Dynamics of dissipative two-level systems in the stochastic approximation

L. Accardi,[1,*] S. V. Kozyrev,[2,†] and I. V. Volovich[3,‡]
[1]*Centro Vito Volterra, Università di Roma, Tor Vergata, 00133 Rome, Italy*
[2]*Institute of Chemical Physics, Kossygina Street 4, 117334 Moscow, Russia*
[3]*Steklov Mathematical Institute of Russian Academy of Sciences, Gubkin Street 8, 117966 Moscow, Russia*
(Received 4 March 1997)

The dynamics of the spin-boson Hamiltonian is considered in the stochastic approximation. The Hamiltonian describes a two-level system coupled to an environment, and is widely used in physics, chemistry, and the theory of quantum measurement. We demonstrate that the method of stochastic approximation, a general method for consideration of dynamics of an arbitrary system interacting with environment, is powerful enough to reproduce qualitatively the striking results by Leggett *et al.* [Rev. Mod. Phys. **59**, 1 (1987)] found earlier for this model system. The results include an exact expression of the dynamics in terms of the spectral density, and show the appearance of two interesting regimes for the system, i.e., pure oscillating and pure damping ones. Correlators describing the environment are also computed. [S1050-2947(97)09409-2]

PACS number(s): 03.65.−w

I. INTRODUCTION

The so-called spin-boson Hamiltonian is widely used in physics and chemistry. In its simplest version it describes a dynamical model of a two-level system coupled to an environment. One of the basic ideas is that the environment induces dissipative effects, but, as we shall see, the picture is much richer. Examples include the motion of defects in some crystalline solids, the motion of the magnetic flux trapped in a rf superconducting quantum interference device ring, some chemical reactions, some approaches to the theory of quantum measurement, and many other approaches quoted in the survey paper [1] which inspired the present work. The "spin-boson" Hamiltonian considered in the present paper is the same as considered in Ref. [1], i.e.,

$$H_\lambda = -\tfrac{1}{2}\Delta\sigma_x + \tfrac{1}{2}\varepsilon\sigma_z + \int dk\ \omega(k)a^\dagger(k)a(k) + \lambda\sigma_z[A(g^*)+A^\dagger(g)], \qquad (1.1)$$

where σ_x and σ_z are Pauli matrixes, and ε and Δ are real parameters interpreted, respectively, as the energy difference of the states localized in the two wells in absence of the tunneling and as the matrix element for tunneling between the wells. We set $\Delta>0$ and denote

$$A^\dagger(g) = \int a^\dagger(k)g(k)dk, \quad A(g^*) = \int a(k)g^*(k)dk,$$

where $a(k)$, and $a^\dagger(k)$ are bosonic annihilation and creation operators,

$$[a(k), a^\dagger(k')] = \delta(k-k'),$$

which describe the environment.

We denote $\omega(k)$ the one-particle energy of the environment, and assume $\omega(k)\geqslant 0$. The function $g(k)$ is a form factor describing the interaction of the system with the environment, and λ is the coupling constant. It is well known that, in times of order t/λ^2, the interaction produces effects of order t. Thus λ provides a natural time scale for the observable effects of the interaction system environment.

Reference [1] showed a very rich behavior of the dynamics of Hamiltonian (1.1) ranging from undamped oscillations, to exponential relaxation, to power-law types of behavior and to total localization. Leggett *et al.* [1] found the remarkable result that the main qualitative features of the system dynamics can be described in terms of the temperature (i.e., the initial state of the environment) and of the behavior, for low frequencies ω, of the spectral function

$$J(\omega) := \int dk |g(k)|^2 \delta[\omega(k)-\omega]. \qquad (1.2)$$

The goal of this paper is to investigate the dynamics of the Hamiltonian (1.1) in the so-called *stochastic approximation*. The overall qualitative picture emerging from this approach is similar to the one described in Ref. [1], and in some cases the quantitative agreement is also good (cf. Sec. IV).

II. STOCHASTIC APPROXIMATION

The basic idea of the stochastic approximation is the following. If one has a Hamiltonian of the form

$$H_\lambda = H_0 + \lambda V, \qquad (2.1)$$

then, by definition, the stochastic limit of the evolution operator

$$U^{(\lambda)}(t) = e^{itH_0}e^{-itH_\lambda} \qquad (2.2)$$

is the following limit [when it exists in the sense specified by Eqs. (2.11) and (2.12) below]:

*Electronic address: accardi@volterra.mat.utovrm.it
†Electronic address: kozyrev@genesis.mian.su
‡Electronic address: volovich@genesis.mi.ras.ru

$$U(t)=\lim_{\lambda\to\infty}U^{(\lambda)}\left(\frac{t}{\lambda^2}\right). \tag{2.3}$$

Notice that, on the right-hand side of Eq. (2.3), there is not $U_t^{(\lambda)}$ but its rescaled version $U^{(\lambda)}(t/\lambda^2)$. Thus the limiting evolution operator $U(t)$, Eq. (2.3), describes the behavior of the model in the time scale described in the introduction. The stochastic approximation is a natural generalization of the Friedrichs–van Hove limit which uses the same time rescaling but allows one to compute only vacuum expectation values of the form $\langle U_{t/\lambda^2}^{(\lambda)}\Lambda U_{t/\lambda^2}^{(\lambda)*}\rangle$ for particular classes of observables Λ. This leads to *irreversible evolution* and to the corresponding *master equation*. Conversely, the stochastic approximation leads to *reversible, unitary evolution* and to the corresponding *quantum stochastic differential equation* from which the master equation is deduced by a now standard procedure which consists of integrating away the environment degrees of freedom.

The stochastic approximation to the original dynamics (2.2) consists in the computation of limit (2.3) in the sense of matrix elements over some states ψ_λ (called ''collective states'') which themselves depend on the parameter λ in a singular way. The fact that one cannot expect limit (2.3) to exist for arbitrary states, but only for a carefully chosen class of states, was already pointed out in the classical paper of van Hove [2]. The effective determination of this class of states was obtained in Ref. [3].

The stochastic approximation could also be considered a new kind of semiclassical approximation in the sense that it studies the *fluctuations* around the classical solution and not the approximation of it. This interpretation however shall not be discussed here (cf. Ref. [4]).

One of the important features of the stochastic method is its *universality*. The restriction to Pauli matrixes in Eq. (1.1) is unnecessary: the theory is applicable whenever the evolution operator $U^{(\lambda)}(t)$, Eq. (2.2), satisfies the equation

$$\frac{dU^{(\lambda)}(t)}{dt}=-i\lambda V(t)U^{(\lambda)}(t), \tag{2.4}$$

where $V(t)=e^{itH_0}Ve^{-itH_0}$ has the form

$$V(t)=\sum_\alpha\,[D_\alpha^\dagger\otimes A_\alpha(t)+D_\alpha\otimes A_\alpha^\dagger(t)], \tag{2.5}$$

and D_α are operators describing the system. The rescaled evolution operator $U^{(\lambda)}(t/\lambda^2)$, associated with Eq. (2.5), satisfies the equation

$$\frac{dU^{(\lambda)}\left(\frac{t}{\lambda^2}\right)}{dt}=-i\sum_\alpha\left[D_\alpha^\dagger\otimes\frac{1}{\lambda}A_\alpha\left(\frac{t}{\lambda^2}\right)+D_\alpha\otimes\frac{1}{\lambda}A_\alpha^\dagger\left(\frac{t}{\lambda^2}\right)\right]U^{(\lambda)}(t/\lambda^2). \tag{2.6}$$

In the spin-boson Hamiltonian (1.1), D_α are Pauli matrixes [cf. formulas (3.10b), (3.12b), and (3.8)], and

$$A_\alpha(t)=\int a(k)e^{-it\omega_\alpha(k)}g^*(k)dk, \tag{2.7a}$$

where the functions $\omega_\alpha(k)$ have the form

$$\omega_\alpha(k)=\omega(k)-\omega_\alpha,\quad \alpha=1,2,3. \tag{2.7b}$$

Here $\omega(k)$ is as in Eq. (1.1), and ω_α are characteristic frequencies given formula (3.12a).

From Eq. (2.7) it is clear that to have a nontrivial limit for $U^{(\lambda)}(t/\lambda^2)$, the limit

$$\lim_{\lambda\to 0}\frac{1}{\lambda}A_\alpha\left(\frac{t}{\lambda^2}\right)=b_\alpha(t) \tag{2.8}$$

should exist. It can be proved (cf. Ref. [4]) that limit (2.8) exists for ''good'' functions $\omega_\alpha(k)$ and $g(k)$ in the sense that

$$\lim_{\lambda\to 0}\left\langle\frac{1}{\lambda}A_{\alpha_1}^{\varepsilon_1}\left(\frac{t_1}{\lambda^2}\right)\cdots\frac{1}{\lambda}A_{\alpha_n}^{\varepsilon_n}\left(\frac{t_n}{\lambda^2}\right)\right\rangle=\langle b_{\alpha_1}^{\varepsilon_1}(t_1)\cdots b_{\alpha_n}^{\varepsilon_n}(t_n)\rangle, \tag{2.9}$$

where the indices ε_i label the creators ($\varepsilon=0$) and the annihilators ($\varepsilon=1$); the brackets in Eq. (2.9) denote mean values over the Fock vacuum or a temperature state, and, for each α, $b_\alpha(t)$ is the Fock Boson quantum field described by Eqs. (2.14) and (2.15) below. In the literature, δ-correlated (in time) quantum fields are often called *quantum noises*. In the present paper only quantum white noises shall appear, but it is important to keep in mind that many other possibilities can arise from different physical models.

From Eq. (2.6), in the limit $\lambda\to 0$ one has,

$$\frac{dU(t)}{dt}=-i\sum_\alpha\,[D_\alpha^\dagger\otimes b_\alpha(t)+D_\alpha\otimes b_\alpha^\dagger(t)]U(t). \tag{2.10}$$

The limit (2.3) means that

$$\lim_{\lambda\to 0}\left\langle\Psi_\lambda,U^{(\lambda)}\left(\frac{t}{\lambda^2}\right)\Psi'_\lambda\right\rangle=\langle\psi,U(t)\psi'\rangle, \tag{2.11}$$

where the collective vectors Ψ_λ are defined by

$$\Psi_\lambda=\frac{1}{\lambda}A_{\alpha_1}^\dagger\left(\frac{t_1}{\lambda^2}\right)\cdots\frac{1}{\lambda}A_{\alpha_n}^\dagger\left(\frac{t_n}{\lambda^2}\right)\Psi^{(0)}, \tag{2.12}$$

and converge to the corresponding n-particle vectors in the noise space, given by

$$\psi=b_{\alpha_1}^\dagger(t_1)\cdots b_{\alpha_n}^\dagger(t_n)\psi^{(0)}. \tag{2.13}$$

$\Psi^{(0)}$ and $\psi^{(0)}$ are the vacuum vectors in the corresponding Fock spaces. If $\omega_\alpha\neq\omega_\beta$ for $\alpha\neq\beta$ [as is the case for Hamiltonian (1.1)] then $b_\alpha,b_\alpha^\dagger$ satisfy the commutation relations

$$[b_\alpha(t),b_{\alpha'}^\dagger(t')]=\delta_{\alpha\alpha'}J_\alpha\delta(t-t'), \tag{2.14}$$

where J_α is the spectral function (1.2),

$$J_\alpha = 2\pi \int dk |g(k)|^2 \delta[\omega_\alpha(k)]. \tag{2.15}$$

Thus, as announced in Sec. I, in the stochastic limit the spectral function emerges naturally as the covariance of the quantum noise. Some care is needed in the interpretation of Eq. (2.10) because, as is clear from Eq. (2.14), $b_\alpha(t)$ are not *bonafide* operators but only operator-valued distributions. In order to give a meaning to Eq. (2.10) (more precisely, to its matrix elements in the n-particle or coherent vectors), we rewrite Eq. (2.10) in normal form by bringing $b_\alpha(t)$ to the right of $U(t)$. This gives rise to a commutator which can be explicitly computed. The result is

$$\frac{dU(t)}{dt} = -i\sum_\alpha [D_\alpha^\dagger U(t) b_\alpha(t) + D_\alpha b_\alpha^\dagger(t) U(t) - i\gamma_\alpha D_\alpha^\dagger D_\alpha U(t)], \tag{2.16}$$

where γ_α are complex numbers given explicitly by

$$\gamma_\alpha = \int_{-\infty}^0 d\tau \int dk\, e^{i\tau\omega_\alpha(k)} |g(k)|^2. \tag{2.17}$$

The connection between the constants γ_α in the last term in Eq. (2.16) (the Ito correction term), and the spectral function (2.15) is obtained by exchanging the $d\tau$ and the dk integral in Eq. (2.17) and using the known formula

$$\int_{-\infty}^0 e^{it\omega} dt = \pi\delta(\omega) - i\mathrm{P}\,\frac{1}{\omega},$$

where P denotes the principal part integral. This shows that *the spectral functions are the real parts of the constants* γ_α, *emerging in the Ito correction term*. This connection is the prototype of the *dispersion relations* widely used in quantum physics since its origins. Since γ_α are complex, Eq. (2.16) *looks like* an equation driven by a non-self-adjoint Hamiltonian. However, this is only an apparent phenomenon due to the normal order. The true Hamiltonian (2.10), although singular, is formally self-adjoint, and this gives an intuitive explanation of the unitarity of the solution of Eq. (2.16) or, equivalently, of Eq. (2.10).

Relations (2.14)–(2.17) define the stochastic approximation to the system (2.4). The term *stochastic* is justified by the fact that the distribution equation (2.16), which has a weak meaning in the n-particle vectors, can be interpreted as a quantum stochastic differential equation (and, in fact, it is in this form that this equation was first derived [3]). The operators $b_\alpha(t)$, $b_\alpha^\dagger(t)$ are called a *quantum white noise*, and the additional term in Eq. (2.16), arising in Eq. (2.16) from normal order, is called the *drift* or the *Ito correction term*. More precisely, in quantum probability one usually writes Eq. (2.16) in the form

$$dU(t) = -i\sum_\alpha [D_\alpha^\dagger dB_\alpha(t) + D_\alpha dB_\alpha^\dagger(t) - i\gamma_\alpha D_\alpha^\dagger D_\alpha dt] U(t), \tag{2.18}$$

where

$$dB_\alpha(t) = \int_t^{t+dt} b_\alpha(\tau) d\tau$$

are called stochastic differentials and satisfy the Ito table

$$dB_t dB_t^\dagger = 2\gamma dt, \quad dt dB_t^\dagger = dB_t dB_t = dB_t^\dagger dB_t^\dagger = dB_t^\dagger dB_t = 0. \tag{2.19}$$

The proof of the Ito table (2.19), as well as its rigorous meaning, was first established in Ref. [5]. This was subsequently applied to several models in quantum optics in Refs. [3] and [4]. Using it, the unitarity of the solution of Eq. (2.18) is easily established.

The advantage of Eq. (2.16) over the original one [Eq. (2.4)] is that it is in some sense completely integrable, and one can easily read the physics from it. For example for the vacuum expectation value one has the equation

$$\frac{d\langle U(t)\rangle}{dt} = -\sum_\alpha \gamma_\alpha D_\alpha^\dagger D_\alpha \langle U(t)\rangle,$$

which gives the damped oscillatory regime (the γ_α are complex number)

$$\langle U(t)\rangle = e^{-\Gamma t}, \quad \Gamma = \sum_\alpha \gamma_\alpha D_\alpha^\dagger D_\alpha.$$

In the following we shall apply this method to the Hamiltonian (1.1) and, in Sec. IV we shall compare our result with those of Refs. [1, 6].

III. STOCHASTIC APPROXIMATION FOR THE "SPIN-BOSON" SYSTEM

In order to apply the stochastic approximation to Hamiltonian (1.1), we write Eq. (1.1) in the form of Eq. (2.1), where

$$H_0 = H_S + H_R. \tag{3.1}$$

The system Hamiltonian H_S is

$$H_S = -\tfrac{1}{2}\Delta\sigma_x + \tfrac{1}{2}\varepsilon\sigma_z \tag{3.2}$$

and the reservoir Hamiltonian H_R is

$$H_R = \int dk\, \omega(k) a^\dagger(k) a(k). \tag{3.3}$$

The evolution operator $U^{(\lambda)}(t)$ satisfies Eq. (2.4), where

$$V(t) = \sigma_z(t)[A(e^{-it\omega}g^*) + A^\dagger(e^{i\omega}g)] \tag{3.4}$$

and

$$\sigma_z(t) = e^{itH_S}\sigma_z e^{-itH_S}. \tag{3.5}$$

To bring Eq. (3.4) into the form of Eq. (2.5), let us compute Eq. (3.5). The eigenvalues of the Hamiltonian (3.2) are

$$H_S|e_\pm\rangle = \lambda_\pm |e_\pm\rangle, \tag{3.6}$$

where

$$\lambda_\pm = \pm\tfrac{1}{2}\Delta\nu \tag{3.7}$$

$$|e_{\pm}\rangle=\frac{1}{\sqrt{1+\mu_{\mp}^2}}\begin{pmatrix}1\\ \mu_{\mp}\end{pmatrix} \tag{3.8}$$

and

$$\mu_{\pm}=\frac{\varepsilon}{\Delta}\pm\nu,\quad \nu=\left[1+\left(\frac{\varepsilon}{\Delta}\right)^2\right]^{1/2}. \tag{3.9}$$

Notice, for future use, that

$$\langle e_{\pm}|\sigma_z|e_{\pm}\rangle=\frac{1-\mu_{\mp}^2}{1+\mu_{\mp}^2},\quad \langle e_{+}|\sigma_z|e_{-}\rangle=\langle e_{-}|\sigma_z|e_{+}\rangle=1/\nu.$$

Therefore

$$\sigma_z(t)=\frac{1-\mu_-^2}{1+\mu_-^2}DD^\dagger+\frac{1-\mu_+^2}{1+\mu_+^2}D^\dagger D+\nu^{-1}e^{it\nu\Delta}D+\nu^{-1}e^{-it\nu\Delta}D^\dagger, \tag{3.10a}$$

where

$$D=|e_+\rangle\langle e_-|. \tag{3.10b}$$

The interaction Hamiltonian (3.4) can now be written in the form of Eq. (2.5):

$$V(t)=\sum_{\alpha=1}^{3}(D_\alpha^\dagger\otimes A(e^{-it\omega_\alpha}g^*)+\text{H.c.}), \tag{3.11}$$

where the three spectral frequencies correspond, respectively, to the down, zero, and up transitions of the two-level system, i.e.,

$$\omega_1(k)=\omega(k)-\nu\Delta,\quad \omega_2(k)=\omega(k),\quad \omega_3(k)=\omega(k)+\nu\Delta, \tag{3.12a}$$

$$D_1=\nu^{-1}D^\dagger,\quad D_2=\frac{1-\mu_-^2}{1+\mu_-^2}DD^\dagger+\frac{1-\mu_+^2}{1+\mu_+^2}D^\dagger D,$$

$$D_3=\nu^{-1}D^\dagger. \tag{3.12b}$$

The corresponding limiting evolution equation therefore has the form of Eq. (2.16). It is important to note however that the constants (2.15) for $\alpha=2$, 3 vanish, i.e.,

$$J_2=J_3=0. \tag{3.13}$$

We shall see that the purely oscillatory regime, first discovered by Leggett *et al.* [1], corresponds to the case when J_1 also vanishes. In this sense it can be interpreted as an off-resonance regime. In this regime a strange (from the point of view of stochastic theory) new phenomenon take place: in the t/λ^2 limit the environment disappears (i.e., the limit on the right-hand side of Eq. (2.8) is zero, corresponding to a quantum white noise of zero variance). However, a remnant of the interaction remains because, after the limit, the system evolves with a new Hamiltonian, equal to the old one plus a shift term depending on the interaction and on the initial state of the field. This is a kind of *Cheshire cat effect*.

Equation (3.13) implies that the operators b_2 and b_3 should be absent in Eq. (2.16). However, the constants γ_2 and γ_3 as well as γ_1 do contribute to Eq. (2.17). We denote $b_1(t)$ by $b(t)$. Thus the operators $b(t)$, $b^\dagger(t)$ satisfy

$$[b(t),b^\dagger(t')]=\gamma\delta(t-t'), \tag{3.14}$$

with γ given by Eq. (3.16) below and ν (in γ) given by Eq. (3.9). The limiting evolution equation can then be written

$$\frac{dU(t)}{dt}=Db^\dagger(t)U(t)-D^\dagger U(t)b(t)-(\gamma+i\sigma)D^\dagger DU(t)-i\varphi U(t), \tag{3.15}$$

where

$$\gamma=\nu^{-2}\pi J(\nu\Delta),$$

$$\sigma=\nu^{-2}[I(-\nu\Delta)-I(\nu\Delta)]+\left[\left(\frac{1-\mu_-^2}{1+\mu_-^2}\right)^2-\left(\frac{1-\mu_+^2}{1+\mu_+^2}\right)^2\right]I(0), \tag{3.16}$$

$$\varphi=\nu^{-2}I(-\nu\Delta)+\left(\frac{1-\mu_-^2}{1+\mu_-^2}\right)^2I(0),$$

and we denote

$$J(\omega)=\int dk|g(k)|^2\delta[\omega(k)-\omega],$$

$$I(\omega)=\mathrm{P}\int_0^\infty\frac{d\omega' J(\omega')}{\omega'-\omega}, \tag{3.17}$$

where P means the principal part of the integral.

In the notations of quantum stochastic equations (3.15) reads

$$dU(t)=[DdB_t^\dagger-D^\dagger dB_t-(\gamma+i\sigma)D^\dagger D-i\varphi]U(t). \tag{3.18}$$

Notice that all parameters γ, σ, and φ in the evolution equation (3.15) are expressed in terms of the spectral density $J(\omega)$, Eq. (3.17), and parameters Δ and ε of the original Hamiltonian (1.1).

IV. ANALYSIS OF THE STOCHASTIC APPROXIMATION. ZERO TEMPERATURE

Let us discuss now in more detail the implications of the results of the previous sections for the ''spin-boson'' Hamiltonian. All the information about the model is encoded into the constants γ, σ, and φ, and these constants are expressed in terms of the spectral density $J(\omega)$, Eq. (3.17), depending on the parameters of the Hamiltonian (ε and Δ) and the temperature (not yet introduced up to now). Thus the method of stochastic approximation confirms the conclusion of Leggett *et al.* [1] that the long-time behavior of the model is expressed in terms of the spectral density $J(\omega)$.

Now let us discuss the dynamics of the system in the stochastic approximation. We are interested in a pure damping or pure oscillating behavior.

For the vacuum expectation value, we have

$$\langle U(t)\rangle = e^{-i\varphi t} + e^{-i\varphi t}(e^{-(\gamma+i\sigma)t}-1)D^\dagger D \tag{4.1}$$

and, taking the trace over the spin variables, one obtains (since $\mathrm{Tr}D^\dagger D=1$)

$$\langle \mathrm{tr}U(t)\rangle = e^{-[\gamma+i(\sigma+\varphi)]t}. \tag{4.2}$$

Since γ, σ, and φ are real [cf. Eqs. (3.15)–(3.17)], one has a purely oscillating behavior, Eq. (4.2), if and only if there is no damping, i.e.,

$$\gamma = 0. \tag{4.3}$$

However, one cannot have a vanishing of oscillations, because the quantity

$$\sigma + \varphi = \nu^{-2} I(-\nu\Delta) + \left(\frac{1-\mu^2}{1+\mu^2}\right)^2 I(0) > 0 \tag{4.4}$$

is strictly positive for positive $J(\omega)$ [because ν, $\Delta > 0$, cf. Eq. (3.9)] and $I(\omega)$ is given by Eq. (3.17).

The stochastic approximation to the vacuum expectation value of the Heisenberg evolution of σ_z is given by

$$P(t) = \langle U^*(t)\sigma_z(t)U(t)\rangle. \tag{4.5}$$

From Eq. (3.15), one obtains the Langevin equation for $P(t)$, whose solution is

$$P(t) = \nu^{-1}e^{-\gamma t}(D^\dagger e^{i(\sigma-\nu\Delta)t} + De^{-i(\sigma-\nu\Delta)t}) + D^\dagger D\left(\frac{1-\mu_+^2}{1+\mu_+^2} - \frac{1-\mu_-^2}{1+\mu_-^2}\right)e^{-2\gamma t} + \frac{1-\mu_-^2}{1+\mu_-^2}. \tag{4.6}$$

Let us discuss separately the simplest case $\varepsilon=0$.

A. Case $\varepsilon=0$, zero temperature

In this case, one has

$$P(t) = e^{-\gamma t}(D^\dagger e^{i(\sigma-\Delta)t} + De^{-i(\sigma-\Delta)t}), \tag{4.7}$$

where γ, σ, and $I(\omega)$ are now

$$\gamma = \pi J(\Delta), \quad \sigma = I(-\Delta) - I(\Delta), \quad I(\omega) = \mathrm{P}\int \frac{d\omega' J(\omega')}{\omega'-\omega}.$$

Two interesting regimes can now appear.

(i) No oscillations. In this case,

$$\sigma - \Delta = 0. \tag{4.8}$$

Equation (4.8) is equivalent to the integral equation

$$\int \frac{dxJ(x)}{x+\Delta} - \mathrm{P}\int \frac{dx\, J(x)}{x-\Delta} = \Delta. \tag{4.9}$$

If Eq. (4.9) is satisfied, then we have pure damping:

$$P(t) = e^{-\gamma t}(D^\dagger + D). \tag{4.10}$$

We will discuss solutions of Eq. (4.9) later.

(ii) Pure oscillations. This regime is defined by the condition

$$\gamma = \pi J(\Delta) = 0. \tag{4.11}$$

Notice that, because of Eq. (3.17) this condition defines an *off-resonance condition*. If Eq. (4.11) satisfied, then

$$P(t) = D^\dagger e^{i(\sigma-\Delta)t} + De^{-i(\sigma-\Delta)t}, \tag{4.12}$$

where

$$\sigma - \Delta = \int \frac{dxJ(x)}{x+\Delta} - \mathrm{P}\int \frac{dxJ(x)}{x-\Delta} - \Delta.$$

This case of pure oscillations is very interesting. If there is a damping, then after a rather short time $P(t)$ becomes a small quantity which is difficult to observe. The case of permanent oscillations looks more promising for observations. This regime is of primary interest in the context of the so-called macroscopic quantum coherence phenomenon [7].

The purely oscillatory regime was discovered in Ref. [1], but the region of parameters there is different from ours. To get pure oscillations we need the only off-resonance condition (4.11), i.e., in terms of the spectral density, what we need is

$$J(\Delta) = \int dk|g(k)|^2\delta[\omega(k)-\Delta] = 0.$$

The difference from Ref. [1] can be attributed to the different boundary conditions on correlators.

Let us present our results on the computation of the correlator

$$C(t) = \tfrac{1}{2}\langle\{U_t^*\sigma_z U_t, \sigma_z\}\rangle = \tfrac{1}{2}\{P(t), P(0)\}.$$

We have

$$C(t) = \tfrac{1}{2}e^{-(\gamma+i\sigma+i\nu\Delta)t}\Bigg[\nu^{-2} + \nu^{-1}D\left(\frac{1-\mu_-^2}{1+\mu_-^2} + \frac{1-\mu_+^2}{1+\mu_+^2}\right) + \mathrm{H.c.} + \frac{1-\mu_-^2}{1+\mu_-^2}\left(\nu^{-1}(D+D^\dagger) + \frac{1-\mu_-^2}{1+\mu_-^2}DD^\dagger + \frac{1-\mu_+^2}{1+\mu_+^2}D^\dagger D\right)\Bigg] + e^{-2\gamma}\left(\frac{1-\mu_+^2}{1+\mu_+^2} - \frac{1-\mu_-^2}{1+\mu_-^2}\right) \times\left(\nu^{-1}(D+D^\dagger) + 2D^\dagger D\frac{1-\mu_+^2}{1+\mu_+^2}\right). \tag{4.13}$$

The trace of $C(t)$ is

$$\mathrm{tr}C(t) = 2\nu^{-2}e^{-\gamma t}\cos(\sigma+\nu\Delta)t + 2e^{-2\gamma t}\left(\frac{1-\mu_+^2}{1+\mu_+^2} - \frac{1-\mu_-^2}{1+\mu_-^2}\right)\left(\frac{1-\mu_+^2}{1+\mu_+^2}\right).$$

The qualitative behavior of $C(t)$ is like that for $P(t)$.

B. Nonzero temperature

For a nonzero temperature we get a stochastic evolution equation of the same form as above Eq. (3.15), only with new constants γ, σ and φ. More precisely,

$$\gamma = \nu^{-2}\pi[J_+(\nu\Delta)+J_-(\nu\Delta)],$$

$$\sigma = \left[\left(\frac{1-\mu_+^2}{1+\mu_+^2}\right)^2 - \left(\frac{1-\mu_-^2}{1+\mu_-^2}\right)^2\right][I_+(0)+I_-(0)] + \nu^{-2}[I_+(-\nu\Delta)-I_+(\nu\Delta)+I_-(-\nu\Delta)-I_-(\nu\Delta)],$$

where the spectral densities are

$$J_+(\omega) = \frac{J(\omega)}{1-e^{-\beta\omega}}, \quad J_-(\omega) = \frac{J(\omega)e^{-\beta\omega}}{1-e^{-\beta\omega}}.$$

Here $J(\omega)$ is the spectral density (3.17), and β is the inverse temperature.

The functions $I_\pm(\omega)$ are defined by

$$I_\pm(\omega) = \mathrm{P}\int \frac{d\omega' J_\pm(\omega')}{\omega'-\omega}.$$

One has the same as for the zero-temperature expressions (4.12) and (4.13) for $P(t)$ and $C(t)$, but now with new constants γ and σ depending on temperature:

$$\gamma = \nu^{-2}\pi J(\nu\Delta)\coth\frac{\beta\nu\Delta}{2}.$$

V. CONCLUSION

To conclude, the following main result are obtained: The theoretical role of the spectral function is explained through its emergence from a canonical limit procedure. Moreover, this function is shown to be real part of a complex function whose imaginary part defines an energy shift in the system Hamiltonian. When the environment free energy depends only on the modulus of momentum $[\omega(k)=\omega(|k|)$ in Eq. (1.1)] the real and imaginary parts of this function are related by a Hilbert transform, thus making a bridge with the standard *dispersion relations* (cf. Sec. II).

In the stochastic approach not only the Heisenberg equation of the system observables is controlled, but also the environment evolution. It is shown that the environment converges to a *quantum noise* (a master field, in particle physicist terminology). This gives a theoretical (i.e., based on a microscopic Hamiltonian description) foundation to the use of classical of quantum noises widely used in several contemporary approaches to quantum measurement theory [1,7]. We can compute the limit matrix elements of Heisenberg evolution for arbitrary n-particle or coherent vectors. The vacuum matrix elements give rise to the master equation. The control of the other matrix elements is a new feature of the stochastic approach.

The purely oscillatory regime, discovered by Leggett *et al.* [1], is related here to a *Cheshire cat effect* in which the environment variables vanish in the limit but, the interaction leaves a track in the system behavior in the form of an operator shift in the system Hamiltonian (cf. Sec. IV).

ACKNOWLEDGMENTS

S.K. and I.V. are grateful to the V. Volterra Center of the Rome University, Tor Vergata, where this work was performed, for the hospitality. S.K. was supported in part by Grant No. RFFI N 95-03-08838. I.V. was supported in part by Grant No. RFFI N 960100312.

[1] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).
[2] L. van Hove, Physica (Amsterdam) **21**, 617 (1955).
[3] L. Accardi, A. Frigerio, and Y. G. Lu, Commun. Math. Phys. **131**, 537 (1990).
[4] L. Accardi, Y. G. Lu, and I. Volovich, *Quantum Theory and Its Stochastic Limit* (Oxford University Press, London, in press).
[5] R. L. Hudson and K. R. Parthasaraty, Commun. Math. Phys. **93**, 301 (1984).
[6] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981).
[7] A. J. Leggett and Anupam Garg, Phys. Rev. Lett. **54**, 857 (1985).