial vector $\mathbf{v}(t=0;m,n)$. With the help of definition (11) we arrive at

$$\|\mathbf{v}(t=0;m,n)\| \leqslant g_5 \max(1,h^2) h^{m+n}\left[\frac{(m+2)(n+2)}{(m+1)!(n+1)!}\right]^{1/2}. \tag{64}$$

The combination of Eqs. (60)–(62) results in the inequalities

$$\frac{\|\mathsf{R}[i\mu_{\eta_1,\eta_2}(m+l,n+l);m+p,n+p]\|}{\prod_{\eta_3,\eta_4} |\mu_{\eta_1,\eta_2}(m+l,n+l)-\mu_{\eta_3,\eta_4}(m+p,n+p)|} \leqslant \frac{2(m+p+1)^{1/2}(n+p+1)^{1/2}}{\kappa u_{m+p}u_{n+p}|p-l|}, \tag{65}$$

$$\|\mathsf{R}[i\mu_{\eta_1,\eta_2}(m+l,n+l);m+l,n+l]\| \leqslant 8(m+l+1)^{1/2}(n+l+1)^{1/2}|\eta_2 u_{m+l}+\eta_1 u_{n+l}|, \tag{66}$$

with $p\neq l$ in Eq. (65) and $m\neq n$ in Eq. (66). Choosing the integer p positive, we replace k by $m+1$ on the rhs and k by p on the lhs of Eq. (63). The ensuing inequality will play an important role in the following, namely, when it comes to simplifying the bounds in Eqs. (65) and (66).

We go to the solution for vector $\mathbf{v}(t;\infty;m,n)$, with $m \neq n$, and set the above-constructed machinery in motion. Owing to results (23), (40), and (64)–(66), one can devise a fairly economical bound on the summand of Eq. (38). It is of crucial importance, however, to treat each of the following cases separately: $k=l=0$; $k\geqslant 1, l=0$; $k\geqslant 1, l\geqslant 1$. The summation over l can be carried out with the aid of the binomial formula. Truncating the summation over k, one ends up with

$$\|\mathbf{v}(t;N;m,n)\| \leqslant \frac{4g_5\max(1,h^2)h^{m+n}\exp[-\kappa t(m+n+1)]}{u_m u_n (m!n!)^{1/2}} \times \sum_{k=0}^{N}(m+k+2)^{1/2}(n+k+2)^{1/2} \times [\xi(t;m,n)]^k/k!. \tag{67}$$

The new function reads

$$\xi(t;m,n) = \frac{4(m+2)^{1/2}(n+2)^{1/2}h^2}{u_{m+1}u_{n+1}}(1+e^{-2\kappa t}). \tag{68}$$

The above inequality can also be established in the case that the integers m and n are equal. This assertion can be proved by means of a simple argument. Regarding m and n as continuous variables, one can verify that Eq. (38) reduces to the solution (39) upon taking the limit $m\rightarrow n$. Therefore, it is sufficient to take the same limit in Eq. (67).

Employing in Eq. (67) the inequality $(m+k+2)^{1/2}\leqslant(k+2)^{1/2}(1+m^{1/2})$, we reach the conclusion that the vector $\mathbf{v}(t;\infty;m,n)$ indeed fulfils condition (20), with constant g_1 taking on the form

$$g_1 = 4u_0^{-2}g_5\max(1,h^2)(2+\xi_0)e^{\xi_0}. \tag{69}$$

The shorthand notation $\xi_0=\xi(t=0;m=0,n=0)$ has been used. Time t may be taken either zero or positive. Thus the assumption made above Eq. (27) appears to be true as well.

The derivation of Eq. (67) can be repeated for the norm of the difference vector $\mathbf{v}(t;\infty;m,n)-\mathbf{v}(t;N;m,n)$ without any problems. Evidently, on the rhs of the corresponding inequality summation index k runs from $N+1$ to infinity. In the square roots of the summand we replace integers m and n by max(m,n). Then one encounters the sum $\Sigma_{k=N}^{\infty}x^k/k!$, which is dominated by the expression $(N+1)x^N/[N!(N+1-x)]$ as long as x does not leave the interval $[0,N+1)$. Consequently, the following inequality can be put forth:

$$\|\mathbf{v}(t;\infty;m,n)-\mathbf{v}(t;N;m,n)\| \leqslant \frac{4g_5\max(1,h^2)h^{m+n}\exp[-\kappa t(m+n+1)]}{u_m u_n(m!n!)^{1/2}} \times \frac{[\max(m,n)+3+N]\xi^{N+1}}{[N+1-\xi]N!}, \tag{70}$$

where the integer $N+1$ may not become smaller than function ξ, defined in Eq. (68).

We calculate the absolute error, as given by the rhs of Eq. (70), for the case that at time zero the field is in a coherent state. Making the choices $|\alpha|^2=10$, $\kappa=0.01$, $m,n\leqslant 100$, and $t\geqslant 0$, we find that for $N\geqslant 240$ the distance between vectors $\mathbf{v}(t;\infty;m,n)$ and $\mathbf{v}(t;N;m,n)$ is smaller than 1.2×10^{-9}. By employing Eqs. (10), (30), (32), and (70), one can also acquire truncation criteria for the series that represent the vector $\mathbf{w}(t;0,n)$ and matrix element $\rho_4(t)_{0,0}$.

V. WEAK DAMPING

The matrix $\mathsf{V}(z;k;m,n)$, defined in Eq. (40), forms the backbone of our solution for the density operator of the damped Jaynes-Cummings model. Unfortunately, the analytic evaluation of the elements of V turns out to be a tough job. At the same time, for the undamped case $\kappa=0$ the structure of the matrix product (40) is expected to become much more transparent. For that reason we shall calculate the atomic density matrix $\rho_A(t)=\mathrm{Tr}_F[\rho(t)]$ in the limit of small damping parameter. As the initial state we choose $\rho(t=0)=\rho_A\otimes|\alpha\rangle\langle\alpha|$, with the coherence parameter α real.

The upper-left element of the atomic density matrix is given by

$$\rho_A(t)_{11} = \sum_{n=0}^{\infty}\mathbf{v}(t;n,n)_1. \tag{71}$$

For the vector component on the rhs we insert the solution (39). By inequality (67), the resulting double series converges uniformly on each closed interval $0\leqslant\kappa\leqslant 2-\epsilon$.

Hence its behavior for small κ may be studied by operating behind the summation signs. Limiting our attention to the first summation of Eq. (39), we meet the expression

$$\frac{(-1)^{k+l+1}n!\exp[-\kappa t(2n+2l+1)]}{4^{k+1}l!(k-l)!(n+k+1)!} \times \mathsf{W}(z=0;k;n)|_{\kappa=0}\mathbf{v}(t=0;n+k,n+k). \tag{72}$$

Contributions of order $\kappa n^{3/2}$ have been discarded. Since terms with $n\simeq\alpha^2$ generate the largest contributions to Eq. (71), we see that the product $\kappa\alpha^3$ is to remain small [32]. There is no restriction on the value of time t. Most important of all, the elements of matrix W can be evaluated now. One has

$$\mathsf{W}(z=0;k;n)_{1j}|_{\kappa=0} = \frac{2(-1)^{k+1}(n+k+1)!(2n+2k+1)!}{(n+k)!(2n+1)!}(\delta_{1,j}+\delta_{4,j}), \tag{73}$$

as follows from Eqs. (40)–(45) by performing an induction proof.

If κ is taken small in the second summation of Eq. (39), only the summand with index k equal to zero survives. Its evaluation does not require any further restrictions on our parameters. The final result for the atomic matrix element reads then

$$\rho_{\mathrm{A}}(t)_{11} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} F_1(t,\alpha,\kappa;n,k) + \sum_{n=0}^{\infty} F_2(t,\alpha,\kappa;n), \tag{74}$$

with the summands given by

$$F_1(t,\alpha,\kappa;n,k) = \frac{1}{2}e^{-\alpha^2-\kappa t}\frac{n!(2n+2k+1)!}{4^k k!(2n+1)![(n+k)!]^2} \times[\rho_{A,11}+\alpha^2\rho_{A,22}(n+k+1)^{-1}] \times[\alpha^2(1-e^{-2\kappa t})]^k[\alpha^2 e^{-2\kappa t}]^n, \tag{75}$$

$$F_2(t,\alpha,\kappa;n) = \frac{1}{2}e^{-\alpha^2-\kappa t}\mathrm{Re}\{\exp[-2it(n+1)^{1/2}] \times[\rho_{A,11}-\alpha^2\rho_{A,22}(n+1)^{-1}-2i\alpha \times \mathrm{Im}\rho_{A,12}(n+1)^{-1/2}]\}\frac{1}{n!}[\alpha^2 e^{-2\kappa t}]^n. \tag{76}$$

The above series are convergent, and for Eq. (74) the choice $t=0$ yields the initial matrix element $\rho_{A,11}$, as expected. By repeating the foregoing computation for the diagonal element $\rho_A(t)_{22}$, one confirms the identity $\mathrm{Tr}_A\rho_A(t)=1$, which is a consequence of Eq. (52). We mention that in Ref. [7] the Jaynes-Cummings model with weak cavity damping has been studied as well, but not on the basis of an exact solution. Results have been reported that are at variance with Eqs. (74)–(76).

The upper-right element of the atomic density matrix must be obtained from

$$\rho_A(t)_{12} = \rho_3(t)_{0,0} + \sum_{n=0}^{\infty}\mathbf{v}(t;n+1,n)_3, \tag{77}$$

where the rhs can be elaborated with the help of relations (13), (38), and (46). The weak-damping limit can be taken in a similar vein as before. The off diagonal comes out as

$$\rho_A(t)_{12} = \frac{1}{2}(\rho_{A,12}-\alpha\rho_{A,22})\exp(-\kappa t/2-\alpha^2+it) + \frac{1}{2}(\rho_{A,12}+\alpha\rho_{A,22})\exp(-\kappa t/2-\alpha^2-it) + \sum_{n=0}^{\infty}F_3(t,\alpha,\kappa;n), \tag{78}$$

with the summand given by

$$F_3(t,\alpha,\kappa;n) = \frac{\alpha}{4n!(n+1)^{1/2}}[\alpha^2 e^{-2\kappa t}]^n \sum_{\eta_1,\eta_2}\exp\{-\alpha^2-2\kappa t + it[\eta_1(n+1)^{1/2}+\eta_2(n+2)^{1/2}]\} \times[\eta_1\rho_{A,11}-\eta_2\alpha^2\rho_{A,22}(n+1)^{-1/2}(n+2)^{-1/2} + \alpha\rho_{A,12}(n+1)^{-1/2}-\eta_1\eta_2\alpha\rho^*_{A,12}(n+2)^{-1/2}]. \tag{79}$$

Again, the choice $t=0$ produces the initial value of the matrix element. Symmetry relations (13) ensure that the lower-left element of the atomic density matrix is equal to the complex conjugate of Eq. (77).

For the case of weak damping, with product $\kappa\alpha^3$ small, the simple results (74) and (78) constitute a solid approximation to the atomic density matrix. They offer us the possibility to study analytically how the two-level atom evolves in time. Formulas for collapse and revival times can be derived along the same lines as in Ref. [33]. A second, and more rewarding, application of Eqs. (74) and (78) consists of investigating the behavior of the exact atomic density matrix in the limit

$$\lim_{\kappa\to 0}{}' f(t,\alpha,\kappa) = \lim_{\kappa\to 0} f(\tilde{t}/\kappa, \tilde{\alpha}/\kappa^{1/4}, \kappa), \tag{80}$$

with f an arbitrary function; $\tilde{t}$ and $\tilde{\alpha}$ are positive constants.

In the undamped case the atomic density matrix almost coincides with the central state $\frac{1}{2}\mathbb{1}_2$ as t becomes large. On the other hand, by setting κ zero in Eq. (74), one sees that the density matrix contains oscillating terms that do not decay to zero and thus inhibit convergence. Below we prove that for the limit (80) all oscillations do decay, so that convergence to the central state does take place.

First of all, we observe that in the limit (80) the parameter κ and product $\kappa\alpha^3$ both go to zero. Hence the difference between a matrix element (74) or (78) and its exact counterpart goes to zero as well. The absolute value of the series figuring in Eq. (78) is smaller than the form $\alpha^3\exp\{-\alpha^2[1-\exp(-2\kappa t)]\}$ multiplied by a positive constant. Since $\tilde{t}=\kappa t$ is greater than zero, it follows that expression (78) vanishes in the limit (80). In other words, the exact atomic density matrix becomes diagonal. A similar argument shows that the second series on the rhs of Eq. (74) does not survive the limit (80) either.

The remaining contribution to Eq. (74) must be handled carefully. For ϵ positive one has

$$\sum_{n=0}^{[\epsilon\alpha^2]}\sum_{k=0}^{\infty}\|F_1(t,\alpha,\kappa;n,k)\| \leqslant 4\alpha^6\exp[-\alpha^2(e^{-2\kappa t}+\epsilon\ln\epsilon-\epsilon)], \tag{81}$$

owing to the inequality

$$\frac{(n!)^2(2n+2k+1)!}{4^k(2n+1)![(n+k)!]^2}\leqslant 1+\frac{k}{n+1}, \tag{82}$$

where integers k and n are non-negative. For the double series with n running from zero to infinity and k from zero to $[\epsilon\alpha^2]$, one derives a bound that is analogous to the one in Eq. (81). If ϵ is taken sufficiently small, both bounds decay exponentially to zero for α large. Thus it is possible to replace the lower boundaries of double series (74) by $[\epsilon\alpha^2]$. Now one can represent all factorials by means of Stirling's expansion [34]. This last step paves the way for making use of the transition

$$L^{-1}\sum_{k=[aL]}^{[bL]} f(k/L)=\int_a^b dx\, f(x)+[f(a)+f(b)]/2L +O(L^{-2}), \tag{83}$$

where L is large and f denotes a continuous function. In the present case one only needs the first contribution on the rhs of Eq. (83).

Upon following the above instructions, Eq. (74) reduces to

$$\rho_A(t)_{11}=\frac{\alpha^2e^{-\kappa t}}{4\pi}\int_\epsilon^\infty dx\int_\epsilon^\infty dy\frac{(x+y)^{1/2}}{x^{1/2}y} \times[\rho_{A,11}+\rho_{A,22}(x+y)^{-1}]\exp[\alpha^2 g(x,y)] +O(\alpha^{-2}), \tag{84}$$

with the definition

$$g(x,y)=-1+x+y-x\ln[x/(1-e^{-2\kappa t})] -y\ln(ye^{2\kappa t}). \tag{85}$$

The integral (84) demonstrates that the introduction of a cutoff parameter ϵ is indispensable, as the integrand diverges at the origin. The limit (80) takes the parameter α to infinity, so in calculating Eq. (84) one can call upon the saddle-point method [35]. The result 1/2 is found.

From the foregoing considerations we learn that the exact atomic density matrix possesses the limit

$$\lim_{\kappa\to 0}{}'\ \rho_A(t)=\tfrac{1}{2}\mathbb{1}_2. \tag{86}$$

The initial state ρ_A of the atom may be chosen arbitrarily. The above result remains valid if the coherence parameter α is chosen to be complex.

We should also investigate how the resonant field mode behaves under the limit (80). The mean photon number and second moment of the photon distribution can be calculated from the relations

$$\overline{n(t)}=\mathrm{Tr}_F[a^\dagger a\rho_F(t)],$$
$$\overline{n^2}(t)=\mathrm{Tr}_F[a^\dagger aa^\dagger a\rho_F(t)]. \tag{87}$$

By adopting the same methodology as earlier, the reader may be convinced of the fact that the following expansions hold true:

$$\overline{n(t)}=(\alpha^2+\tfrac{1}{2}-\rho_{A,22})e^{-2\kappa t}+O(\alpha^{-2}), \tag{88}$$

$$\overline{n^2}(t)=\alpha^4e^{-4\kappa t}+\alpha^2e^{-2\kappa t}+\alpha^2(1-2\rho_{A,22})e^{-4\kappa t}+O(\alpha^0). \tag{89}$$

The parameter κ is small, whereas α and t satisfy the constraints belonging to Eq. (80). Notice that not only the leading but also the next-to-leading terms have been evaluated. Hence, in making the transition (83), one has to cope with contributions that arise from the correction term of order $1/L$. All of these turn out to be of exponential decay.

Often, one characterizes a photon distribution by means of a factor $Q=\overline{n^2}/\overline{n}-\overline{n}-1$ [36]. The above results reveal that

$$\lim_{\kappa\to 0}{}'\ Q(t)=0. \tag{90}$$

The Q factor takes on the value that belongs to a coherent state. In assessing Eq. (90), one should be aware of the fact that the density operator itself may not be replaced by a coherent state. To enlighten this statement we assume the existence of a complex function $\beta(\alpha)$ for which the norm $|\mathrm{Tr}_F[a\rho_F(t)]-\beta(\alpha)|$ vanishes in the limit (80). It can be verified that the expectation value $\mathrm{Tr}_F[a\rho_F(t)]$ is of exponential decay as $|\alpha|$ gets large, so we conclude that the same must be true for function β. By identity $\overline{n}=|\beta|^2$, the last requirement contradicts Eq. (88).

As a final point, we observe that the limits (86) and (90) are not affected by a transformation of the density operator to the Schrödinger picture. The unperturbed Hamiltonian of the Jaynes-Cummings model commutes with both the unit operator in Eq. (86) and the number operator in Eq. (87).

VI. SUMMARY

In the Jaynes-Cummings model with cavity damping the evolution of all observables is dictated by the master equation (3) for the density operator $\rho(t)$, at least if the cavity does not contain any thermal photons. By adopting a representation based on direct products of two-dimensional Cartesian vectors $\{\mathbf{e}_1,\mathbf{e}_2\}$ and photon-number states $\{|n\rangle\}$, the master equation can be converted into an infinite set of ordinary differential equations. Subsequently, upon going over to Laplace language and distributing all matrix elements $\langle \mathbf{e}_i \otimes m | \rho(t) | \mathbf{e}_j \otimes n \rangle$ over two- and four-dimensional vectors in a suitable manner, two algebraic recursion relations are found. As shown in Sec. II, these can be solved exactly by performing an iteration *ad infinitum*. Convergence is secured with the help of a few inequalities, the verification of which is carried out at a later stage.

The iterative procedure provides us with solutions for all matrix elements $\langle \mathbf{e}_i \otimes m | \rho(t) | \mathbf{e}_j \otimes n \rangle$, each of these being an infinite series. Explicit results are presented in Sec. III. They obey important physical constraints on the density operator, namely, Hermiticity, conservation of trace, convergence to the initial state for small times, as well as convergence to the state of lowest energy for large times. The last three properties can be proved on the basis of techniques from function theory. At this point the advantages of Laplace transformation become manifest.

Invoking some methods from linear algebra, we demonstrate in Sec. IV that our solutions for the matrix elements of the density operator are consistent with all of the inequalities employed earlier. Besides that, we estimate for each matrix element how fast its solution converges. This information is useful when undertaking a comprehensive numerical study on the evolution of observables. The conclusions of such a study will be discussed in a forthcoming paper [37].

The results of Sec. IV can also serve to find out how cavity damping affects the magnitude of matrix elements. For instance, together with Eq. (38) the inequality (67) tells us the following: As soon as damping parameter κ has a finite value, the modulus of product $\exp(\kappa t)\langle \mathbf{e}_1 \otimes m | \rho(t) | \mathbf{e}_1 \otimes n \rangle$ decreases by a factor of $\exp(-\kappa t)$ for each photon making up states $|m\rangle$ and $|n\rangle$. As pointed out in the Introduction, this can be understood from the fact that the photons inside the cavity experience a finite transparency of the cavity mirrors simultaneously. Similar reasoning can be held for the other matrix elements of the density operator.

The foregoing observations have an important consequence for the work performed in Sec. V. There the atomic density matrix is evaluated for the case that damping is weak and the field is initially in a coherent state. Due to factor $h^{m+n}(m!n!)^{-1/2}$ on the right-hand side of Eq. (67), the chief contributions to Eqs. (71) and (77) possess a summation index n of order α^2. Therefore, if the coherence parameter α increases, one has to face more factors of $\exp(-\kappa t)$ in evaluating Eqs. (71) and (77), so that the impact of cavity damping on the atomic density matrix becomes stronger. The last statement has been advanced before, on the basis of a numerical computation of the atomic inversion [13].

As seen from a theoretical standpoint, one advantage of the present model over its undamped counterpart resides in the limit (49). This result permits us to make a clear-cut statement on the behavior of observables for long times, a possibility that is lacking for the Jaynes-Cummings model itself. On the other hand, by taking the limit (49) one completely disregards the interesting asymptotic behavior of the atomic density matrix in the undamped case. For κ equal to zero and t large, the atom hovers about the central state $\frac{1}{2}\mathbb{1}_2$ and thus attempts to maximize its von Neumann entropy $S_A = -\mathrm{Tr}_A[\rho_A(t)\ln\rho_A(t)]$ [38].

In Sec. V we prove that the exact atomic density matrix truly converges to the central state if the parameters t, α, and κ^{-1} are taken to infinity such that the products κt and $\kappa\alpha^4$ remain constant. Numerical work shows that the new limit is not only of mathematical interest. For example, if the choices $\kappa = 0.001$ and $\alpha = 5$ are made, then on the interval $250 \leqslant t \leqslant 1250$ the entropy S_A differs from its maximal value ln2 by an amount of 10^{-3} at most. This long stage of quasiequilibrium is preceded by the usual dynamics of collapse and revival in the Rabi oscillations of the inversion. For $t \simeq 1500$ dissipation sets in and the atom decays toward its ground state.

We point out that the remarkable crossover from quasireversible to fully irreversible behavior persists if the initial condition for the field is modified. In the Appendix we establish an asymptotic limit, which is similar to Eq. (80), for the case that the field starts from a photon-number state. The various stages in the time evolution of the atomic density matrix will be discussed at length in a future paper [37].

Finally, it should be mentioned that one can try to apply the strategy developed in this paper to the case in which atomic detuning and spontaneous emission are present. On the contrary, the handling of thermal photons might prove difficult because then one meets an operator $\mathcal{L}_2$ of a different structure as before.

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APPENDIX: PROOF OF A LIMIT

Opting for the initial condition

$$\rho(t=0) = \rho_A \otimes |m\rangle\langle m|, \tag{A1}$$

where $|m\rangle$ denotes a photon-number state (2), we set out to prove a limit that is akin to Eq. (86). Like before, we take the time, the square of the initial em energy density, and the inverse of the cavity-damping parameter to infinity in such a manner that all of the ratios between these quantities remain constant. The limit (80) is thus replaced by

$$\lim_{\kappa\to 0}{}'' f(t,m,\kappa) = \lim_{\kappa\to 0} f(\tilde{t}/\kappa, \tilde{m}/\kappa^{1/2}, \kappa), \tag{A2}$$

with $\tilde{t}$ and $\tilde{m}$ positive constants.

Following the same route as in Sec. V and choosing $m \geqslant 1$, we obtain from Eqs. (71), (77), and (A1) the following results for the atomic density matrix:

$$\begin{aligned}\rho_A(t)_{11} = {} & \tfrac{1}{2}\rho_{A,11}e^{-\kappa t}C(\kappa t;m) + \tfrac{1}{2}\rho_{A,22}e^{-\kappa t}C(\kappa t;m-1) \\ & + \tfrac{1}{2}\rho_{A,11}e^{-\kappa t(2m+1)}\cos[2t(m+1)^{1/2}] \\ & - \tfrac{1}{2}\rho_{A,22}e^{-\kappa t(2m-1)}\cos[2tm^{1/2}],\end{aligned} \tag{A3}$$

$$\rho_A(t)_{12}=\rho_{A,12}e^{-2\kappa tm}\cos[tm^{1/2}]\cos[t(m+1)^{1/2}]. \quad (A4)$$

We have defined a series

$$C(t;m)=b_m\sum_{k=0}^{m}b_k^{-1}\binom{m}{k}[e^{-2t}]^k[1-e^{-2t}]^{m-k}, \quad (A5)$$

$$b_m=\frac{(2m+1)!}{4^m(m!)^2}. \quad (A6)$$

The damping parameter κ must be chosen such that the product $\kappa m^{3/2}$ is small.

All contributions that contain a cosine factor do not survive the limit (A2). For series (A5) things work out differently because we can establish the bounds

$$1\leqslant\lim_{\kappa\to 0}{}''C(\kappa t;m)\leqslant e^{2\kappa t}, \quad (A7)$$

by employing the inequalities

$$1\leqslant b_m/b_k\leqslant(m+1)/(k+1), \quad (A8)$$

which are valid for $m\geqslant k$. In evaluating the limit in Eq. (A7), we replace boundaries m and 0 of the sum (A5) by $m-[\epsilon m]$ and $[\epsilon m]$, respectively. If ϵ is chosen sufficiently close to zero, the ensuing errors vanish in the limit (A2), as one can demonstrate with the help of Eq. (A8).

Upon inserting Stirling's expansion for all factorials and making the transition (83), we see that in leading order of m the series (A5) becomes equal to

$$C(\kappa t;m)=\left(\frac{m}{2\pi}\right)^{1/2}\int_{\epsilon}^{1-\epsilon}dx\frac{\exp[-mh(x)]}{x(1-x)^{1/2}}, \quad (A9)$$

with the function

$$h(x)=(1-x)\ln[(1-x)/(1-e^{-2\kappa t})]+x\ln(xe^{2\kappa t}). \quad (A10)$$

Application of the saddle-point method gives the result $\exp(\kappa t)$ for the integral (A9). Hence we have proved the limit

$$\lim_{\kappa\to 0}{}''\rho_A(t)=\tfrac{1}{2}\mathbb{1}_2. \quad (A11)$$

If the field is in a photon-number state at time zero, the atomic density matrix converges to the central state under the limit (A2).

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