Quantization of the motion of a particle on an n-dimensional sphere

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We develop here a simple formalism that converts second-class constraints into first-class ones for a particle moving on an n-dimensional sphere. The Poisson algebra generated by the Hamiltonian and the constraints closes and by quantization transforms into a Lie algebra. The observable of the theory is given by the Casimir operator of this algebra and coincides with the square of the angular momentum. [S1050-2947(97)04710-0]

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I. INTRODUCTION

The quantization of classical Hamiltonians, when the canonical coordinates are not completely independent, is a long-standing problem in quantum mechanics. The constraints, i.e., a set of functions

$$\varphi_i(q,p)=0, \quad i=1,2,\ldots,p, \tag{1.1}$$

restrict the motion of the classical system to a manifold embedded in the initial Euclidean phase space. This has as a consequence that the canonical quantization rules

$$[q_i,p_j]=i\hbar\delta_{ij}$$

are no longer sufficient for the quantum description of the physical system.

When the manifold is a proper subspace of a Euclidean space Podolski [1] gave a solution by postulating that the Euclidean Laplacian should be replaced by the Laplace-Beltrami operator acting on this manifold. Applied to the motion of a point particle on an n-dimensional sphere S^n of radius R, this gives for the Laplace-Beltrami operator the result $L^2/2R^2$, where L is the angular momentum of the particle.

The eigenvalues of the Casimir operator L^2 are $l(l+n-1)$, $l=0,1,\ldots$, and in deriving this result one makes use of the Lie algebra of the angular momentum, forgetting completely about the canonical variables initially entering in the problem, avoiding in this way any trouble that could appear. Doubts concerning the correctness of this spectrum have arisen in people who derive the Schrödinger equation by Feynman's path-integral method; see, for example, [2–4]. They found an extra energy term proportional to the Riemann scalar curvature of the manifold, but they do not agree upon the value of the proportionality factor. Another type of doubt has arisen recently [5], namely, that the Dirac's quantization method [6] has to be rejected because, at least for this problem, the resulting energy spectrum is physically incorrect. The last statement [5] comes from a misunderstanding of the subtlety of the problem: The constraints, like Eq. (1), reduce the dimension of the original phase space. The mechanism found by Dirac was the introduction of a different symplectic structure to handle the second-class constraints. The first-class constraints are used to drop some pairs of dynamical variables (q_i,p_i) from the naive Hamiltonian, whose effect is that the nonphysical degrees of freedom are eliminated. See Ref. [7] for a treatment of the rigid rotator in the Dirac-bracket quantization formalism.

The purpose of this paper is to look at the Dirac formalism from a slightly modified point of view and to show that the present proposal leads to correct results. In fact, we propose an alternative method for converting the second-class constraints into first-class constraints.

The Dirac quantized theory [6] is patterned after the corresponding classical theory: The observables representing constraints *must* have zero expectation values. This requirement is inconsistent with the fact that there are dynamical variables whose Poisson brackets with the constraints fail to vanish. To solve the problem Dirac constructed a different type of bracket, the Dirac bracket, which vanishes whenever one of the two factors is a second-class constraint.

We develop here a formalism that converts the second-class constraints into first-class ones and leads directly to group properties of the Poisson brackets. By quantization the Poisson algebra goes into a Lie algebra. The *observables* of the theory will be the *Casimir operators* of this algebra and the operators generated by the constraints will commute with the observables in the whole Hilbert space. With this interpretation Dirac's theory of constrained systems gives correct results and in the particular case of a point particle on S^n it confirms the Podolski result.

Our idea is to separate all the constants terms that may appear on the left-hand side of Eq. (1.1) and push them to the right-hand side. Thus we prefer to write Eq. (1.1) in the form

$$\varphi_i(q,p)=a_i, \quad i=1,2,\ldots,p, \tag{1.1$'$}$$

where a_i are some complex constants.

One reason is that the Poisson bracket structure does not discriminate between φ and $\varphi+C$, with C a constant. The main reason is that it now becomes possible to write the Poisson algebra in a closed form such as

$$\{\varphi_i(q,p),\varphi_j(q,p)\}=C_{ij}^k\varphi_k(q,p),$$

$$\{H(q,p),\varphi_i(q,p)\}=C_i^j\varphi_j(q,p), \tag{1.2}$$

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where $H(q,p)$ is the Hamiltonian, $\varphi_i(q,p)$ are the constraints appearing in Eq. (1.1′), and C_{ij}^k and C_i^j are *constant* structure coefficients.

Almost all that is found here is in Dirac's book [6], the only difference being the redefinition of the dynamical variables φ_i. But this different form has the advantage of transforming at least some second-class constraints into first-class constraints, as we will show in Sec. II. In our opinion the Poisson algebra (1.2) generated by the Hamiltonian and the constraints *is the best of the Dirac method.*

The Poisson algebra, by the quantization procedure, transforms into a Lie algebra. The true observables of the physical systems are given by the Casimir operators of the corresponding Lie algebras. In other words, none of the initial operators transforms into a veritable observable. The observables are given at least by quadratic functions of the old operators: Hamiltonian plus constraints together.

The quantum description of the physical model is given by a representation of this algebra onto a Hilbert space. In this way it becomes possible to avoid the canonical variables, which appear also in the Dirac formalism and sometimes cause problems since their Dirac brackets are not always canonical; for a discussion in this sense see the treatment of the three-dimensional rotator in Ref. [7].

Another consequence of the above idea is a solution of the embarrassing situation of forcing the operators generated by the constraints to vanish on the whole Hilbert space, as a Dirac dixit. Because now the constraints are no longer observables, the above problem disappears. What we can say is that there exists a representation of the Lie algebra into an operator algebra acting on the Hilbert space of the associated physical system such that the operators $\hat{\varphi}_i$ generated by the constraints should have the numbers a_i in their spectra.

Of course there are cases when the above procedure does not work. An example of such a constraint is

$$\chi_1=\sum_i c_i q_i = a,$$

which is linear in coordinates. Its Poisson bracket with a quadratic free Hamiltonian gives the secondary constraint

$$\chi_2=\{\chi_1,H\}=\sum_i c_i p_i,$$

which is linear in momenta. The Poisson bracket of these two constraints is a constant Σc_i^2; hence the linear constraints in coordinates and/or momenta generate another type of algebra. In these cases relations (1.2) are supplemented with at least a few relations of the form

$$\{\chi_i,\chi_j\}=a_{ij}$$

where a_{ij} are constants. These cases have to be treated by the Dirac formalism. With this mild interpretation of Dirac theory, the difficulties are overcome and the present theory is ready for applications.

In Sec. II we treat the point particle on an n-dimensional sphere showing that the ''Hamiltonian'' of the problem is the square of the angular momentum and is obtained as the Casimir operator of the Lie algebra obtained by quantization of the Poisson algebra generated by the Hamiltonian and constraints. In Sec. III we consider a related problem: an $(n+1)$-dimensional harmonic oscillator constrained to move on a hypersurface. The paper concludes in Sec. IV.

II. POINT PARTICLE ON AN S^N SPHERE

We shall consider a point particle moving on an n-dimensional sphere S^n whose equation is

$$r^2=(q,q)=\sum_{i=1}^{n+1} q_i^2,$$

where (q,q) denotes the Euclidean scalar product in $(n+1)$-dimensional space, i.e., we view the particle moving into a subspace of $(n+1)$-dimensional Euclidean space. Phase-space degrees of freedom (q_i,p_i) take values over the entire real axis and possess a canonical Poisson bracket structure.

The primary constraint is usually written as

$$\varphi=r^2-R^2=0$$

and the Hamiltonian has the form

$$H=\frac{1}{2}(p,p)=\frac{p^2}{2}.$$

We define $U=r^2$ as a dynamical variable and by taking the Poisson brackets we get

$$\{U,H\}=2\sum q_i p_i=2(q,p)=2V,$$
$$\{V,H\}=p^2=2H, \tag{2.1}$$
$$\{U,V\}=2U.$$

If we use φ as a dynamical variable instead of U, as it is usually done [5,7], we find

$$\{\varphi,H\}=2V,$$
$$\{V,H\}=2H,$$
$$\{\varphi,V\}=2r^2=2(\varphi+R^2).$$

The last relation is usually written as $\{\varphi,V\}=2r^2=2R^2$, which has the consequence that after quantization one gets the commutation relation $[\varphi,V]=2i\hbar R^2$. Here we use the same notation for operators on Hilbert space and for dynamical variables on phase space. The last relation is in conflict with the conditions $\varphi=V=0$ imposed on operators acting on Hilbert space. How we will solve this conflict will be seen later.

The relation $\{\varphi,V\}=2(\varphi+R^2)$ suggested to us the introduction of U as a dynamical variable because

$$\{\varphi,V\}=\{\varphi+R^2,V\}=2(\varphi+R^2).$$

Taking $U=\varphi+R^2=r^2$ as a dynamical variable has the advantage of closing the algebra as relations (2.1) show and, more important, by this procedure both U and V become first-class constraints. In the following we shall take $\hbar=1$.

By the correspondence principle we obtain from Eqs. (2.1) the commutation relations

$$[U,H]=2iV,$$
$$[V,H]=2iH, \tag{2.2}$$
$$[U,V]=2iU.$$

In order to solve the problem we state our first postulate: All the relevant physics concerning the problem is contained in the Lie algebra (2.2). This algebra reminds us of the known Lie algebra of the SU(2) group, so we can proceed as in that case. The single observable is the Casimir operator, which is easily seen to be

$$C=V^2-UH-HU. \tag{2.3}$$

Indeed, by a trivial calculation we obtain that

$$[C,H]=[C,U]=[C,V]=0.$$

C is the true observable of the theory and it commutes with the operators generated by the classical constraints and the classical Hamiltonian.

Like in the SU(2) case, we can look for a common basis of eigenvectors for C and one of the operators H, U, or V. We choose V as the ''third'' component because it is singled out by the algebra (2.2), as we shall see later.

From Eq. (2.1), $V=(q,p)$ is the scalar product of q and p giving the projection of the momentum along the radius. The classical requirement $V=0$ means that the motion of the point particle has to be such that there should be no energy or momentum flow across the sphere surface.

If we change from Euclidean to spherical coordinates V has the form

$$V=\frac{1}{i}r\frac{\partial}{\partial r},$$

where $\partial/\partial r$ is the normal derivative or the gradient at the point of the sphere determined by its spherical coordinates.

Usually one imposes $V=0$ as an operator equation on the Hilbert space. In our approach V is not an observable, so the equation $V=0$ is senseless. Our point of view is expressed by the second postulate: In the Hilbert space associated with our physical problem there exists one vector that is annihilated by V. Thus the common eigenvector of C and V has to satisfy the equation

$$r\left.\frac{\partial\Psi(r,\Omega)}{\partial r}\right|_{r=R}=0, \tag{2.4}$$

where by Ω we denote the angular variables. Equation (2.4) tell us that the eigenvector Ψ may depend on the radial variable r, but its dependence is such that Ψ is stationary at $r=R$. If we require that Eq. (2.4) should be valid for an arbitrary value of R, we get that Ψ does not depend on r.

By using the commutation relations (2.2) we find that

$$C=V^2+2iV-2UH.$$

On the other hand, in spherical coordinates, H has the form

$$2H=-\left(\frac{\partial^2}{\partial r^2}+\frac{n}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}L^2\right),$$

where L^2 is the Laplace-Beltrami operator on the unit sphere S^n and coincides with the Casimir operator of the orthogonal group SO$(n+1)$ [8].

Putting together the previous information, we find that C has the form

$$C=L^2+(n+1)r\frac{\partial}{\partial r}. \tag{2.5}$$

Taking into account Eq. (2.4) we find that

$$C\Psi=\left(L^2+(n+1)r\frac{\partial}{\partial r}\right)\Psi|_{r=R}=L^2\Psi=l(l+n-1)\Psi.$$

Although C is not a Hermitian operator, because V is not, the action of both C and V on the eigenvector Ψ reduces to the action of the Hermitian operator L^2. In this way the eigenvalue problem is quantum mechanically well posed and the spectrum is $l(l+n-1)$, $l=0,1,\ldots$ [8].

Thus the Dirac formalism in the present interpretation tells us that the observable of a particle moving on an n-dimensional sphere is its angular momentum, a result that was expected. This result can be tested directly at the classical and consequently at the quantum level.

For simplicity we take $n=2$ and in this case the components of the angular momentum are

$$L_1=q_2p_3-q_3p_2,\quad L_2=q_3p_1-q_1p_3,\quad L_3=q_1p_2-q_2p_1.$$

Let us introduce its projection on an arbitrary ray

$$S=q_1L_1+q_2L_2+q_3L_3=lR,$$

where l is a constant, the length of the projection, and consider S as a new constraint.

Taking into account the Poisson bracket relations

$$\{L_i,p_j\}=\epsilon_{ijk}p_k,\quad \{L_i,q_j\}=\epsilon_{ijk}q_k,$$
$$\{L_i,L_j\}=\epsilon_{ijk}L_k,\quad \{x_i,p_j\}=\delta_{ij},\quad i,j,k=1,2,3,$$

we find that

$$\{S,H\}=\{S,U\}=\{S,V\}=0,$$

a relation that shows that S is a conserved quantity at the classical level since it commutes with the Hamiltonian and both constraints. At the quantum level this relation shows that $S=f(C)$, i.e., it is a function of the Casimir operator of the Lie algebra. We think that the above correct quantization of this simplest non-Euclidean system will have a fundamental theoretical interest, showing us the route to follow in much more complicated cases.

Now we show that V is the analog of L_3 of the SU(2) group. We make the notation

$$H_1=-\frac{V}{2i},\quad E_+=-\frac{U}{2i},\quad E_-=\frac{H}{2i}$$

and the commutation relations (2.2) take the form

$$[H_1,E_+]=E_+,$$
$$[H_1,E_-]=-E_-, \tag{2.6}$$
$$[E_+,E_-]=H_1,$$

i.e., the well-known Cartan-Chevalley form of the most simple Lie algebra. From Eq. (2.6) we see that $E_\pm$ is the analog of $L_\pm$ and H_1 is the analog of L_3, where $L_\pm$ and L_3 are the usual generators of the SU(2) group.

III. CONSTRAINED HARMONIC OSCILLATOR

In the following we give an argument in favor of our interpretation. For this we will consider another simple system: namely, the quantization of the ℓ-dimensional oscillator whose Hamiltonian is

$$H_0=\frac{1}{2}\sum_{i=1}^{\ell}(p_i^2+q_i^2).$$

We suppose that its movement is confined to the hypersurface given by the constraint

$$V_0=\sum_{i=1}^{\ell}q_ip_i=a, \tag{3.1}$$

where a is a constant.

If we proceed as above, we find the Poisson algebra

$$\{H_0,V_0\}=\sum_{i=1}^{\ell}(q_i^2-p_i^2)=2U_0,$$
$$\{U_0,V_0\}=2H_0,$$
$$\{U_0,H_0\}=2V_0.$$

After quantization we find the Casimir operator, which commutes with H_0, U_0, and V_0,

$$C_0=H_0^2-V_0^2-U_0^2.$$

We denote by $L_{ij}=q_ip_j-q_jp_i$ $(i<j,\ i,j=1,2,\ldots,\ell)$ the components of the angular momentum and by using the above expressions for H_0, U_0, and V_0 we find that

$$C_0=\sum_{i<j}L_{ij}^2=L^2,$$

i.e., the true Hamiltonian of the problem is again the square of the angular momentum. In fact, this problem is a reformulation of the previous one. Indeed H_0, U_0, and V_0 are related to H, U, and V by the relations

$$H_0=\frac{1}{2}U+H,$$
$$U_0=\frac{1}{2}U-H,$$
$$V_0=V.$$

Our procedure can be applied whenever the Poisson algebra closes after a finite number of secondary constraints. This may not be the usual situation as a ‘‘small perturbation’’ of the previous problem shows. We deform the constraint (3.1) to

$$\mathcal{V}_1=\sum_{i=1}^{i=\ell}c_iq_ip_i=a,$$

where c_i are constants. Then one easily finds that the Poisson algebra never closes. If we define the sequence of Hamiltonians

$$\mathcal{H}_n=\frac{1}{2}\sum_{i=1}^{i=\ell}c_i^{2n}(q_i^2+p_i^2),\quad n=0,1,\ldots,$$

and constraints

$$\mathcal{V}_n=\sum_i c_i^{2n-1}q_ip_i=0,\quad n=2,3,\ldots$$

$$\mathcal{U}_n=\frac{1}{2}\sum_i c_i^{2n-1}(q_i^2-p_i^2)=0,\quad n=1,2,\ldots,$$

after quantization we find an infinite-dimensional algebra. Its commutation relations are

$$[\mathcal{U}_m,\mathcal{H}_n]=2i\mathcal{V}_{m+n},\quad m=1,2,\ldots,\quad n=0,1,\ldots$$
$$[\mathcal{H}_m,\mathcal{V}_n]=2i\mathcal{U}_{m+n},\quad m=0,1,\ldots,\quad n=1,2,\ldots$$
$$[\mathcal{U}_m,\mathcal{V}_n]=2i\mathcal{H}_{m+n-1},\quad m,n=1,2,\ldots \tag{3.2}$$
$$[\mathcal{H}_m,\mathcal{H}_n]=0,\quad [\mathcal{U}_m,\mathcal{U}_n]=0,\quad [\mathcal{V}_m,\mathcal{V}_n]=0,$$

where in the last three equations the indices of $\mathcal{H}_m$ take the values $m=0,1,2,\ldots,$ the range of the others being $m=1,2,\ldots,$ according with the notation of the first three equations.

A similar algebra is obtained by deforming the sphere $(q,q)=R^2$ into an ellipsoid by the change $q_i\rightarrow q_i/a_i$. This shows that the problem of quantization with constraints is not a simple one, its natural place being the representation theory of infinite-dimensional algebras.

IV. CONCLUSION

The main difficulty appearing in the quantization of constrained systems is caused by the second-class constraints. To overcome it Dirac invented a symplectic structure, the Dirac bracket. However, its use is not straightforward and we have to be careful when using it. This was a sufficient reason to looking for alternative methods of quantization.

The best known method is one of Abelian conversion that transforms a second-class constrained system into an Abelian gauge theory [9,10]. The idea is to extend the original phase space by introducing new canonical coordinates and to convert the original Hamiltonian into a new one obtained by solving some equations. Upon quantization all nonphysical degrees of freedom are removed by the restriction of the Hilbert space to a physical subspace formed by gauge-invariant states.

In this paper we observed that there are some cases when the conversion of second-class constraints into first-class ones is very simple, namely, we have seen that the obstacle was caused by the presence of constant terms within the functions defining the constraints. Because the Poisson bracket of a dynamical variable with a constant vanishes, this opens the possibility to rewrite the original Poisson brackets in another form by a simple redefinition of some of the dynamical variables. In this way it becomes possible to write the Poisson brackets in the form of Poisson algebra (1.2), which, after quantization, transforms into a Lie algebra. Once this algebra is obtained we can use the known powerful machinery of the representation theory to find the *observables* of the physical theory formalized by this algebra and to obtain the spectra of the physically relevant operators.

In this respect we consider that our proposal will have a large applicability to the problem treated here. In general, it will be successful in those cases where it will be possible to convert the second-class constraints into first-class ones by our method and possibly others.

The lesson to be learned is that for constrained systems none of the initial dynamical variables transforms into an observable. The observables are given by Casimir operators of Lie algebras, i.e., at least by quadratic functions of dynamical variables. As a consequence, we are no longer constrained to impose operators equations on the Hilbert space, like in our case $\varphi = V = 0$, because the constraints do not become observables of the quantum theory.

It will be an interesting exercise to find a physically interesting constrained system for which the Cartan subalgebra of the corresponding Lie algebra is two dimensional because in this case it will be possible to obtain two observables of the system. An interesting example would be the study of the free particle motion on a pseudosphere. Preliminary results show that the free Hamiltonian and the constraint generate a six-dimensional Lie algebra that is isomorphic to the D_2 algebra. Work in this direction is in progress. The construction of an operator representation of the infinite-dimensional algebra (3.2) on Hilbert space that will solve, for example, the quantization of the point particle motion on an ellipsoid is a big challenge.

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