Measuring the cyclotron state of a trapped electron

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We propose the cyclotron state retrieval of an electron trapped in a Penning trap by using different measurement schemes based on suitable modifications of the applied electromagnetic fields, and exploiting the axial degree of freedom as a probe. A test for the matter-antimatter symmetry of the quantum state is proposed. [S1050-2947(97)01110-4]

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I. INTRODUCTION

The quantum state of a system is a fundamental concept in quantum mechanics, because the density matrix describing it contains the complete information we can obtain about that system. Recently, in quantum measurement theory, a great interest has been devoted to the possibility of reconstructing the density matrix by measuring a complete set of probability distributions over a range of different operator representations.

A tomographic approach to the problem of state retrieval was first introduced by Bertrand and Bertrand [1], and some measurement schemes proposed to implement quantum state tomography are reviewed by Royer [2]. Later on Vogel and Risken [3] showed that s-parametrized quasiprobability distributions can be obtained from the probability distributions of rotated quadrature phases, a technique which allows practical implementation within the field of quantum optics [4].

The method relies on the possibility of making homodyne measurements of the field of interest by scanning the phase of the added local oscillator field, which is called optical homodyne tomography (OHT). Recently, another method [5,6] was introduced, based on direct photon counting by scanning both the phase and the amplitude of a reference field. The latter method, also called photon number tomography (PNT) [7], has the advantage of avoiding sophisticated computer processing of the recorded data [5] and is applicable when direct access to the system is inhibited [7].

Although the reconstruction of the phase-space distribution was already proposed for nonoptical systems, i.e., particles [8,9], it is based on optical measurements performed on the field radiated by these particles, exploiting the resonance fluorescence phenomenon. These methods would not be suitable in systems without an internal electronic structure such as a trapped electron (or proton) [10]. The purpose of this work, instead, is to show how to reach the characterization of the quantum state of a trapped "elementary" particle, not having an electronic structure, by using tomographiclike measurements without the use of the radiated field. In particular, we shall consider an electron trapped in a Penning trap [10] developing techniques resembling both OHT and PNT, which allow one to obtain the state of the cyclotron motion by probing its axial degree of freedom.

II. MODEL

We consider the motion of an electron in a uniform magnetic field B along the positive z axis and a static quadrupolar potential. As is well known [11], the motions of that electron in the trap are well separated in energy scale. In what follows we shall consider only the cyclotron and the axial degrees of freedom, which radiate in the GHz and MHz ranges, respectively, neglecting the slow magnetron motion in the kHz region. To simplify our presentation, we assume an *a priori* knowledge of the electron's spin [12], then we neglect all the spin-related terms in the Hamiltonian that, for an electron of rest mass m and charge $-|e|$, can be written as the quantum counterpart of the classical one

$$\hat{H}=\frac{1}{2m}\left[\hat{\mathbf{p}}-\frac{e}{c}\hat{\mathbf{A}}\right]^2+eV_0\,\frac{\hat{x}^2+\hat{y}^2-2\hat{z}^2}{4d^2}, \qquad (1)$$

where $\hat{\mathbf{A}}=(-\hat{y}B/2,\hat{x}B/2,0)$, c is the speed of light, d characterizes the dimensions of the trap and V_0 is the potential applied to its electrodes.

It is convenient to introduce the rising and lowering operators for the cyclotron motion,

$$\hat{a}_c=\frac{1}{2}\left[\beta(\hat{x}-i\hat{y})+\frac{1}{\beta\hbar}(\hat{p}_y+i\hat{p}_x)\right], \qquad (2)$$

$$\hat{a}_c^{\dagger}=\frac{1}{2}\left[\beta(\hat{x}+i\hat{y})+\frac{1}{\beta\hbar}(\hat{p}_y-i\hat{p}_x)\right], \qquad (3)$$

with $\beta=(m\omega_c/2\hbar)^{1/2}$, and $\omega_c=|e|B/mc$ being the cyclotron angular frequency. For the axial motion we define

$$\hat{a}_z=\left[\frac{m\omega_z}{2\hbar}\right]^{1/2}\hat{z}+i\left[\frac{1}{2m\hbar\omega_z}\right]^{1/2}\hat{p}_z, \qquad (4)$$

$$\hat{a}_z^{\dagger}=\left[\frac{m\omega_z}{2\hbar}\right]^{1/2}\hat{z}-i\left[\frac{1}{2m\hbar\omega_z}\right]^{1/2}\hat{p}_z, \qquad (5)$$

with $\omega_z^2 = |e| V_0 / m d^2$. Thus, by using these new operators, Hamiltonian (1) simply becomes

$$\hat{H} = \hbar \omega_c (\hat{a}_c^\dagger \hat{a}_c + \tfrac{1}{2}) + \hbar \omega_z (\hat{a}_z^\dagger \hat{a}_z + \tfrac{1}{2}), \tag{6}$$

where all terms not containing the raising or lowering operators have been omitted. The obtained Hamiltonian (6) is decomposed into two indipendent parts, each forming a complete Hilbert space with their own basis. The state of the electron is thus the inner product of the two states we shall call the cyclotron and axial states, $|\psi\rangle = |\psi_c\rangle |\psi_a\rangle$.

To go further, we also remark that the best way of observing the microscopic system from the outside world is through the measurement of the current due to the induced charge on the cap electrodes of the trap, as a consequence of the axial motion of the electron along the symmetry axis, because there are not good detectors in the GHz range to measure the cyclotron radiation [11]. Therefore, since the motions are completely decoupled, only the axial one is easily detectable [11]. In the following we shall consider some interaction Hamiltonians suitable for an indirect characterization of the cyclotron motion.

III. DETECTION TECHNIQUES

The OHT technique is based on the possibility of measuring different quadratures of the field of interest; let us then define in our case the generic cyclotron quadrature

$$\hat{X}_c(\phi) = \hat{a}_c e^{-i\phi} + \hat{a}_c^\dagger e^{i\phi}, \tag{7}$$

where ϕ is the angle in the phase space of the cyclotron motion. Furthermore, as mentioned above, the axial motion could be considered as a meter; then, in order to couple the meter with the system (cyclotron motion), we may consider

$$\hat{H}_{int} = \hbar g (\hat{a}_c e^{-i\phi + i\omega_c t} + \hat{a}_c^\dagger e^{i\phi - i\omega_c t}) \hat{z}, \tag{8}$$

where g is the strenght of interaction. This interaction Hamiltonian could be obtained by applying the following fields on the trapped particle

$$\hat{\mathbf{A}} = \left(\frac{mc}{|e|\beta} g \hat{z} \sin(\phi - \omega_c t) - \frac{B}{2}\hat{y}, \frac{mc}{|e|\beta} g \hat{z} \cos(\phi - \omega_c t) + \frac{B}{2}\hat{x}, 0 \right), \tag{9}$$

$$\hat{V} = V_0 \, \frac{\hat{x}^2 + \hat{y}^2 - 2\hat{z}^2}{4d^2} - \frac{m}{2e\hbar^2\beta^2} g^2 \hat{z}^2,$$

which differ from the usual ones [Eq. (1)] for time-dependent terms added at the preexistent components. By considering Eqs. (6), (7), and (8), the evolution in the (cyclotron) interaction picture will be determined by the Hamiltonian

$$\hat{H} = \hbar \omega_z (\hat{a}_z^\dagger \hat{a}_z + \tfrac{1}{2}) + \hbar g \hat{X}_c(\phi) \hat{z}\ , \tag{10}$$

where the cyclotron quadrature phase ϕ can be set by the experimentalist as shown in Eq. (9).

We now assume that the interaction time is much shorter than the axial period, i.e. $\tau << 2\pi/\omega_z$, so that the free evolution of the electron in Eq. (10) can be neglected. This assumption allows us to disregard a possible detuning of the applied field with respect to the cyclotron frequency. Indeed, in the usual experimental setup the detuning is of the order of a few kHz [11]. Hence the effect of the interaction is determined by applying a unitary kick

$$\hat{K}(\tau) = \exp[-ig\tau \hat{X}_c(\phi)\hat{z}]. \tag{11}$$

By measuring the current due to the induced charge variation on the cap electrodes of the trap, one obtains the axial momentum [11], then the value of quadrature $\hat{X}_c(\phi)$ by means of

$$\begin{aligned} \langle \hat{p}_z(t+\tau) \rangle &= \langle \hat{K}^\dagger(\tau) \hat{p}_z(t) \hat{K}(\tau) \rangle \\ &= \langle \hat{p}_z(t) \rangle + \hbar g \tau \langle \hat{X}_c(\phi) \rangle \cos(\omega_z t). \end{aligned} \tag{12}$$

Repeated measurements allow us to recover the probability distribution $\mathcal{P}(X_c, \phi)$ for that cyclotron quadrature (marginal distribution). However, the measurement process is state destructive; hence the initial state of the electron has to be reset prior to each new measurement. After one has performed a large set of measurements for each phase angle ϕ, the s-parametrized Wigner function can be obtained from the probability of experimental data $\mathcal{P}(X_c, \phi)$ through the inverse radon transform [3]

$$\begin{aligned} W(\alpha, \alpha^*, s) = & \int_{-\infty}^{+\infty} \frac{dr|r|}{4} \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dX_c \ \ \mathcal{P}(X_c, \phi) \\ & \times \exp\{ sr^2/8 + ir[x - \mathrm{Re}(\alpha e^{-i\phi})]\}. \end{aligned} \tag{13}$$

It is evident from this expression that convergence problems arise for $s \geqslant 0$.

In analogy with Refs. [5,7] we now show how one can probe the quantum cyclotron phase space by measuring the number of cyclotron excitations. In Ref. [13] the quantum nondemolition measurement of the latter is shown to be accessible by only considering the coupling between the axial and the cyclotron motions induced by the relativistic correction to the electron's mass; however, in order to obtain a stronger coupling we shall consider the magnetic bottle configuration, which leads to [11]

$$\hat{H}_{\mathrm{int}} = \hbar \kappa \hat{a}_c^\dagger \hat{a}_c \hat{z}^2, \tag{14}$$

where the constant κ is related to the strength b of the field characterizing the magnetic bottle. As a consequence of Eq. (14), the axial angular frequency will be affected by the number of cyclotron excitations; in fact, it will result in

$$\hat{\Omega}_z^2=\omega_z^2+2\frac{\hbar\kappa}{m}\hat{a}_c^\dagger\hat{a}_c\,. \tag{15}$$

Then, probing the resonance frequency of the output electric signal, one can obtain the number of cyclotron excitations.

The axial motion relaxes much faster than the cyclotron one [11], so, when Hamiltonian (14) is added to the Hamiltonian of Eq. (1), by considering the axial steady state, and neglecting a very small anharmonicity term, we can write

$$\hat{H}_{\rm cyc}=\hbar(\omega_c+\kappa\langle\hat{z}^2\rangle)\hat{a}_c^\dagger\hat{a}_c\,, \tag{16}$$

with $\langle\hat{z}^2\rangle=k_BT/m\omega_z^2$ the thermal equilibrium value of the free axial motion, from which we may recognize a shift effect also on the cyclotron angular frequency due to the coupling.

Furthermore, to perform a PNT-like scheme, according to the optical case of Ref. [7], we need a reference field which displaces the state one wants to recover or, equivalently, one which is mixed to it by means of a beam splitter [5,6] that, however, in this case is not applicable. To this end we can use a driving field with amplitude ϵ acting immediately before the measurement process induced by Hamiltonian (14), and given by a Hamiltonian term of the type

$$\hat{H}_{\rm drive}=-i\hbar(\epsilon e^{-i\omega_c t}\hat{a}_c^\dagger-\epsilon^* e^{i\omega_c t}\hat{a}_c). \tag{17}$$

Hence, in the interaction picture we may write

$$\hat{\rho}_{cyc}(\tau)=\hat{D}^\dagger(E)\hat{\rho}_{cyc}(0)\hat{D}(E), \tag{18}$$

with

$$\hat{D}(E)=\exp[E\hat{a}_c^\dagger-E^*\hat{a}_c],\quad E=\epsilon\tau, \tag{19}$$

where τ represents the driving time interval. One can again get rid of the detuning with respect to the cyclotron frequency, provided one chooses the interaction time τ to be much shorter than the axial period.

Once one has ‘‘displaced’’ the initial state, the measurement process of the axial frequency allows one to obtain the number of cyclotron excitations present in the state $\hat{\rho}_c(\tau)$; hence the probability $P(n_c)=\langle n_c|\hat{D}^\dagger(E)\hat{\rho}_{cyc}(0)\hat{D}(E)|n_c\rangle$, which becomes $\forall\ n_c$ after a large set of measurements. Then, by referring to the expression of the s-parametrized Wigner function introduced in Ref. [14], we may write

$$W(E,E^*,s)=\frac{2}{1-s}\sum_{n_c=0}^{\infty}\left(\frac{s+1}{s-1}\right)^{n_c}\times\langle n_c|\hat{D}^\dagger(E)\hat{\rho}_{\rm cyc}(0)\hat{D}(E)|n_c\rangle, \tag{20}$$

where the quasiprobability distribution corresponds to the state $\hat{\rho}_{\rm cyc}(0)$, and can be entirely obtained by varying the complex parameter E (i.e., the reference field).

This scheme requires the use of the following fields:

$$\hat{\mathbf{A}}=\begin{cases}\left(2\dfrac{mc}{\beta|e|}{\rm Im}\{\epsilon e^{-i\omega_c t}\}-\dfrac{B}{2}\hat{y},2\dfrac{mc}{\beta|e|}{\rm Re}\{\epsilon e^{-i\omega_c t}\}+\dfrac{B}{2}\hat{x},0\right), & t\leqslant\tau\\[2ex] \left(-\dfrac{B}{2}\hat{y}-\dfrac{b}{2}[\hat{y}\hat{z}^2-\hat{y}^3/3],\dfrac{B}{2}\hat{x}+\dfrac{b}{2}[\hat{x}\hat{z}^2-\hat{x}^3/3],0\right), & t>\tau,\end{cases} \tag{21}$$

which means turning on the magnetic bottle as soon as the driving field is switched off. The latter may consist of an electromagnetic field circularly polarized in the x-y plane and oscillating at ω_c. The scalar potential remains the same as Eq. (1), while Eqs. (14) and (17) can be obtained from the above fields by considering $b\ll B$ and making the rotating-wave approximation and dipole approximation; in doing that, one also obtains $\kappa=|e|b/2mc$. We wish to point out that these fields are commonly used in the experimental setup of the Penning trap, differently from an OHT-like scheme, where nontrivial modifications of the fields are required.

IV. CONCLUDING REMARKS

In conclusion, we have shown the possibility of reconstructing the cyclotron state of a trapped electron using different measurement schemes based on suitable modifications of the external electromagnetic fields. In particular the PNT-like scheme could be considered more powerful with respect to the OHT-like scheme, since it does not need any filtered back-projection process, directly giving the desired phase-space distribution from measured data. We confined our treatment to the case of undamped cyclotron motion, which, however, could be reasonable for a perfectly off-resonant situation [15]. To simplify the presentation of the measurement schemes we also assumed unity efficiency in the detection process. For nonunity efficiency it is possible to show [16,6] that one never reconstructs the full Wigner function but only a smoothed version of it. Moreover, the present model can be applied to reconstruct the quantum state of a trapped antiparticle such as the positron (or antiproton), and

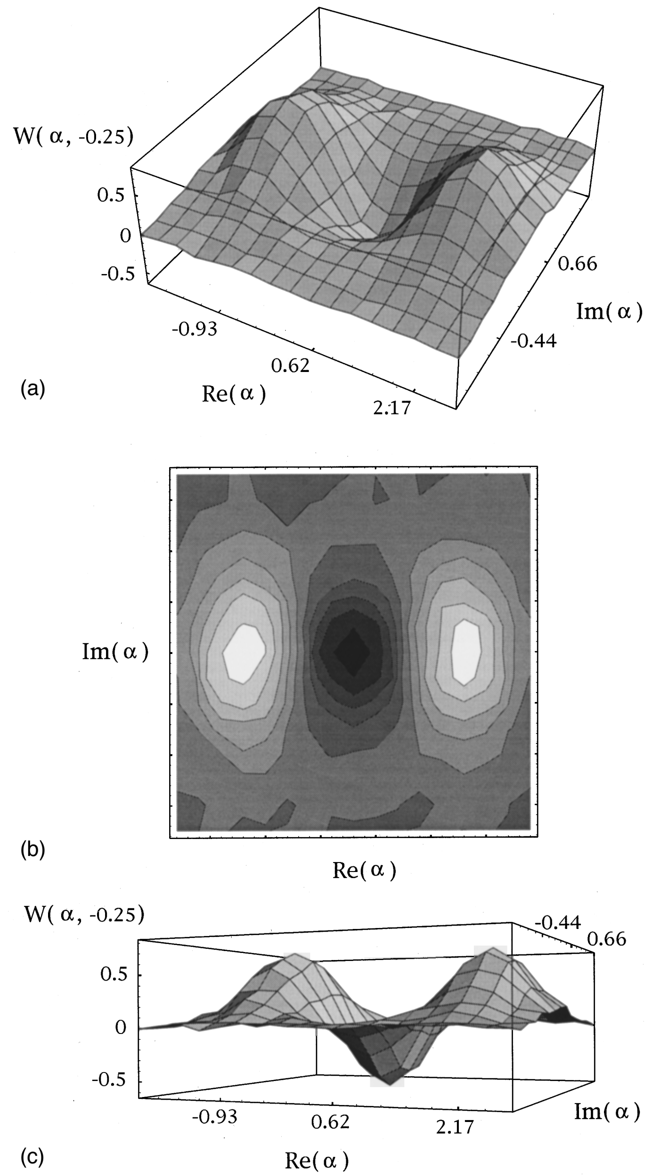

FIG. 1. Simulations of the reconstruction of the s-parametrized Wigner function ($s=-0.25$) for an odd cat state with $\alpha=1.5$ by means of OHT; 27 phases are scanned with 10^3 data each.

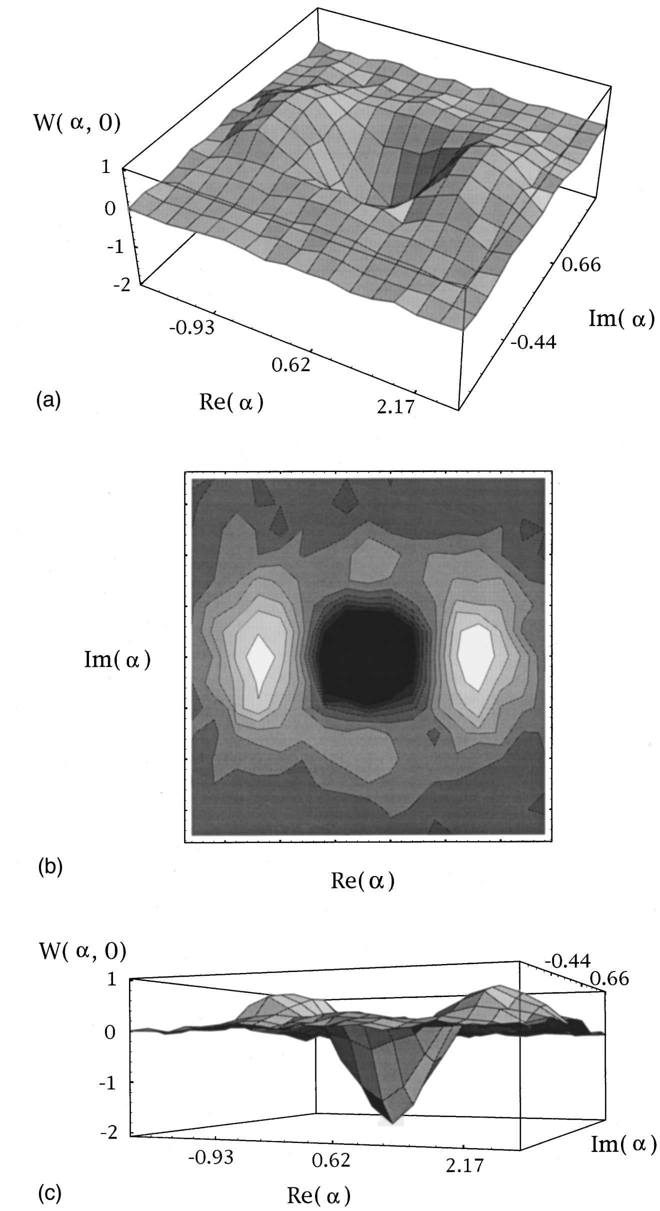

FIG. 2. Simulations of the reconstruction of the Wigner function for an odd cat state with $\alpha=1.5$ by means of PNT; 10^3 events are sampled for each of the 255 points of the grid.

to test whether particle and antiparticle have the same quantum state under charge conjugation transformation.

Let us give some numerical simulations to emphasize the discussed possibilities of reconstructing quasiprobability distributions. We demonstrate the methods assuming that the cyclotron state is an odd coherent state state [17] (or Schrödinger cat state)

$$|\alpha_-\rangle = N_-(|\alpha\rangle - |-\alpha\rangle), \tag{22}$$

with $|\alpha\rangle$ being a coherent state and N_- a normalization constant. States of this type exhibit quantum interferences, giving rise to negative values and sharp structures in the Wigner function, so that their reconstruction necessitates particular care.

In Fig. 1 we show a smoothed version of the Wigner function ($s=-0.25$) reconstructed by using OHT; the negativity of the parameter s makes the numerical algorithm for the inversion easier. The full Wigner function ($s=0$), instead, can be readily reached by means of PNT (Fig. 2), where no convergence problems arise and cumbersome numerical algorithms can be avoided.

Cat states could be created in a Penning trap by considering relativistic effects (which introduce an anharmonicity), having a macroscopic character whenever strong excitations are considered, and they can be displayed with the aid of the discussed methods [18]. From the experimental point of view the difficulty in using PNT is connected with the measurement of the cyclotron excitation number, and the need to distinguish among the Landau level n_c and its nearest $n_c \pm 1$. It was shown in Ref. [13] that with the sensibility already reached for the axial resonance frequency, this measurement is feasible when the electron's spin is known.

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