Stability and collective excitations of a two-component Bose-Einstein condensed gas: A moment approach

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The dynamics of a two-component dilute Bose-Einstein gas of atoms at zero temperature is described in the mean-field approximation by a two-component Gross-Pitaevskii equation. We solve this equation assuming a Gaussian shape for the wave function, where the free parameters of the trial wave function are determined using a moment method. We derive equilibrium states and the phase diagrams for the stability for positive and negative s-wave scattering lengths, and obtain the low-energy excitation frequencies corresponding to the collective motion of the two Bose-Einstein condensates. [S1050-2947(97)04110-3]

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I. INTRODUCTION

Since the first observations of Bose-Einstein condensation in an alkali-metal vapor [1–3] considerable effort has been made to characterize these systems both experimentally and theoretically. One of the latest advances is the experimental preparation of two-component Bose-Einstein condensates in different internal states, exhibiting a spatial overlap [4]. In this experiment ^{87}Rb atoms in two different hyperfine states, $|1\rangle=|1,-1\rangle$ and $|2\rangle=|2,2\rangle$, are loaded into a magneto-optical trap, are cooled down to effectively zero temperature, and subsequently undergo a phase transition toward a Bose-Einstein condensate [5].

The Gross-Pitaevskii equation (GPE), a nonlinear Schrödinger equation (NLSE) for the macroscopic wave function of the Bose-Einstein condensed gas, provides an accurate description of the ground state and of the excitation spectrum of a dilute Bose-Einstein condensate at zero temperature [6–20]. Recently, several groups have solved the GPE for single-component Bose-Einstein condensates, and found excellent agreement with experiment [6,11]. The aim of this paper is to study solutions of the *two-component* GPE for a harmonic trapping potential. In particular we will derive the Bose-Einstein ground state, investigate its stability properties, and determine the low-lying excitation frequencies for positive and negative scattering lengths. Our technique of solving the two-component GPE is based on a Gaussian ansatz for the condensate wave function where the open parameters are determined by moment methods [21] (which is equivalent to a variational technique [14]). This allows one to obtain essentially analytical solutions of the GPE, and thus complements numerical studies of this equation [24].

The paper is organized as follows. In Sec. II of this paper we first define our model, and derive the equations of motion for the parameters of our Gaussian wave function. In particular, we study in detail the simple case of of isotropic traps and isotropic condensates. In Sec. III we investigate first the case where the centers of the two trapping potentials for the two condensates coincide. For this configuration we calculate the equilibrium points and the low-energy eigenmodes. In addition, we analyze the stability of the system. Finally, in Sec. IV we consider the case of displaced trap centers, and compare with the results of Sec. III.

II. BASIC EQUATIONS

A. The model

Zero-temperature mean-field theory provides an accurate theoretical description of dilute Bose-Einstein condensed systems. In particular, the dynamics of a two-component Bose-Einstein gas can be modeled by the coupled Gross-Pitaeveskii equations

$$i\partial_t\Psi_1=\left[-\frac{1}{2M}\nabla^2+V_1(\vec r)+\sum_{n=1}^{2}U_{1n}|\Psi_n|^2\right]\Psi_1, \quad (1a)$$

$$i\partial_t\Psi_2=\left[-\frac{1}{2M}\nabla^2+V_2(\vec r)+\sum_{n=1}^{2}U_{2n}|\Psi_n|^2\right]\Psi_2 \quad (1b)$$

which are a set of nonlinear Schrödinger equations for the macroscopic wave functions Ψ_1 and Ψ_2 for the two components of the condensate. In Eqs. (1) $V_n,(n=1$, and 2) denotes the trap potentials which we assume to be harmonic in agreement with common experimental situations

$$V_n(\vec r)=\tfrac{1}{2}M\omega^2[\lambda_x^2(x-x_n)^2+\lambda_y^2y^2+\lambda_z^2z^2]. \quad (2)$$

The parameters λ_η ($\eta=x,y,z$) account for the anisotropy of the trap, and the trap centers for the first and second components are displaced in the x direction by x_1 and x_2, respectively. The coupling constants U_{nm} are related to the scattering lengths a_{nm} by $U_{nm}=4\pi\hbar^2a_{nm}/M$. In this paper we allow only for the following elastic scattering processes: $|1\rangle|1\rangle\rightarrow|1\rangle|1\rangle$, $|2\rangle|2\rangle\rightarrow|2\rangle|2\rangle$, and $|1\rangle|2\rangle\rightarrow|1\rangle|2\rangle$. Up to now, Eqs. (1) have been solved in the Thomas-Fermi approximation, exploring the spatial densities of the mixtures [22] and the long-wavelength excitations [23]. Numerical studies of these systems have been carried out in Ref. [24].

For noninteracting particles, Eqs. (1) are solved for the ground state by Gaussian functions, which are no more solutions in the general case due to the nonlinear self-interaction and coupling terms. The basic assumption behind the following derivations is that, in the case of weak interactions, the wave function is still well approximated by a Gaussian,

$$\Psi_n(\vec{r},t)=A_n(t)e^{-(1/2)[\beta_{nx}(t)(x-x_{n0})^2+\beta_{ny}(t)y^2+\beta_{nz}(t)z^2]}, \tag{3}$$

where $\beta_{n\eta}=\beta_{n\eta}^R+i\beta_{n\eta}^I$ are complex numbers. The adjustable parameters in it can be interpreted as the amplitudes A_n, the width $w_{n\eta}=1/\sqrt{\beta_{n\eta}^R}$, and the curvature $(M_{n\eta}\sqrt{\beta_{n\eta}^R})^{-1/2}=(\beta_{n\eta}^I)^{-1/2}$ of the wave function. To derive reliable results for the corresponding equations of motion, it is necessary to take also the imaginary part of the wave function into account, as shown in Ref. [25] for a similar case. One has to keep in mind when applying this ansatz to problems where losses are included that the loss terms usually induce aberrations, thus making the profile non-Gaussian. When nonlinear loss or gain terms are substantially important, these aberrations make the information that can be obtained form an ansatz like Eqs. (3) only qualitative. However, in the present case when losses are small corrections and do not determine the dynamics the solutions of Eq. (1) can be well approximated by Gaussian functions.

In the following we find it convenient to work with scaled variables. We will measure energies in units of $\hbar\omega$, and will scale lengths with respect to $a_0\sqrt{2}$, where $a_0=[\hbar/(M\nu)]^{1/2}$ is the size of the ground state of the bare harmonic oscillator.

B. Moment equations

In Ref. [14], a solution of the single-component Gross-Pitaevskii equation was formulated as a variational problem, where a Lagrangian density $\mathcal{L}$ is derived so that the stationary point of the corresponding action gives the nonlinear Schrödinger equation. An approximate solution of the GPE was then derived by finding the extremum of the action within a set of Gaussian trial functions which give ''Newton-type'' equations for the variational parameters. In contrast to such a variational analysis, we will solve the two-component GPE by means of a moment method. In the case of real U_{nm}, i.e. no losses [26], this method is equivalent to the variational approach. In principle, this approach is more general, as it can also be applied in a situation where loss terms are included in Eqs. (1).

Using the Gaussian ansatz for a system where one condensate is centered at $x_1=0$ and the second one at $x_2=\alpha$, it is possible to calculate the number of particles in each condensate $n=1$ and 2 as a function of the parameters

$$N_n=\int_{-\infty}^{\infty}d^3\vec{r}|\Psi_n(\vec{r},t)|^2=|A_n(t)|^2\pi^{3/2}w_{nx}w_{ny}w_{nz}. \tag{4}$$

In a similar way, the second moments of the spatial coordinates $\eta=x,y,z$ and the second moments of the momenta are given by

$$\langle\eta^2\rangle_1=\int_{-\infty}^{\infty}d^3\vec{r}\ \eta^2|\Psi_1(\vec{r},t)|^2=\tfrac{1}{2}N_1w_{1\eta}^2, \tag{5a}$$

$$\langle\eta^2\rangle_2=\int_{-\infty}^{\infty}d^3\vec{r}\ \eta^2|\Psi_2(\vec{r},t)|^2=\tfrac{1}{2}N_2w_{2\eta}^2\left(1+\frac{2\alpha}{w_{2\eta}^2}\right) \tag{5b}$$

and

$$\langle\partial_\eta^2\rangle_n=\int_{-\infty}^{\infty}d^3\vec{r}\ \Psi_n(\vec{r},t)^*\partial_\eta^2\Psi_n(\vec{r},t)=-\frac{N_n}{2}(w_{n\eta}^{-2}+M_{n\eta}^2), \tag{6}$$

respectively. These 14 variables provide a complete description of the system within the previously mentioned approximation. To obtain a closed set of equations of motion for $N_n(t)$, $w_{n\eta}(t)$, and $M_{n\eta}(t)$, we take the time derivative of Eqs. (4), (5), and (6), eliminate the time derivatives of the wave function with the help of the Schrödinger equations (1), and work out the resulting integrals over $\vec{r}$ with the help of the Gaussian ansatz (3). We obtain

$$\dot{N}_1=\frac{2q_{11}^IN_1^2}{w_{1x}w_{1y}w_{1z}}+2\sqrt{8}q_{12}^IN_1N_2e^{-\alpha^2/w_{1x}^2+w_{2x}^2}[(w_{1x}^2+w_{2x}^2)\times(w_{1y}^2+w_{2y}^2)(w_{1z}^2+w_{2z}^2)]^{-1/2}, \tag{7a}$$

$$\dot{w}_{1x}=-M_{1x}-\frac{N_1q_{11}^I}{2w_{1y}w_{1z}}+\sqrt{8}q_{12}^IN_2[(w_{1x}^2+w_{2x}^2)(w_{1y}^2+w_{2y}^2)\times(w_{1z}^2+w_{2z}^2)]^{-1/2}\frac{w_{1x}^3}{(w_{1x}^2+w_{2x}^2)^2}[w_{1x}^2-w_{2x}^2-2\alpha^2]\times e^{-\alpha^2/w_{1x}^2+w_{2x}^2}, \tag{7b}$$

$$\dot{w}_{1y}=-M_{1y}-\frac{N_1q_{11}^I}{2w_{1x}w_{1z}}+\sqrt{8}q_{12}^IN_2[(w_{1x}^2+w_{2x}^2)(w_{1y}^2+w_{2y}^2)\times(w_{1z}^2+w_{2z}^2)]^{-1/2}\frac{w_{1y}^3}{w_{1y}^2+w_{2y}^2}e^{-\alpha^2/w_{1x}^2+w_{2x}^2}, \tag{7c}$$

$$\begin{aligned}\dot{M}_{1x}={}&w_{1x}\lambda_x^2-\frac{1}{w_{1x}^3}-\frac{N_1}{w_{1x}w_{1y}w_{1z}}\left[\frac{3}{2}q_{11}^IM_{1x}+\frac{q_{11}^R}{w_{1x}}\right]\\&+\sqrt{8}N_2[(w_{1x}^2+w_{2x}^2)(w_{1y}^2+w_{2y}^2)(w_{1z}^2+w_{2z}^2)]^{-1/2}\\&\times\frac{1}{(w_{1x}^2+w_{2x}^2)^2}\Bigg[q_{12}^I\Bigg(\frac{2}{M_{1x}}(w_{1x}^2+w_{2x}^2-2\alpha^2)\\&-M_{1x}(w_{1x}^4+2w_{2x}^4+3w_{1x}^2w_{2x}^2+2\alpha^2w_{1x}^2)\Bigg)\\&-2q_{12}^Rw_{1x}[w_{1x}^2+w_{2x}^2-2\alpha^2]\Bigg]e^{-\alpha^2/(w_{1x}^2+w_{2x}^2)},\end{aligned} \tag{7d}$$

$$\dot{M}_{1y}=w_{1y}\lambda_y^2-\frac{1}{w_{1y}^3}-\frac{N_1}{w_{1x}w_{1y}w_{1z}}\left[\frac{3}{2}q_{11}^I M_{1y}+\frac{q_{11}^R}{w_{1y}}\right] + \sqrt{8}N_2[(w_{1x}^2+w_{2x}^2)(w_{1y}^2+w_{2y}^2)(w_{1z}^2+w_{2z}^2)]^{-1/2} \times\frac{1}{(w_{1x}^2+w_{2x}^2)^2}\left[q_{12}^I\left(\frac{2}{M_{1x}}(w_{1x}^2+w_{2x}^2)-M_{1x}(w_{1x}^4 + 2w_{2x}^4+3w_{1x}^2w_{2x}^2)\right) -2q_{12}^R w_{1x}[w_{1x}^2+w_{2x}^2]\right]e^{-\alpha^2/w_{1x}^2+w_{2x}^2}, \tag{7e}$$

where we replaced $q_{nm}=U_{nm}\sqrt{8\pi^3}$ [26]. The corresponding equations for the second condensate are obtained by exchanging every index 1 and 2, and the corresponding equations for the z direction are obtained by exchanging the indices y and z in the equations for the y direction.

III. ISOTROPIC TRAPS AND IDENTICAL CONDENSATES: NO DISPLACEMENT OF TRAP CENTERS

A. Equations

The simplest case which is amenable to analytical treatment corresponds to an isotropic trapping potential $\lambda_x=\lambda_y=\lambda_z=1$ with (nondisplaced) trap centers $x_n=0$ ($n=1$ and 2) [27]. We will study the case of a fixed number of particles in each condensate, $U_{11}^I=U_{22}^I=U_{12}^I=0$, and, in particular, consider them equal, $N_1=N_2=N$. With these assumptions, and the initial condition of two isotropic condensates $w_{nx}=w_{ny}=w_{nz}\equiv w_n$ ($n=1$ and 2), the equations of motion for the widths of the condensates have the form

$$\ddot{w}_1=-w_1+\frac{1}{w_1^3}+\frac{Nq_{11}}{w_1^4}+2\sqrt{8}\,\frac{Nq_{12}w_1}{(w_1^2+w_2^2)^{5/2}}, \tag{8}$$

whereby exchanging the indices we obtain the corresponding one for w_2.

Similar to Ref. [14], we can find a ''potential'' for these equations. Hence the evolution of the width of the condensates can be viewed as the coordinates of a fictitious classical particle moving in a three-dimensional potential

$$V_{\mathrm{eff}}(w_1,w_2)=\tfrac{1}{2}(w_1^2+w_2^2)+\frac{1}{2}\left(\frac{1}{w_1^2}+\frac{1}{w_2^2}\right) + \tfrac{1}{3}N\left(\frac{q_{11}}{w_1}+\frac{q_{22}}{w_2}\right)+\tfrac{2}{3}\sqrt{8}\,\frac{Nq_{12}}{(w_1^2+w_2^2)^{3/2}}, \tag{9}$$

where the first term on the right-hand side stems from the harmonic trapping potential, the second term results from the kinetic-energy term, and the third and fourth terms are due to the interaction between alike particles and between particles in different states. Below, we further reduce the number of open parameters by assuming the scattering lengths a_{11} and a_{22} to be equal.

B. Equilibrium points

If we now examine the state in equilibrium of the system, we find that the requirement of equal scattering lengths implies that the condensates must have equal widths $w_1=w_2\equiv w$. Equilibrium points are obtained from the extrema of the potential (9), which gives

$$w=\frac{1}{w^3}+\frac{N}{w^4}(q_{11}+q_{12}). \tag{10}$$

In the hydrodynamic approximation this equation of motion reduces further, and can be analytically solved as $w_{\mathrm{eq}}^{\mathrm{ha}}=N^{1/5}(q_{11}+q_{12})^{1/5}$. To test for stability of these equilibrium points requires us to check for minima with the help of the Hessian matrix $H=[\partial^2V_{\mathrm{eff}}/(\partial w_{1n}\partial w_{2n})]_{w_{\mathrm{eq}}}$. This analysis will be carried out in Sec. III C.

C. Low-energy eigenmodes and stability

To obtain the frequencies of small-amplitude oscillations around the equilibrium positions we linearize the full equations (8) around the equilibrium points, $w_{n\eta}=w_{n\eta}^{\mathrm{eq}}+\delta w_{n\eta}$. We find the following four frequencies:

$$\omega_a=\left(1+\frac{3}{w^4}+\frac{N}{w^5}(q_{11}-q_{12})\right)^{1/2}, \tag{11a}$$

$$\omega_b=\left(1+\frac{3}{w^4}+\frac{N}{w^5}(q_{11}+q_{12})\right)^{1/2}, \tag{11b}$$

$$\omega_c=\left(1+\frac{3}{w^4}+\frac{N}{w^5}(4q_{11}+4q_{12})\right)^{1/2}, \tag{11c}$$

$$\omega_d=\left(1+\frac{3}{w^4}+\frac{N}{w^5}(4q_{11}-q_{12})\right)^{1/2}, \tag{11d}$$

which can be interpreted as collective oscillations depicted in Fig. 1. The modes (a) and (b) are twofold degenerate, corresponding to oscillation, e.g., in the x-y plane and the x-z plane [28]. Modes (c) and (d) correspond to isotropic oscillations.

The order in which these motions appear with increasing energy depends strongly on the values of the scattering lengths involved. In Fig. 2 we display the spectrum for a positive value of a_{11}. For large, negative values of a_{12}, the two modes with the lowest energies are those with the maximum spatial overlap, namely, modes (b) and (c). If we now consider the region where a_{12} is positive and large, we naturally find that the modes (a) and (d), for which the spatial overlap is minimized, are the lowest-energy modes. The point at which this crossing happens changes with changing values of a_{11}. Again we find, as in Ref. [14], that the widths of the condensates remain finite up to the point of collapse.

Eigenfrequencies (11) provide the stability region of the equilibrium points (10),

$$N(q_{11}+q_{12})=-\tfrac{4}{5}\left(\tfrac{1}{5}\right)^{1/4} \tag{12a}$$

$$-\left(\tfrac{5}{4}\right)^5N^4q_{11}^5+\tfrac{1}{4}q_{11}=q_{12}. \tag{12b}$$

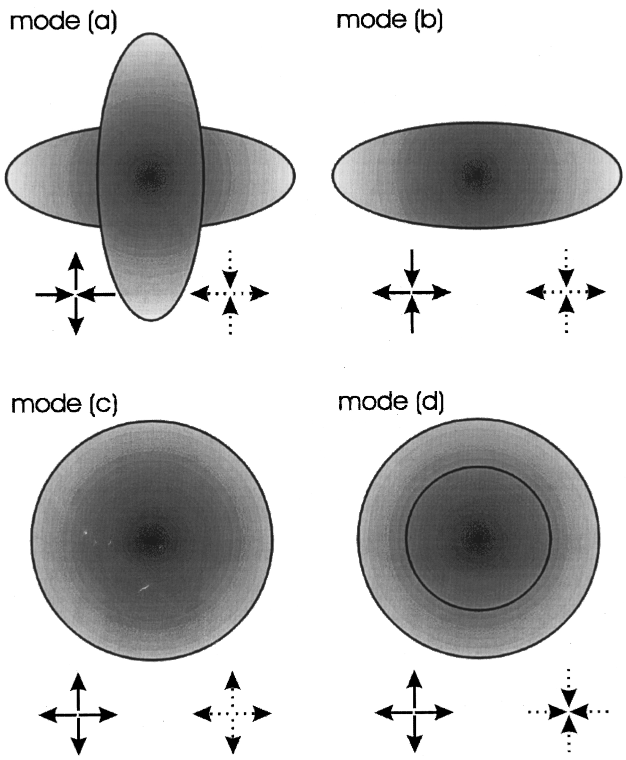

FIG. 1. Diagrammatical representation in the x-y plane of the four collective excitations with the lowest energy. Oscillations of particles in state $|1\rangle$ are visualized by solid line arrows, and oscillations of particles in state $|2\rangle$ by dashed line arrows.

Since for $a_{12}=0$ we recover the single-component results [14] where only particles in one state are present, we see that for positive values of a_{12} the region of stability increases and for negative ones it decreases. This is due to contributions to the total system energy from either the repulsive or attractive interaction. However, this effect is nonlinear due to the nature of this interaction. It is illustrated in Fig. 3.

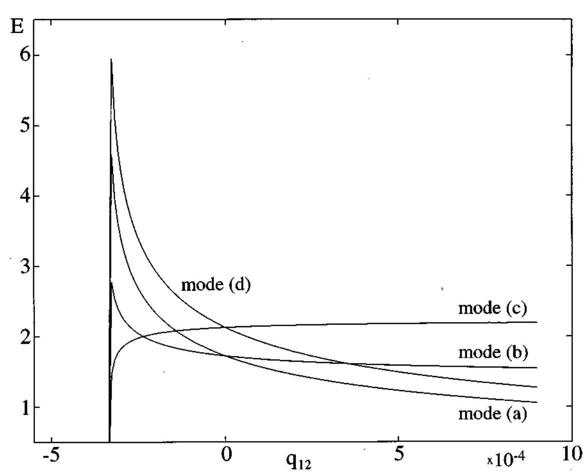

FIG. 2. Spectrum of the low-lying excitation frequencies for a fixed, positive value $q_{11}=2.3\times 10^{-4}$. With q_{12} increasing from negative values to positive ones, the two low-lying modes and the two high-lying ones are exchanged.

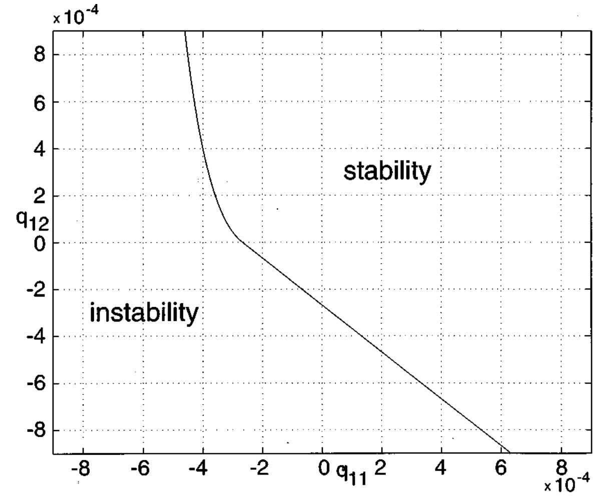

FIG. 3. Phase diagram for the stability of the two-condensate system for $N=2000$ particles. For positive a_{12} the region of stability is increased due to the positive energy contribution from the repulsive interaction. However, it does not expand linearly with increasing a_{12}.

IV. ISOTROPIC TRAPS AND IDENTICAL CONDENSATES: DISPLACED TRAP CENTERS

A. Equations

Let us now consider the case of two spatially displaced traps, and work out the differences from the above situation. We assume a displacement in the x direction and make an ansatz analogous to Eqs. (3) and (1), where one condensate is centered at $x_1=0$ and the second at $x_2=\alpha$. Also in this case we assume that the number of particles is conserved, which implies that the imaginary parts of the coupling constants are again zero. We apply the moment method as in the previous case and arrive, since the spatial symmetry is now broken in the x direction, at the following set of equations of motion for the widths of the two condensates in the x direction, w_{nx}, and in the y and z directions, w_n:

$$\ddot{w}_{1x}=-w_{1x}\lambda_{1x}^2+\frac{1}{w_{1x}^3}+\frac{N_1 q_{11}}{w_{1x}^2 w_1^2}+2\sqrt{8}N_2\,\frac{q_{12}w_{1x}(w_{1x}^2+w_{2x}^2-2\alpha^2)}{(w_{1x}^2+w_{2x}^2)^{5/2}(w_1^2+w_2^2)}e^{-\alpha^2/(w_{1x}^2+w_{2x}^2)}, \tag{13a}$$

$$\ddot{w}_1=-w_1\lambda_1^2+\frac{1}{w_1^3}+\frac{N_1 q_{11}}{w_{1x}w_1^3}+2\sqrt{8}N_2\,\frac{q_{12}w_1}{(w_{1x}^2+w_{2x}^2)^{1/2}(w_1^2+w_2^2)^2}e^{-\alpha^2/(w_{1x}^2+w_{2x}^2)}. \tag{13b}$$

Again one obtains the equations for w_{2x} and w_2 by simply exchanging the indices. We can also, as in the nondisplaced case, find a potential of the form of Eq. (9), which is suitable for the derivation of these equations. However, in this case the term originating from the interaction of the particles in the different states is multiplied by an additional factor,

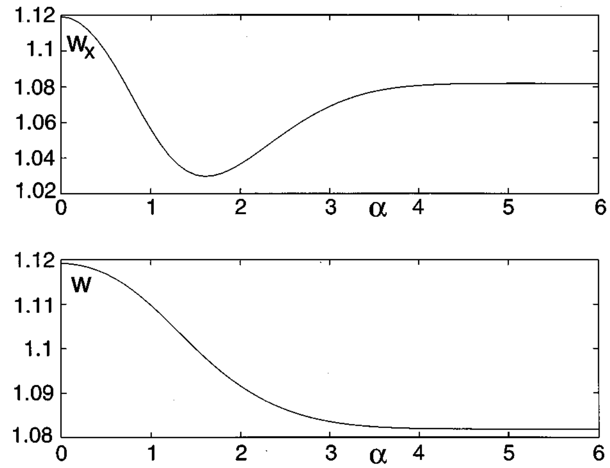

FIG. 4. Width of the condensates in the x direction and in the y and z directions for $q_{11}=0.2\times10^4$ and $q_{12}=0.2\times10^4$. Due to the repulsive interaction between particles from different states, the condensate is squeezed in the x direction at a certain distance, and then expands once the region of overlap diminishes further.

$\exp[-\alpha^2/(w_{1x}^2+w_{2x}^2)]$, which ensures that the influence of these interactions becomes weaker the more the condensates are displaced. This then implies an even smaller region of spatial overlap. Note that the widths of the condensates are included in this factor, and that it therefore has a non-negligible effect on the physics of the system in addition to weakening the magnitude of effects found in the nondisplaced case.

Let us start by calculating the equilibrium states for this case from Eqs. (13). In Fig. 4 we plotted the widths w_x and w of the condensates for a_{11} and a_{12} to be positive and equal. Once the condensates are separated by a distance $\alpha=\sqrt{(w_{1x}^2+w_{2x}^2)/2}$, the repulsive interaction between particles from different states becomes an effectively attractive one (and vice versa), as can be seen from the fact that the width in the x direction becomes smaller than the width in the y and z directions. The explanation for this effect is that the direction of the repulsive force, with respect to the surfaces of the condensates, changes in these regions where they no longer overlap. At large distances, when the interaction between the particles in different states again becomes negligible, the case of two independent, isotropic condensates is again recovered.

B. Low-energy eigenmodes and stability

Looking at the excitation spectrum of the two-component system by analyzing it again via a linear expansion around the equilibrium points, we find, as can be seen from Fig. 5, that the degeneracies existing in the nondisplaced case are lifted with increasing distance. Bringing the condensates that far from each other that they can be considered as two independent condensates [14], the spectrum again reduces to two threefold-degenerate values, since the condensates again become isotropic.

In this Gaussian ansatz we can again easily interpret these modes to be of the following types:

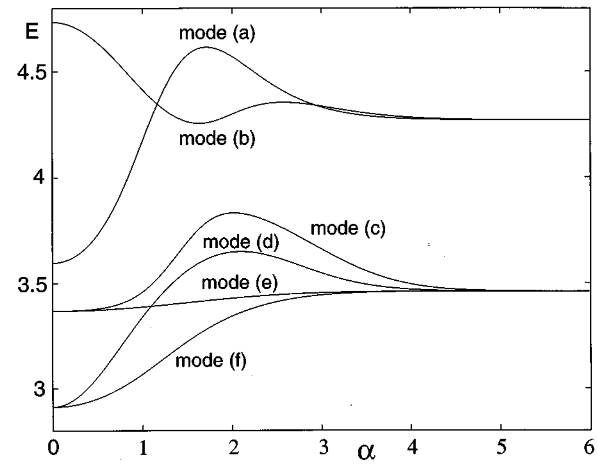

FIG. 5. Spectrum of the low-lying excitation energies for $q_{11}=q_{12}=6\times10^{-4}$. The degeneracies are lifted with increasing α, and for $\alpha>2$ it reduces to the case of two isotropic, single condensates again.

$$\begin{array}{lr}
(-\gamma_a\ \ -1\ \ -1\ \ \gamma_a\ \ 1\ \ 1\ ) & \text{mode(a)},\\
(\ \ \gamma_b\ \ 1\ \ 1\ \ \gamma_b\ \ 1\ \ 1\ ) & \text{mode(b)},\\
(\ \ \gamma_c\ \ -1\ \ -1\ \ \gamma_c\ \ -1\ \ -1\ ) & \text{mode(c)},\\
(-\gamma_d\ \ 1\ \ 1\ \ \gamma_d\ \ -1\ \ -1\ ) & \text{mode(d)},\\
(\ \ 0\ \ -1\ \ 1\ \ 0\ \ -1\ \ 1\ ) & \text{mode(e)},\\
(\ \ 0\ \ 1\ \ -1\ \ 0\ \ -1\ \ 1\ ) & \text{mode(f)}.
\end{array} \tag{14}$$

The members of these vectors represent the oscillation amplitude in the directions $(\delta_{1x}\ \delta_{1y}\ \delta_{1z}\ \delta_{2x}\ \delta_{2y}\ \delta_{2z})$, where γ_l $(l=a,b,c,d)$ depends on the parameters of the system. The lifting of the degeneracies is mainly due to the fact that the spatial displaced system is not invariant under an exchange of the variables w_{1y} and w_{1z} and not of w_{2y} and w_{2z}, as can be seen from Eqs. (13).

Let us now look at the stability of the stationary states, which we again retrieve from the analysis of the Hessian matrix. As can be seen from Fig. 6, the further the condensates separate from each other, the weaker the effect of additional stabilization (destabilization) becomes, since it re-

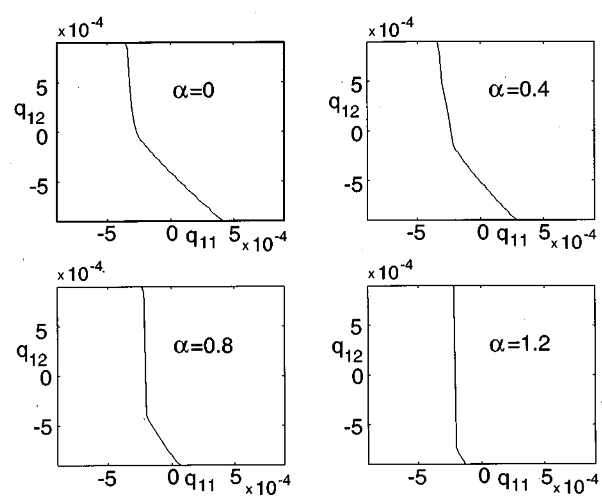

FIG. 6. Stability region for displaced traps with a displacement parameter α. For far displaced traps the result for independent condensates is recovered.

sults from the repulsive (attractive) interaction between the condensates. For $\alpha \approx > 1.5$, one recovers the stability condition for the case of two separate condensates.

V. CONCLUSIONS

We used a moment method to describe the behavior of two overlapping Bose-Einstein condensates in different internal states, assuming that the shapes of the individual wave functions do not deviate too much from a Gaussian shape. With this method it is straightforward to derive equations of motion for the open parameters, defining the exact form of the Gaussian function, especially the width of the condensates. They can, for the general cases, easily be treated numerically, and for special cases even analytical solutions are possible.

For the case of isotropic traps and isotropic condensates, we calculated the equilibrium points for the system and analyzed their stability. We found that, in comparison to the case of one pure condensate, the region of stability can be extended (shrunken) if the interaction energy between particles in different states is positive (negative). Due to the nonlinear character of the interaction, this effect is also of a nonlinear nature. It appears strongest for nondisplaced traps, and becomes weaker the farther apart the centers of the two traps are from each other. We were also able to calculate the energies of the first excited states and interpret the collective motions. For the case of displaced traps, the degeneracies existing in the nondisplaced case are lifted.

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[27] In the experiment of Ref. [4], the trap is anisotropic ($\nu_x=\nu_z=400$ Hz and $\nu_y=11$ Hz), and since the spring constants of the traps are different for both condensate states, the centers of the traps are slightly displaced due to gravity.
[28] Oscillations in the y-z plane can be expressed as a superposition of x-y and x-z oscillations.