Approximate quantum error correction can lead to better codes

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We present relaxed criteria for quantum error correction that are useful when the specific dominant quantum noise process is known. As an example, we provide a four-bit code that corrects for a single amplitude damping error. This code violates the usual Hamming bound calculated for a Pauli description of the error process and has no simple explanation in terms of the usual Pauli basis GF(4) codes. [S1050-2947(97)04410-7]

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I. INTRODUCTION

Quantum error correction is the reversal of part of the adverse changes due to an undesired and unavoidable quantum process. Code criteria for perfect error correction have been developed [1–4] and all presently known codes satisfy these criteria exactly. Furthermore, most of these codes belong to a general GF(4) classification [5]. In these schemes, quantum bit (qubit) errors are described using a Pauli (I,X,Y,Z) error basis and coding is performed to allow correction of arbitrary unknown errors.

However, in the usual case in the laboratory, one works with a specific apparatus with a particular dominant quantum noise process. The current GF(4) quantum codes do not take advantage of this specific knowledge and may thus be suboptimal in terms of transmission rate and code complexity. Unfortunately, there is no general method to construct quantum codes in a non-Pauli basis and few such codes are known [6–10]. The code criteria are generally very difficult to satisfy and without the Pauli basis no way is known to apply classical coding techniques for quantum error correction.

In this paper we develop an alternative approach to quantum error correction based on approximate satisfaction of the existing quantum error correction criteria. These approximate criteria are simpler and less restrictive and in certain cases, such as for amplitude damping, codes can be found relatively more easily. The approximate criteria also admit more codes; therefore, codes requiring shorter block lengths may also be possible.

For example, using this approach we have discovered a four-bit code that corrects for single-qubit *amplitude damping* [10,11] errors. Such a short *nondegenerate* code is impossible using the Pauli basis. The reason is that the effects [12,13] to be corrected in a nondegenerate code have to map the codeword space to orthogonal spaces if the syndrome is to be detected unambiguously. Hence the minimum allowable size for the encoding space is the product of the dimension of the codeword space and the number of effects to be corrected. The single-qubit amplitude damping effect operators expressed in the Pauli basis are

$$A_0=\frac{1}{2}[(1+\sqrt{1-\gamma})I+(1-\sqrt{1-\gamma})\sigma_z], \tag{1}$$

$$A_1=\frac{\sqrt{\gamma}}{2}[\sigma_x+i\sigma_y]. \tag{2}$$

These describe the changes due to the loss of zero or one excitation to the environment. To first order in the scattering probability γ, $n+1$ possible effects may happen to an n-qubit code using the A_0,A_1 error basis, so it follows that $n\geqslant 3$ is required. In contrast, in the Pauli basis, any σ_x or σ_y error must be corrected by the code, so that $2n+1$ possible effects must be dealt with. It follows that at least $n\geqslant 5$ is required for a nondegenerate Pauli basis code, in contrast to the four-bit code that we demonstrate in this paper.

The lessons are that (i) better codes may be found for specific error processes and (ii) approximate error correction simplifies code construction and admits more codes. Approximate error correction is a property with no analog in classical digital error correction because it makes use of slight nonorthogonalities possible only between quantum states. We describe our approach to this problem by first exhibiting our four-bit example code in detail. We then generalize our results to provide relaxed error correction criteria and specific procedures for decoding and recovery. We conclude by discussing possible extensions to our work.

II. FOUR-BIT AMPLITUDE DAMPING CODE

Consider the single-qubit quantum noise process defined by

$$\mathcal{E}(\rho)=\sum_{k=0,1}A_k\rho A_k^\dagger, \tag{3}$$

$$A_0=\begin{bmatrix}1 & 0\\ 0 & \sqrt{1-\gamma}\end{bmatrix},\quad A_1=\begin{bmatrix}0 & \sqrt{\gamma}\\ 0 & 0\end{bmatrix}, \tag{4}$$

known as *amplitude damping*. The probability of losing a photon γ is assumed to be small. To correct errors induced by this process, we encode one qubit using four, with the logical states

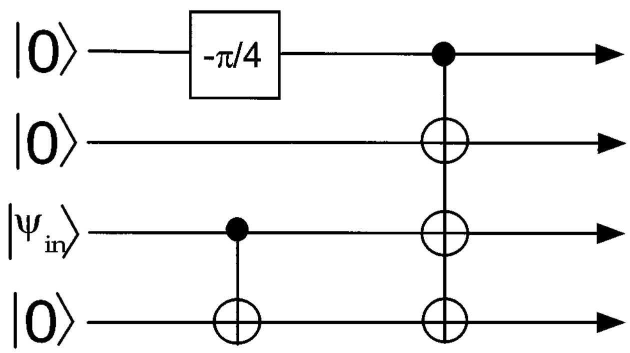

FIG. 1. Circuit for encoding a qubit. The third mode contains the input qubit. The rotation gate in the first qubit performs $\exp(-i\pi\sigma_y/4)$.

$$|0_L\rangle = \frac{1}{\sqrt{2}}[|0000\rangle + |1111\rangle] \tag{5}$$

$$|1_L\rangle = \frac{1}{\sqrt{2}}[|0011\rangle + |1100\rangle]. \tag{6}$$

A circuit for encoding the logical state is shown in Fig. 1.

The possible outcomes after amplitude damping may be written as

$$[\psi_{out}\rangle = \bigoplus_{\tilde{k}} |\phi_{\tilde{k}}\rangle \equiv \bigoplus_{\tilde{k}} A_{\tilde{k}}|\psi_{in}\rangle, \tag{7}$$

where $\tilde{k}$ are strings of 0's and 1's serving as indices for each error, and we use the notation $A_{010\cdots} = A_0 A_1 A_0 \cdots$. $[\rangle$ is convenient shorthand [11] for a mixed state (tensor sum of unnormalized pure states). In the following, the squares of the norm of $|\rangle$ states will give their probabilities for occurring in a mixture. For the input qubit state

$$|\psi_{in}\rangle = a|0_L\rangle + b|1_L\rangle, \tag{8}$$

all possible final states occurring with probabilities $O(\gamma)$ or above are

$$|\phi_{0000}\rangle = a\left[\frac{|0000\rangle + (1-\gamma)^2|1111\rangle}{\sqrt{2}}\right] + b\left[\frac{(1-\gamma)[|0011\rangle + |1100\rangle]}{\sqrt{2}}\right], \tag{9}$$

$$|\phi_{1000}\rangle = \sqrt{\frac{\gamma(1-\gamma)}{2}}\,[a(1-\gamma)|0111\rangle + b|0100\rangle],$$

$$|\phi_{0100}\rangle = \sqrt{\frac{\gamma(1-\gamma)}{2}}\,[a(1-\gamma)|1011\rangle + b|1000\rangle],$$

$$|\phi_{0010}\rangle = \sqrt{\frac{\gamma(1-\gamma)}{2}}\,[a(1-\gamma)|1101\rangle + b|0001\rangle],$$

$$|\phi_{0001}\rangle = \sqrt{\frac{\gamma(1-\gamma)}{2}}\,[a(1-\gamma)|1110\rangle + b|0010\rangle].$$

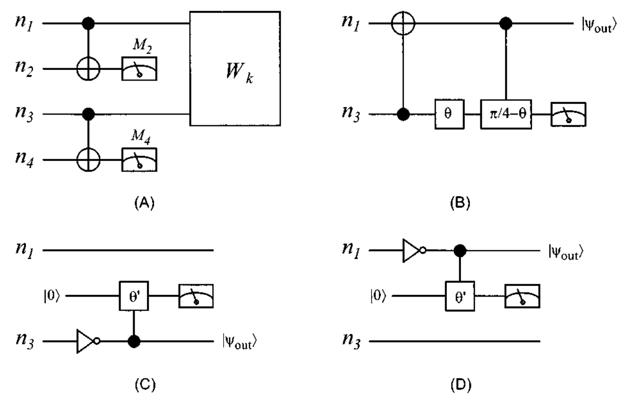

FIG. 2. (a) Circuit for error syndrome detection. The measurement result is used to select W_k out of three actions. If the result (M_2,M_4) is 00, 10, or 01, circuits (b), (c), and (d) are applied, respectively, to recover the state. The angles θ,θ' are given by $\tan\theta=(1-\gamma)^2$ and $\cos\theta'=1-\gamma$. The rotation gate and controlled-rotation gate specified by the angle $\tilde{\theta}$ perform the functions $\exp(i\tilde{\theta}\sigma_y)$ and $\Lambda_1(\exp(i\tilde{\theta}\sigma_y))$, respectively in the notation of [14].

The usual criteria used to study quantum codes (reviewed in Sec. III) require that $\langle 0_L|A_{\tilde{k}}^\dagger A_{\tilde{k}}|0_L\rangle = \langle 1_L|A_{\tilde{k}}^\dagger A_{\tilde{k}}|1_L\rangle$, but

$$\langle 0_L|A_{0000}^\dagger A_{0000}|0_L\rangle = 1 - 2\gamma + 3\gamma^2 + O(\gamma^3), \tag{10}$$

$$\langle 1_L|A_{0000}^\dagger A_{0000}|1_L\rangle = 1 - 2\gamma + \gamma^2. \tag{11}$$

So the code we have constructed does not satisfy the usual criteria. We will demonstrate that the code satisfies relaxed approximate error correction conditions later on and revisit the recovery procedure afterward. First, we exhibit how it works.

A. Decoding and recovery circuit

Let us denote each of the four qubits by $|n_1\rangle,\ldots,|n_4\rangle$. Error correction is performed by distinguishing the five possible outcomes of Eq. (9) and then applying the appropriate correction procedure. The first step is *syndrome calculation*, which may be done using the circuit shown in Fig. 2(a). There are three possible measurement results from the two meters: $(M_2,M_4)=(0,0)$, (1,0), and (0,1). Conditioned on (M_2,M_4), recovery processes W_k implemented by the other three circuits of Fig. 2 can be applied to the output $|n_1n_3\rangle$ from the syndrome calculation circuit.

If $(M_2,M_4)=(0,0)$, then $|n_1n_3\rangle$ is

$$a\left[\frac{|00\rangle + (1-\gamma)^2|11\rangle}{\sqrt{2}}\right] + b\left[\frac{(1-\gamma)(|01\rangle + |10\rangle)}{\sqrt{2}}\right]. \tag{12}$$

To regenerate the original qubit, the circuit of Fig. 2(b) is used: A controlled-NOT is applied using $|n_3\rangle$ as the control, giving

$$a|0\rangle\left[\frac{|0\rangle + (1-\gamma)^2|1\rangle}{\sqrt{2}}\right] + b|1\rangle\left[\frac{(1-\gamma)(|1\rangle + |0\rangle)}{\sqrt{2}}\right]. \tag{13}$$

$|n_1\rangle$ can now be used as a control to rotate $|n_3\rangle$ to be parallel to $|0\rangle$. We obtain as the final output

$$|n_1 n_3\rangle = \left[a\sqrt{\frac{(1-\gamma)^4+1}{2}}\,|0\rangle + b(1-\gamma)|1\rangle \right]|0\rangle \quad (14)$$

$$= [(1-\gamma)(a|0\rangle + b|1\rangle) + O(\gamma^2)|0\rangle]|0\rangle, \quad (15)$$

with the corrected and decoded qubit left in $|n_1\rangle$ as desired.

If $(M_2,M_4)=(1,0)$, the inferred state before syndrome measurement is ϕ_{1000} or ϕ_{0100}. In either case, $|n_1 n_3\rangle$ is in a product state and the third qubit has the distorted state

$$|n_3\rangle = \sqrt{\frac{(1-\gamma)\gamma}{2}}\,[a(1-\gamma)|1\rangle + b|0\rangle]. \quad (16)$$

To undo the distortion, we apply the *nonunitary* transformation in Fig. 2(c). The combined operation on $|n_3\rangle\langle n_3|$ due to the NOT gate, the controlled-rotation gate, and the measurement of the ancilla bit can be expressed in the operator sum representation $\mathcal{N}(\rho)=N_0\rho N_0^\dagger + N_1\rho N_1^\dagger$, where

$$N_0 = |0\rangle\langle 1| + (1-\gamma)|1\rangle\langle 0|, \quad N_1 = \sqrt{\gamma(2-\gamma)}|1\rangle\langle 0|. \quad (17)$$

The N_0 and N_1 operators correspond to measuring the ancilla to be in the $|0\rangle$ and $|1\rangle$ states, respectively. If the ancilla state is $|0\rangle$, we obtain the state

$$|\psi_{out}\rangle = \sqrt{\frac{(1-\gamma)^3\gamma}{2}}\,[a|0\rangle + b|1\rangle] \quad (18)$$

in the third mode because N_0 preferentially damps out the $b|0\rangle$ component in Eq. (16). We get an error message if the ancilla is in the $|1\rangle$ state. Finally, if $(M_2,M_4)=(0,1)$ the same procedure can be applied as in the $(M_2,M_4)=(1,0)$ case, with n_1 and n_3 swapped.

The fidelity, defined as the worst (over all input states) possible overlap between the original qubit and the recovered qubit is

$$\mathcal{F} = (1-\gamma)^2 + 4\left[\frac{(1-\gamma)^3\gamma}{2}\right] = 1 - 5\gamma^2 + O(\gamma^3). \quad (19)$$

Note that the final state Eq. (15) is slightly distorted. This occurs because the recovery operation is not exact, due to the failure to satisfy the code criteria exactly. Furthermore, the circuits in Figs. 2(c) and 2(d) have a finite probability for failure. However, these are second-order problems and do not detract from the desired fidelity order.

III. APPROXIMATE SUFFICIENT CONDITIONS

We now explain why our code works despite its violation of the usual error correction criteria. The reason is simple: Small deviations from the criteria are allowed as long as they do not detract from the desired fidelity order. This section presents a simple generalization of the usual error correction criteria that makes this idea mathematically concrete. These *approximate error correction criteria* are *sufficient* to do an approximate error correction. We expect that a more complete theory giving necessary and sufficient conditions to do approximate error correction is possible, but have not obtained such a theory. Nevertheless, we hope that the simple sufficient conditions presented here will inspire other researchers to develop a general theory of approximate error correction.

Quantum error correction is performed by encoding logical basis states in a subspace $\mathcal{C}$ of the total Hilbert space $\mathcal{H}$ so that some effects of $\mathcal{E}$ can be reversed on $\mathcal{C}$. Let the noise process be described in some *operator sum representation* $\mathcal{E}(\rho)=\Sigma_{n\in\mathcal{K}} A_n\rho A_n^\dagger$, where $\mathcal{K}$ is the index set of $\mathcal{A}$, the set of all effects A_n appearing in the sum. We denote by $\mathcal{A}_{re}\subset\mathcal{A}$ the reversible subset on $\mathcal{C}$ and let $\mathcal{K}_{re}=\{n|A_n\in\mathcal{A}_{re}\}$ be the index set of $\mathcal{A}_{re}$. In other words, the process $\mathcal{E}'(\rho) =\Sigma_{n\in\mathcal{K}_{re}} A_n\rho A_n^\dagger$ is reversible on $\mathcal{C}$. $\mathcal{A}_{re}$ includes all the effects satisfying the condition [1–4]

$$P_C A_m^\dagger A_n P_C = g_{mn} P_C \quad \forall m,n\in\mathcal{K}_{re}, \quad (20)$$

where P_C is the projector onto $\mathcal{C}$ and g_{mn} are entries of a positive matrix. When Eq. (20) is true for some subset of effects A_n that form an operator-sum representation for the operation $\mathcal{E}$, there exists some operator-sum representation of $\mathcal{E}$ such that $\mathcal{E}'(\rho)=\Sigma_{n\in\tilde{\mathcal{K}}_{re}}\tilde{A}_n\rho\tilde{A}_n^\dagger$ and

$$P_C \tilde{A}_m^\dagger \tilde{A}_n P_C = p_n \delta_{mn} P_C \quad \forall m,n\in\tilde{\mathcal{K}}_{re}, \quad (21)$$

where p_n are non-negative c-numbers. Equation (21) [or equivalently Eq. (20)] must be satisfied for some subset of effects $\tilde{A}_n$ [A_n] that form an operator-sum representation for the operation $\mathcal{E}$. We emphasize that the A_n operators in Eq. (20) and the $\tilde{A}_n$ operators in Eq. (21) are not necessarily the same and similar distinction holds for the reversible subsets. For simplicity of notation, we will drop the tilde even when we refer to the operators and reversible subset in Eq. (21).

Equation (21) is equivalent to the usual (exact) *orthogonality* and *nondeformation* conditions for a *nondegenerate* code with logical states $|c_i\rangle$ and these conditions can be stated concisely as, for all i,j,

$$\langle c_i|A_m^\dagger A_n|c_j\rangle = \delta_{ij}\delta_{mn}p_n \quad \forall m,n\in\mathcal{K}_{re}, \quad (22)$$

where p_n are the non-negative c-numbers in Eq. (20) and they represent the *error detection probability* of the effects A_n.

When Eq. (21) is satisfied, the effects A_n have polar decompositions

$$A_n P_C = \sqrt{p_n}U_n P_C \quad \forall n\in\mathcal{K}_{re}, \quad (23)$$

where the U_n's are unitary. Note that $P_C U_n^\dagger U_m P_C = \delta_{nm}P_C$ is required for syndrome detection to be unambiguous. This is simple to see by writing the recovery operation $\mathcal{R}$ as

$$\mathcal{R}(\rho) = \sum_{k\in\mathcal{K}_{re}} R_k\rho R_k^\dagger + P_E\rho P_E, \quad (24)$$

where $R_k = P_C U_k^\dagger$ is the appropriate reversal process for each effect A_k $(k\in\mathcal{K}_{re})$ and $P_E\equiv I-\Sigma_{k\in\mathcal{K}_{re}}U_k P_C U_k^\dagger$. When we apply $\mathcal{R}$ on $\mathcal{E}(|\psi_{in}\rangle\langle\psi_{in}|)$, where $|\psi_{in}\rangle$ is a pure input state, the first term in Eq. (24) is proportional to $|\psi_{in}\rangle\langle\psi_{in}|$

and represents the recovered state. $\mathcal{R}(\rho)$ is trace preserving as a result of orthogonality of different $U_k P_C$.

It is convenient to define $P^{det}=\Sigma_{n\in\mathcal{K}_{re}}p_n$ to be the total detection probability for the reversible subset. It is immediate that to achieve a desired fidelity $\mathcal{F}$, we have to include in $\mathcal{A}_{re}$ a sufficient number of highly probable effects so that $P^{det}\geqslant\mathcal{F}$. Thus $\mathcal{A}_{re}$ is also a high probability subset.

Now we generalize Eqs. (21)–(23) based on the following assumption: The error is parametrized by a certain number of small quantities with physical origins such as the strength and duration of the coupling between the system and the environment. For simplicity, we consider only one-parameter processes and let ϵ be the small parameter. For example, ϵ can be the single-qubit error probability. Suppose the aim is to find a code for a known $\mathcal{E}$ with fidelity

$$\mathcal{F}\geqslant 1-O(\epsilon^{t+1}). \tag{25}$$

In the relaxed criteria, it is still necessary that $P^{det}\geqslant\mathcal{F}$, that is, $\mathcal{A}_{re}$ has to include all effects A_n with maximum detection probability $\max_{|\psi_{in}\rangle\in\mathcal{C}}\operatorname{tr}(|\psi_{in}\rangle\langle\psi_{in}|A_n^\dagger A_n)\approx O(\epsilon^s)$, $s\leqslant t$. However, it is *not* necessary to recover the exact input state; only a good overlap between the input and output states is needed. In terms of the condition on the codeword space, it suffices for the effects to be *approximately* unitary and mutually orthogonal. These observations can be expressed as relaxed *sufficient* conditions for error correction. Suppose

$$A_n P_C = U_n\sqrt{P_C A_n^\dagger A_n P_C} \tag{26}$$

is a polar decomposition for A_n. We define c-numbers p_n and λ_n so that p_n and $p_n\lambda_n$ are the largest and the smallest eigenvalues of $P_C A_n^\dagger A_n P_C$, considered as an operator on $\mathcal{C}$. The relaxed conditions for error correction are that

$$p_n(1-\lambda_n)\leqslant O(\epsilon^{t+1}) \quad \forall n\in\mathcal{K}_{re}, \tag{27}$$

$$P_C U_m^\dagger U_n P_C = \delta_{mn}P_C. \tag{28}$$

Note that when $\lambda_n=1$, Eqs. (26)–(28) reduce to the exact criteria. In the approximate case, $P^{det}=\Sigma_{n\in\mathcal{K}_{re}}\operatorname{tr}(|\psi_{in}\rangle\langle\psi_{in}|A_n^\dagger A_n)$ is not a constant but depends on the input state $|\psi_{in}\rangle$. Since $\mathcal{A}_{re}$ includes enough effects so that $P^{det}\geqslant 1-O(\epsilon^{t+1})$, when Eq. (27) is satisfied, we also have $\Sigma_{n\in\mathcal{K}_{re}}p_n\geqslant 1-O(\epsilon^{t+1})$ and $\Sigma_{n\in\mathcal{K}_{re}}p_n\lambda_n\geqslant 1-O(\epsilon^{t+1})$.

We now prove that $\Sigma_{n\in\mathcal{K}_{re}}p_n\lambda_n$ is a lower bound on the fidelity. Defining the *residue operator*

$$\pi_n=\sqrt{P_C A_n^\dagger A_n P_C}-\sqrt{p_n\lambda_n}P_C, \tag{29}$$

we find, for the operator norm of π_n,

$$0\leqslant|\pi_n|\leqslant\sqrt{p_n}-\sqrt{p_n\lambda_n} \tag{30}$$

and

$$A_n P_C = U_n(\sqrt{p_n\lambda_n}I+\pi_n)P_C. \tag{31}$$

The sufficiency of our conditions to obtain the desired fidelity may be proved as follows. Though Eq. (23) is not satisfied, as long as Eq. (28) is true, we can still define the *approximate* recovery operation

$$\mathcal{R}(\rho)=\sum_{k\in\mathcal{K}_{re}}R_k\rho R_k^\dagger+P_E\rho P_E, \tag{32}$$

with $R_k=P_C U_k^\dagger$ as the *approximate* recovery operation for A_k and P_E defined as in the case of exact error correction. For a pure input state $|\psi_{in}\rangle\langle\psi_{in}|$, applying $\mathcal{R}(\rho)$ on $\mathcal{E}(|\psi_{in}\rangle\langle\psi_{in}|)$ and ignoring the last term that is positive definite produces an output with fidelity

$$\begin{aligned}\mathcal{F}&\equiv\min_{|\psi_{in}\rangle\in\mathcal{C}}\operatorname{tr}[|\psi_{in}\rangle\langle\psi_{in}|\mathcal{R}(\mathcal{E}(|\psi_{in}\rangle\langle\psi_{in}|))]\\ &\geqslant\min_{|\psi_{in}\rangle\in\mathcal{C}}\sum_{k,n\in\mathcal{K}_{re}}|\langle\psi_{in}|U_k^\dagger A_n|\psi_{in}\rangle|^2.\end{aligned} \tag{33}$$

Omitting all terms for which $k\neq n$ and applying Eq. (31) gives

$$\mathcal{F}\geqslant\min_{|\psi_{in}\rangle\in\mathcal{C}}\sum_{n\in\mathcal{K}_{re}}|\langle\psi_{in}|\sqrt{p_n\lambda_n}+\pi_n|\psi_{in}\rangle|^2\geqslant\sum_{n\in\mathcal{K}_{re}}p_n\lambda_n, \tag{34}$$

where in the last step we have used Eq. (30). Hence the fidelity is at least $\Sigma_{n\in\mathcal{K}_{re}}p_n\lambda_n\geqslant 1-O(\epsilon^{t+1})$ and the desired fidelity order is achieved as claimed.

An explicit procedure for performing this recovery is as follows. First, a measurement of the projectors $P_k\equiv U_k P_C U_k^\dagger$ is performed. Conditional on the result k of the measurement, the unitary operator U_k is applied to complete the recovery.

Note that when the exact criteria hold, it is not necessary for the set $\mathcal{A}$ to form an operator sum representation for the error process; they may instead give a generalized description of the quantum process such as

$$\mathcal{E}(\rho)=\sum_{mn\in\mathcal{K}}A_m\rho A_n^\dagger\chi_{mn}, \tag{35}$$

where χ_{mn} is a matrix of c-numbers [15]. In this case, approximate criteria can be obtained straightforwardly along the lines we have described for the operator sum representation (where χ_{mn} is diagonal). The main issue is to ensure that interference terms coming from off-diagonal terms in χ_{mn} are sufficiently small. We will not describe this calculation here. Instead, we return to our four-bit code and analyze it using the relaxed criteria.

IV. FOUR-BIT CODE REVISITED

In terms of the approximate quantum error correction criteria Eqs. (27) and (28), we may understand why our four-bit amplitude damping quantum code works as follows. Let the orthonormal basis for the Hilbert space be ordered as

$$|0000\rangle,|0011\rangle,|1100\rangle,|1111\rangle,|0111\rangle,|0100\rangle,\ \ldots\ . \tag{36}$$

The projection operator $|0_L\rangle\langle 0_L|+|1_L\rangle\langle 1_L|$ onto the two-dimensional codeword space $\mathcal{C}$ is

$$P_C = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \tag{37}$$

where we include only the first few nonzero rows and columns in the above matrix. We are interested in the restriction of the effect operators A_k to $\mathcal{C}$; therefore, we will exhibit only the rows and columns in the matrices that have nontrivial contributions to $A_k P_C$. The first effect operator (no loss of a quantum to the environment) is

$$A_{0000} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 \\ 0 & 0 & 1-\gamma & 0 \\ 0 & 0 & 0 & (1-\gamma)^2 \end{bmatrix}. \tag{38}$$

The eigenvalues of $P_C A_{0000}^\dagger A_{0000} P_C$ are $(1-\gamma)^2$ and $\frac{1}{2}[1+(1-\gamma)^4]$. Interested readers can check for themselves that

$$A_{0000} P_C = U_{0000}\{(1-\gamma)I + (\gamma^2 + O(\gamma^4))]\tilde{\pi}_{0000}\} P_C \tag{39}$$

(the order of γ in π_{0000} is factored out of $\tilde{\pi}_{0000}$) with the choice

$$U_{0000} = \begin{bmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & 0 & 0 & -\sin\left(\theta - \frac{\pi}{4}\right) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\left(\theta - \frac{\pi}{4}\right) & 0 & 0 & \cos\left(\theta - \frac{\pi}{4}\right) \end{bmatrix}, \tag{40}$$

$$\tilde{\pi}_{0000} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{41}$$

where $\tan\theta = (1-\gamma)^2$ and again we have truncated the matrices U_{0000} and $\tilde{\pi}_{0000}$ to the first few rows and columns contributing nontrivially to the restriction to $\mathcal{C}$. The exact quantum error correction criteria are not satisfied, as $P_C A_{0000}^\dagger A_{0000} P_C$ has different eigenvalues. However, the difference is of $O(\gamma^2)$ and thus the relaxed condition Eq. (27) is satisfied.

For the second effect (loss of one quantum from the n_1 mode), we have

$$A_{1000} = (1-\gamma)^{1/2}\sqrt{\gamma}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

The eigenvalues of $P_C A_{1000}^\dagger A_{1000} P_C$ are $\gamma(1-\gamma)$ and $\gamma(1-\gamma)^3$. The difference is $(2\gamma^2 - \gamma^3)(1-\gamma)$. We have the decomposition

$$A_{1000} P_C = \sqrt{\frac{(1-\gamma)\gamma}{2}}\, U_{1000}[(1-\gamma)I + \gamma\tilde{\pi}_{1000}] P_C, \tag{43}$$

$$U_{1000} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{\pi}_{1000} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{44}$$

Other one-loss cases are similar. For different k, the $U_k P_C$ matrices have nonzero entries in different rows and are orthogonal to each other. Hence all the approximate code criteria are satisfied. Using these explicit matrices, we obtain

$$\mathcal{R}(\mathcal{E}(|\psi_{in}\rangle\langle\psi_{in}|)) \approx \sum_{k\in\mathcal{K}_{re}} P_C U_k^\dagger A_k |\psi_{in}\rangle\langle\psi_{in}| A_k^\dagger U_k P_C$$
$$= (1-3\gamma^2)|\psi_{in}\rangle\langle\psi_{in}| + \cdots . \tag{45}$$

The fidelity is thus at least $1-3\gamma^2$ and is of the desired order.

The recovery procedure suggested in Eq. (45) contrasts with the decoding and recovery circuits in Sec. II A. It is an interesting exercise to check that the composition of the operations in Figs. 2(a) and 2(b), followed by re-encoding the recovered qubit has the same effect on $\mathcal{C}$ as applying $U^{\dagger}_{0000}$ for recovery. For the case in which an emission occurs in the first qubit, the composition of operations in Figs. 2(a) and 2(c), followed by re-encoding the recovered qubit, has the same effect on $\mathcal{C}$ as preferentially damp out the $|n_3\rangle=|0\rangle$ component followed by applying $U^{\dagger}_{1000}$ for recovery. Note that it costs $2\gamma^2$ in the fidelity for removing the distortion.

V. APPLICATIONS TO OTHER CODES

Our approximate criteria may also be used to simplify code construction using a non-Pauli error description basis. For example, consider the bosonic quantum codes for amplitude damping [10] (these are codes which utilize bosonic states $|0\rangle,\dots,|n\rangle$ instead of qubits). For logical states $|c_1\rangle,|c_2\rangle,\dots$ of the form

$$|c_l\rangle=\sqrt{\mu_1}|n_{11}n_{12}\cdots n_{1m}\rangle+\sqrt{\mu_2}|n_{21}n_{22}n_{2m}\rangle+\cdots + \sqrt{\mu_{N_l}}|n_{N_l1}n_{N_l2}\cdots n_{N_lm}\rangle, \tag{46}$$

the original nondeformation conditions for correcting up to one photon loss require the following to be constant for all logical states:

$$\langle c_l|A_0^{\dagger}A_0|c_l\rangle=\sum_{i=1}^{N_l}(1-\gamma)^{RS_i}\mu_i, \tag{47}$$

$$\langle c_l|A^{\dagger}_{0\cdots1\cdots0}A_{0\cdots1\cdots0}|c_l\rangle=\sum_{i=1}^{N_l}(1-\gamma)^{RS_i-1}\gamma\mu_i n_{ij}. \tag{48}$$

In the above, $RS_i=\Sigma_{j=1}^{m}n_{ij}$ is the total number of excitation in the ith quasiclassical state (QCS) in $|c_l\rangle$. It is difficult to find a solution for Eq. (48) when RS_i is not constant for all i. That is the reason why previous work [6,9] suggests using only QCSs with the same number of excitations.

With the relaxed criteria, it suffices for the following to be constant for all logical states:

$$\sum_{i=1}^{N_l}\gamma\mu_i n_{ij} \quad \forall j. \tag{49}$$

That is, instead of requiring the individual number of excitations in each QCS to be balanced, it is sufficient to balance the *average* number of excitations over the QCS in each codeword. This provides an alternative explanation of why the known five-bit perfect quantum codes [3,16]

$$|0_L\rangle=|00000\rangle+|11000\rangle-|10011\rangle-|01111\rangle+|11010\rangle +|00110\rangle+|01101\rangle+|10101\rangle,$$
$$|1_L\rangle=|11111\rangle-|00011\rangle+|01100\rangle-|10000\rangle-|00101\rangle +|11010\rangle+|10010\rangle-|01010\rangle \tag{50}$$

work for amplitude damping errors [as described in Eq. (4)]: Although the codes do not satisfy the exact nondeformation criterion, Eq. (48) for the error representation of Eqs. (1) and (2), they do satisfy the *approximate* ones leading to Eq. (49).

VI. CONCLUSION

We have shown by an example that choosing an appropriate error basis can potentially reduce the number of qubits and other requirements in coding schemes. We also suggested a method to enable code construction without the Pauli basis to be done more easily. We believe that much more can be done along these two directions. The relaxed sufficient criteria for error correction are far from being necessary. Our results show that it is worthwhile to look for better approximate conditions or conditions that are both necessary and sufficient. Approximate error correction is particularly interesting because it is a property with no analog in classical digital error correction, as it makes use of slight nonorthogonalities possible only between quantum states. It also extends the current scope of quantum error correction, which is closely related to classical digital error correction. It would be especially useful to develop a general framework for constructing codes based on approximate conditions, similar to the group-theoretic framework now used to construct codes that satisfy the exact conditions.

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